Differential Equations in Physics

(I.)

$$\boxed{\text{Schrödinger's Eq}^{\underline{n}}: \quad \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi = i\hbar \frac{\partial \Psi}{\partial t}} \quad \boxed{\text{Eq}^{\underline{n}}\ 1}$$

is the starting point for non relativistic Quantum Mechanics here the sol$^{\underline{n}}$ $\Psi(x,t)$ is the wave function. This is a partial differential eq$^{\underline{n}}$ because there are at least two independent variables x and t. There is a large class of seperable sol$^{\underline{n}}$'s where

$$\Psi(x,t) = \psi(x)\,\varphi(t)$$

If we substitute into Eq$^{\underline{n}}$ 1 we get

$$\frac{-\hbar^2}{2m}\psi''\varphi + V\psi\varphi = i\hbar\psi\varphi'$$

Divide by $\psi\varphi$ to get

$$\underbrace{\frac{-\hbar^2}{2m}\frac{\psi''}{\psi} + V}_{\text{function of } x} = \underbrace{i\hbar\frac{\varphi'}{\varphi}}_{\text{function of } t} = \text{constant} \equiv E$$

So we get two ordinary differential eq$^{\underline{n}}$'s

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \quad : \text{Time independent Schrödinger Eq}^{\underline{n}}$$

$$i\hbar\frac{d\varphi}{dt} = E\varphi \quad : \text{Energy Eigenvalue Eq}^{\underline{n}}$$

we'll learn how to solve these soon

My Point? At some point it usually ends up being a problem of solving ODEs. (ma 341 = ODEs)

(II) Maxwell's Eq^{n}'s:

$$\nabla \cdot \vec{E} = \rho/\epsilon_0 \quad : \text{Gauss' Law}$$

$$\nabla \cdot \vec{B} = 0 \quad : \text{No magnetic monopoles}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad : \text{Faraday's Law}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad : \text{Ampere's Law}$$

Again these are Partial Differential eq^{n}'s since the sol^{n}'s are the electric and magnetic fields $\vec{E}(x,y,z,t)$ and $\vec{B}(x,y,z,t)$. If you study a particular problem these will reduce to several Ordinary Differential Eq^{n}'s.

Example: In vacuum $\rho = J = 0$. Maxwell's Eq^{n}'s can be rewritten as $\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ and $\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$. Each of these is 3 PDE's and in case you're wondering $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\nabla^2 E_x = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad \text{for } \vec{E} = (E_x, E_y, E_z)$$

$$\nabla^2 E_y = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

$$\nabla^2 E_z = \mu_0 \epsilon_0 \frac{\partial^2 E_z}{\partial t^2}$$

Lets pick one and pretend our system is one-dimensional so $\nabla^2 = \partial^2/\partial x^2$ (the real story is more complicated)

We should solve

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$$

The way to solve this is to suppose $E(x,t) = X(x)T(t)$

(II) Continued, we're trying to solve $\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2}$

we **guessed** $E(x,t) = X(x)\,T(t)$ so substitute into

$$X''(x)\,T(t) = \mu_0 \epsilon_0\, X(x)\,T''(t)$$

Divide by $X(x)\,T(t)$,

$$\underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x} = \underbrace{\mu_0 \epsilon_0 \frac{T''(t)}{T(t)}}_{\text{function of } t} = \text{constant} = a$$

So we have to solve a pair of Ordinary Differential Eq^{n}'s

$$\frac{d^2 X}{dx^2} = a X \quad \& \quad \frac{d^2 T}{dt^2} = \frac{a}{\mu_0 \epsilon_0}$$

If $a = -k^2$ we find sol^{n}'s, let $\omega \equiv k/\sqrt{\mu_0 \epsilon_0}$

$$X(x) = A_1 \cos(kx + \varphi_1)$$

$$T(t) = A_2 \cos(\omega t + \varphi_2)$$

Then a little algebra reveals

$$\boxed{E(x,t) = A \cos(kx - \omega t)}$$

This is a wave with speed $V = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = C = 3 \times 10^8 \frac{m}{s}$.

MAXWELL'S EQ^{n}S $\Rightarrow$ SPEED OF LIGHT IS CONSTANT!

(III) $\boxed{\text{Newton's Law: } m\frac{d^2\vec{r}}{dt^2} = \vec{F}_{net}}$ (assuming $\frac{dm}{dt} = 0$)

In this case we have <u>one</u> independent variable, the time t. Again to solve particular problem it boils down to solving ODEs.

<u>Example</u>: Gravity near earth's surface.

$$m\frac{d^2\vec{r}}{dt^2} = -mg\hat{k} \quad \longrightarrow \quad \left.\begin{aligned} mx'' &= 0 \\ my'' &= 0 \\ mz'' &= -mg \end{aligned}\right\} \text{three ordinary diff. eq}^{\underline{o}}\text{s.}$$

<u>QUESTION</u>: where will we use ordinary differential eq$^{\underline{o}}$s in the real world?

<u>ANSWER</u>: MOST EVERYWHERE THAT AN EQUATION CAN MODEL AN ACTIVITY. MY EXAMPLES ARE FROM FOUNDATIONAL PHYSICS, BUT THERE ARE MANY MANY MORE EXAMPLES IN APPLICATIONS OF PHYSICS, ENGINEERING, BIOLOGY, ETC... SO ORDINARY DIFFERENTIAL EQUATIONS ARE BASIC KNOWLEDGE YOU SHOULD ADD TO YOUR ARSENAL OF MATHEMATICAL TOOLS.