PARAMETRIC CURVES IN PLANE

To parametrize a curve in $\mathbb{R}^2$ we provide

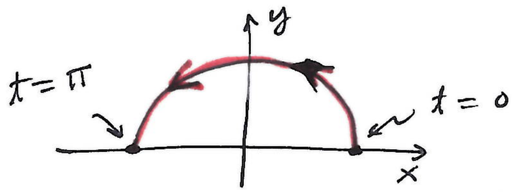
$$\boxed{\left.\begin{aligned} x &= f_1(t) \\ y &= g_2(t) \end{aligned}\right\} \text{ for all } t \in I \subseteq \mathbb{R}}$$

where I is the domain of the parametrization.

In other words, $\vec{r} : I \longrightarrow \mathbb{R}^2$ where $\text{dom}(\vec{r}) = I$ and $\vec{r} = (f_1, f_2)$ defines a parametrized curve or path which covers $\vec{r}(I) = \{ \vec{r}(t) = (f_1(t), f_2(t)) \mid t \in I \}$.

$\boxed{E1}$ $\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right\}$ $0 \leq t \leq \pi$

parametrizes a circle radius 1, well to be precise it only covers the upper-half since $0 \leq \sin t \leq 1$ for $0 \leq t \leq \pi$



Notice $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$.

In CARTESIAN FORM; $x^2 + y^2 = 1$ and $y \geq 0$.

Def^n/ A level curve or implicitly defined curve C is given by an equation $F(x, y) = L$. If $x = f_1(t)$, $y = f_2(t)$ has $F(f_1(t), f_2(t)) = L$ $\forall t \in I$ then $\vec{r} = (f_1, f_2) : I \longrightarrow \mathbb{R}^2$ parametrizes C. Notice

$$C = F^{-1}\{L\} = \{ (x, y) \in \mathbb{R}^2 \mid F(x, y) = L \}$$

level function

Def^{n} A graph $y = f(x)$ for $x \in \mathrm{dom}(f)$ is
$$\{ (x,y) \mid y = f(x),\ x \in \mathrm{dom}(f) \} \subseteq \mathbb{R}^2$$

We can view a graph either implicitly (2) or explicitly (1),

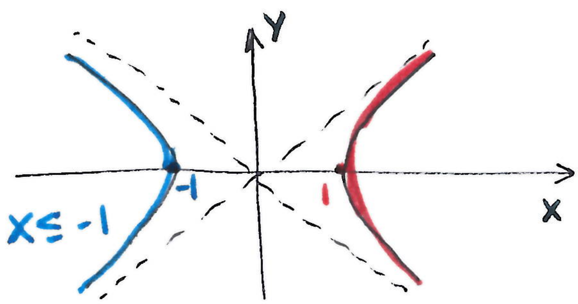
(1) $x = t$
$y = f(t)$
$t \in \mathrm{dom}(f)$

in words, a graph is a parametrization where we use x as the parameter

(2) $F(x,y) = y - f(x)$ gives $F(x,y) = 0$ as a level-curve formulation of $y = f(x)$

Generally, a graph is a parametrization where we use one of the Cartesian coordinates as a parameter.

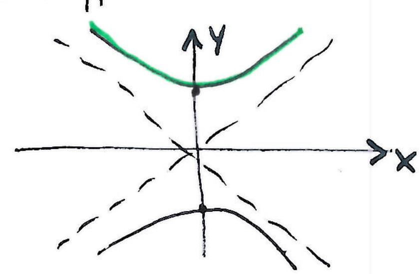
$\boxed{E2}$ $x^2 - y^2 = 1$ is a hyperbola



$\boxed{\begin{aligned} x &= \cosh t \\ y &= \sinh t \\ t &\in \mathbb{R} \end{aligned}}$ parametrizes the right $(x \geq 1)$ branch

$\boxed{\begin{aligned} x &= -\cosh t \\ y &= \sinh t \\ t &\in \mathbb{R} \end{aligned}}$ parametrizes left $(x \leq -1)$ branch of the hyperbola above.

