EXAMPLES OF PARTIAL FRACTIONS

1.) To calculate $\int \frac{dx}{x^2-3x+2}$ we use a bit of <u>algebra</u> known as <u>partial fractions</u>. Factor $x^2-3x+2 = (x-1)(x-2)$ then we seek A, B constants s.t.

$$\frac{1}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

multiply by $x^2-3x+2 = (x-1)(x-2)$,

$$\frac{x^2-3x+2}{x^2-3x+2} = \left(\frac{A}{x-1} + \frac{B}{x-2}\right)(x-1)(x-2)$$

$$\Rightarrow \quad 1 = A(x-2) + B(x-1)$$

$$\underline{x=2} \quad 1 = A(0) + B(2-1) \quad \therefore \quad \underline{B = 1}.$$

$$\underline{x=1} \quad 1 = A(1-2) + B(1-1) \quad \therefore \quad \underline{A = -1}.$$

Thus
$$\underbrace{\frac{1}{x^2-3x+2}}_{\text{no clue how to integrate}} = \underbrace{\frac{-1}{x-1} + \frac{1}{x-2}}_{\text{easily integrated by } u=x-1 \text{ and } u=x-2 \text{ subst.}}$$

To conclude,

$$\int \frac{dx}{x^2-3x+2} = \int \left[\frac{1}{x-2} - \frac{1}{x-1}\right] dx$$

$$= \boxed{\ln|x-2| - \ln|x-1| + C}$$

<u>Remark</u>: $\int \left(\frac{3x+10}{x^2-3x+2}\right) dx$ could be calculated in the same fashion as 1. The partial fractions $\frac{A}{x-1} + \frac{B}{x-2}$ is given to us by the structure of the <u>denominator</u>. As long as degree (numerator) ≤ 1 we can use $\frac{A}{x-1} + \frac{B}{x-2}$ for $\frac{\text{numerator}}{x^2-3x+2}$.

2.) $$\frac{3x+2}{x^3-4x^2} = \frac{3x+2}{x^2(x-4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-4}$$

Multiplying by denominator yields,

$$: \; 3x+2 = Ax(x-4) + B(x-4) + Cx^2$$

$\boxed{0=x}$ $$2 = A(0) - 4B + C(0) \Rightarrow \underline{B = -1/2}.$$

$\boxed{4=x}$ $$14 = A(0) + B(0) + 16C \Rightarrow \underline{C = \frac{14}{16} = \frac{7}{8}}.$$

$\boxed{1=x}$ $$5 = A(1)(-3) + B(-3) + C$$

$$\Rightarrow 3A = C - 3B - 5$$

$$\Rightarrow \underline{A = \frac{1}{3}\left(\frac{7}{8} + \frac{3}{2} - 5\right) = \frac{-7}{8}}.$$

Consequently,

$$\frac{3x+2}{x^3-4x^2} = \frac{-7}{8x} - \frac{1}{2x^2} + \frac{7}{8(x-4)}$$

With this algebra in place we can integrate $\frac{3x+2}{x^3-4x^2}$,

$$\int \left(\frac{3x+2}{x^3-4x^2}\right) dx = \int \left(\frac{-7}{8x} - \frac{1}{2x^2} + \frac{7}{8(x-4)}\right) dx$$

$$= \boxed{\frac{-7}{8}\ln|x| + \frac{1}{2x} + \frac{7}{8}\ln|x-4| + C}$$

3.) $$\frac{8x+x^2+2}{x^3+4x^2+5x} = \frac{8x+x^2+2}{x(x^2+4x+5)} = \frac{8x+x^2+2}{x[\underbrace{(x+2)^2+1}_{\text{irreducible quadratic.}}]}$$

Thus we use following partial fractions decomposition,

$$\frac{8x+x^2+2}{x((x+2)^2+1)} = \frac{A}{x} + \frac{B(x+2)+C}{(x+2)^2+1}$$

Multiply by denominator,

$$8x+x^2+2 = A((x+2)^2+1) + (B(x+2)+C)x$$

Often evaluation at appropriate x yield good eq^{n}'s for A, B, C,

$\underline{x=0}$ $\quad 2 = 5A + (2B+C)(0) \Rightarrow \underline{A = 2/5}.$

$\underline{x=-2}$ $\quad -16+4+2 = A(0+1) + (B(0)+C)(-2)$

$$\Rightarrow -10 = A - 2C$$

$$\Rightarrow 2C = A+10 = \frac{2}{5}+10 = \frac{52}{5} \quad \therefore \underline{C = \frac{26}{5}}.$$