$\boxed{E3}$ $y^2 - x^2 = 1$ is also a hyperbola

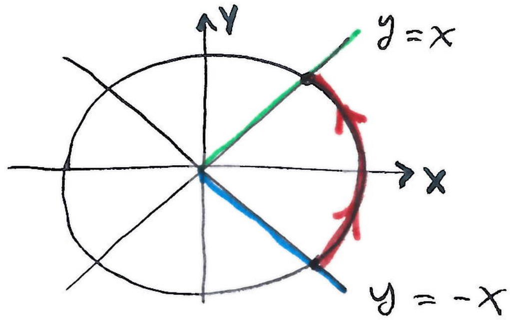


$\boxed{\begin{aligned} y &= \cosh t \\ x &= \sinh t \\ t &\in \mathbb{R} \end{aligned}}$ param. upper branch of hyperbola.

$\boxed{E4}$ $y = \sqrt{1 + x^2}$
$x \in \mathbb{R}$ describes upper-branch of $\boxed{E3}$ as a <u>graph</u> of $f(x) = \sqrt{1+x^2}$.

$\boxed{E5}$ $x^2+y^2=4$ and $x \geq 0$ with $-x \leq y \leq x$

gives quarter-circle of radius 2,



parametrically, $\begin{cases} x = 2\cos t \\ y = 2\sin t \\ -\pi/4 \leq t \leq \pi/4 \end{cases}$

$\boxed{E6}$ Line-Segment from $\overbrace{(P_1, P_2)}^{P}$ to $\overbrace{(Q_1, Q_2)}^{Q}$

given by $\vec{r}(t) = P + t(Q-P)$ using vector notation.

Notice $\vec{r}(0) = P$ whereas $\vec{r}(1) = P + (Q-P) = Q$.

Here $\underbrace{Q - P = \overrightarrow{PQ}}_{\text{direction vector of path } t \longmapsto \vec{r}(t).}$ of high-school geometry.

$\boxed{E7}$ Line-segment from $(2,3)$ to $(8,6)$ given by

$\vec{r}(t) = (2,3) + t\big((8,6) - (2,3)\big) = (2,3) + t(6,3)$

thus $\vec{r}(t) = (2+6t, 3+3t)$ which we typically

denote by $\begin{cases} x = 2+6t \\ y = 3+3t \\ 0 \leq t \leq 1 \end{cases}$

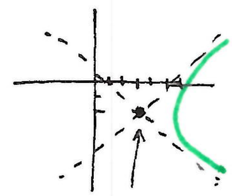
Remark: I've much more written in my Lecture Notes & your text offers further insight...

$\boxed{E8}$ find Cartesian form of curve given by

$x = 3 + \cosh(t)$ and $y = -2 + \sinh t$.

I remember $\cosh^2 t - \sinh^2 t = 1$. Hence study $x - 3 = \cosh t$ and $y + 2 = \sinh t$ to see

$$\boxed{(x-3)^2 - (y+2)^2 = 1}$$



to be precise, $x - 3 = \cosh t \geq 1$ so the boxed eq$^{\underline{n}}$ has the condition $\boxed{x \geq 4}$ attached.

$\boxed{E9}$ Again find Cartesian form of path $x = e^{2t}$

$y = e^{3t} - 7$.

This time I notice common e^t term;

$$x = e^{2t} = (e^t)^2 \xrightarrow{\text{cubing}} x^3 = (e^t)^6$$

$$y = e^{3t} - 7 \Rightarrow (e^t)^3 = y + 7 \xrightarrow[\text{squaring}]{} (y+7)^2 = (e^t)^6$$

thus $\boxed{(y+7)^2 = x^3}$. I can even find a graph of this path, $y + 7 = \pm\sqrt{x^3}$ $\therefore$ $\boxed{y = -7 + \sqrt{x^3}}$

(must choose + since $y > -7$ is clear from given parametrization) (since $y + 7 = e^{3t} \geq 0$)