I usually pick another value which makes nice arithmetic, here $x=1$ seems wise,

$\underline{x=1}$ $\quad 11 = A(10) + (B(3)+C)(1)$

$$\Rightarrow 3B = 11 - 10A - C$$

$$\Rightarrow \underline{B = \frac{1}{3}\left(11 - 10\left(\frac{2}{5}\right) - \frac{26}{5}\right) = \frac{3}{5}}.$$

Thus,

$$\int \frac{8x+x^2+2}{x^3+4x^2+5x}\,dx = \int\left(\frac{2}{5x} + \frac{\frac{3}{5}(x+2)+\frac{26}{5}}{(x+2)^2+1}\right)dx$$

$$= \frac{2}{5}\ln|x| + \frac{3}{5}\int\frac{(x+2)\,dx}{(x+2)^2+1} + \frac{26}{5}\int\frac{dx}{(x+2)^2+1}$$

$$= \frac{2}{5}\ln|x| + \frac{3}{10}\int\frac{dw}{w} + \frac{26}{5}\int\frac{du}{u^2+1} \qquad \begin{pmatrix} w = (x+2)^2+1, & u = x+2 \\ dw = 2(x+2)dx, & du = dx \end{pmatrix}$$

$$= \boxed{\frac{2}{5}\ln|x| + \frac{3}{10}\ln|x^2+4x+5| + \frac{26}{5}\tan^{-1}(x+2) + C}$$

4.) The last example was nicer in the integration than is generally the case.

$$\int \frac{(x+13)\,dx}{x^2+6x+25} = \int \frac{(x+13)\,dx}{(x+3)^2+16}$$

$$= \int \frac{(x+3)\,dx}{\underbrace{(x+3)^2+16}_{w}} + \int \frac{10\,dx}{(x+3)^2+16}$$

$dw = 2(x+3)dx$

$x+3 = 4\tan\theta$ & $dx = 4\sec^2\theta\,d\theta$

$(x+3)^2+16 = 16(\tan^2\theta+1) = 16\sec^2\theta$

$$= \int \frac{dw}{2w} + \int \frac{10(4\sec^2\theta\,d\theta)}{16\sec^2\theta}$$

$$= \frac{1}{2}\ln|w| + \frac{5}{2}\int d\theta$$

$$= \frac{1}{2}\ln|x^2+6x+25| + \frac{5}{2}\theta + C$$

$\tan\theta = \frac{x+3}{4}$

$$= \boxed{\frac{1}{2}\ln|x^2+6x+25| + \frac{5}{2}\tan^{-1}\left(\frac{x+3}{4}\right) + C}$$

5.) $$\frac{3x+17}{2x^2+12x+22} = \frac{3x+17}{2(x^2+6x+11)}$$

$$= \frac{3x+17}{2((x+3)^2+2)}$$

$$= \frac{1}{2}\left[\frac{3(x+3-3)+17}{(x+3)^2+2}\right]$$

$$= \frac{1}{2}\left[\frac{3(x+3)+8}{(x+3)^2+2}\right]$$

$x+3 = \sqrt{2}\tan\theta$

$$\int \frac{(3x+17)\,dx}{2x^2+12x+22} = \frac{3}{4}\ln|x^2+6x+11| + 4\int \frac{dx}{(x+3)^2+2}$$

$$= \boxed{\frac{3}{4}\ln|x^2+6x+11| + 2\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C}$$

<u>Remark</u>: I've shown some detailed complete sol^{n}s in 1 – 5. Now let me just indicate <u>how</u> to set-up the partial fractions w/o doing the complete algebra or integration. The point here is you <u>could</u> do the algebra to find A, B, C, ... and then you <u>could</u> integrate the resulting expressions. That said, we just SET-UP PARTIAL FRACTAL DECOMPOSITION WITHOUT EXPLICITLY FINDING A, B, C etc. in what follows.

6.) $$\frac{13x^2+2x+1}{x^4-16} = \frac{13x^2+2x+1}{(x^2-4)(x^2+4)} = \frac{13x^2+2x+1}{\underbrace{(x+2)(x-2)(x^2+4)}_{\text{Completely factored over } \mathbb{R}}}$$

$$\boxed{\frac{13x^2+2x+1}{x^4-16} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+4}}$$

7.) $$\frac{3x^3-7}{(x^2-1)^2} = \frac{3x^3-7}{(x-1)^2(x+1)^2} = \boxed{\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}}$$

8.) $$\frac{x^2+14+x^3}{(x^2+2x+1)(x^2+3x+2)} = \frac{x^3+x^2+14}{(x+1)^2(x+1)(x+2)}$$

$$= \frac{x^3+x^2+14}{(x+1)^3(x+2)} = \boxed{\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2}}$$

9.) $$\frac{3x^3+10}{x^4+2x^2+1} = \frac{3x^3+10}{(x^2+1)^2} = \boxed{\frac{A+Bx}{x^2+1} + \frac{C+Dx}{(x^2+1)^2}}$$

<u>Remark</u>: $(x^2+1)^2$ and $(x+1)^2$ give <u>very</u> different terms. Compare 9. and 7.

10.) $$\frac{3x+2}{x-3} = \frac{3(x-3+3)+2}{x-3} = \frac{3(x-3)}{x-3} + \frac{11}{x-3} = 3 + \frac{11}{x-3}$$

Alternatively, you could use long division to derive the result above. Notice,

$$\int\left(\frac{3x+2}{x-3}\right)dx = \int\left(3 + \frac{11}{x-3}\right)dx$$
$$= \boxed{3x + 11\ln|x-3| + C}$$

11.) If the degree of the numerator is larger than or matching the degree of the denominator then long-division is 1st required before partial fractions can be arranged. For example,

$$\frac{2x^3+3 \leftarrow \text{degree } 3}{x^2-3x+2 \leftarrow \text{degree } 2}$$

$$\begin{array}{r|l} & 2x + 6 \\ x^2-3x+2 & 2x^3 + 3 \\ & \underline{2x^3 - 6x^2 + 4x} \\ & \quad 6x^2 - 4x + 3 \\ & \quad \underline{6x^2 - 18x + 12} \\ & \qquad 14x - 9 \end{array}$$

remainder term has degree smaller than the divisor.

Thus,

$$\frac{2x^3+3}{x^2-3x+2} = 2x + 6 + \underbrace{\frac{14x-9}{x^2-3x+2}}_{\text{we use par. frac. on this.}}$$

11.) continued,

$$\frac{14x-9}{x^2-3x+2} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$\Rightarrow \quad 14x-9 = A(x-2) + B(x-1)$$

$\underline{x=2}$ $\quad 28-9 = A(0) + B(2-1) \Rightarrow \underline{B = 17}$.

$\underline{x=1}$ $\quad 14-9 = A(-1) + B(0) \Rightarrow \underline{A = -5}$.

Put it all together,

$$\int\left(\frac{2x^3+3}{x^2-3x+2}\right)dx = \int\left(2x+6-\frac{5}{x-1}+\frac{17}{x-2}\right)dx$$

$$= \boxed{x^2+6x-5\ln|x-1|+17\ln|x-2|+C}$$

12.) $$\frac{3x^5+x^2}{x^2+x} = \frac{x^2(3x^3+1)}{x(x+1)} = \frac{x(3x^3+1)}{x+1} = \frac{3x^4+x}{x+1}$$

$$\begin{array}{r|l} & 3x^3-3x^2+3x-2 \\ \hline x+1 & 3x^4+x \\ & \underline{3x^4+3x^3} \\ & -3x^3+x \\ & \underline{-3x^3-3x^2} \\ & 3x^2+x \\ & \underline{3x^2+3x} \\ & -2x \\ & \underline{-2x-2} \\ & 2 \end{array}$$

$$\Rightarrow \frac{3x^5+x^2}{x^2+x} = 3x^3-3x^2+3x-2+\frac{2}{x+1}$$

$$\therefore \int\frac{3x^5+x^2}{x^2+x}\,dx = \int\left(3x^3-3x^2+3x-2+\frac{2}{x+1}\right)dx$$

$$= \boxed{\frac{3}{4}x^4-x^3+\frac{3}{2}x^2-2x+2\ln|x+1|+C}$$