Printed Name: ______________________________.

PHYSICS 331	MISSION 8: RETARDED E&M TO RADIATION (38+2)PTS)

Problems mentioned refer to David Griffith's *Introduction to Electrodynamics*, 5th edition. Skip two problems. Each problem worth 5pts.

Problem 78 Problem 10.13 (Coulomb's Law holding in a particular context)

Problem 79 Problem 10.14 (Biot-Savart Law approximately holding)

Problem 80 Problem 10.17 (deriving Lienard Weichart potential for constant velocity charge)

Problem 81 Problem 10.20 (algebra to support conclusions of §10.3)

Problem 82 Problem 10.21 (electric and magnetic fields of moving charge)

Problem 83 Problem 10.22 (moving flux calculation!)

Problem 84 Problem 10.26 (summary diagram)

Problem 85 Problem 11.1 (retarded dipole problem)

Problem 86 Problem 11.14 (lifetime of classical electron)

Problem 87 Problem 11.18(a) (point charge on a spring)

Mission 8 Solution

P78 Problem 10.13

Suppose $\vec{J}(\vec{r})$ is constant in time then $\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)t$

Show that
$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}',t)}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'$$
that is, Coulomb's Law holds with the charge density evaluated at nonretarded time.

Let's use Jefimenko's Eq$^{\underline{n}}$, where $t_r = t - \mathscr{r}/c$

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',t_r)}{c\mathscr{r}}\hat{\mathscr{r}} - \cancelto{0}{\frac{\dot{\vec{J}}(\vec{r}',t_r)}{c^2\mathscr{r}}}\right]d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{r}',0) + \dot{\rho}(\vec{r}',0)t_r}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',0)}{c\mathscr{r}}\hat{\mathscr{r}}\right]d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{\mathscr{r}^2}\left(\rho(\vec{r}',0) + \dot{\rho}(\vec{r}',0)\left(t - \frac{\mathscr{r}}{c}\right)\right)\hat{\mathscr{r}} + \frac{\dot{\rho}(\vec{r}',0)}{c\mathscr{r}}\hat{\mathscr{r}}\right]d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}',0) + \dot{\rho}(\vec{r}',0)t}{\mathscr{r}^2}\, d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{r}',t)}{\mathscr{r}^2}\hat{\mathscr{r}}\, d\tau'.$$

$\boxed{P79}$ Problem 10.14

Suppose current density changes slowly enough to approximate

$$\vec{J}(t_r) = \vec{J}(t) + (t_r - t)\,\dot{\vec{J}}(t) + \cdots$$

Then show that, given this approximation,

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t)\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

Jefimenko's equation,

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}^2} + \frac{\dot{\vec{J}}(\vec{r}\,',t_r)}{c\mathscr{r}}\right]\times\hat{\mathscr{r}}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}^2}\left(\vec{J}(t) + (t_r - t)\,\dot{\vec{J}}(t)\right) + \frac{1}{c\mathscr{r}}\,\dot{\vec{J}}(t)\right]\times\hat{\mathscr{r}}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}^2}\left(\vec{J}(t) - \frac{\mathscr{r}}{c}\,\dot{\vec{J}}(t)\right) + \frac{1}{c\mathscr{r}}\,\dot{\vec{J}}(t)\right]\times\hat{\mathscr{r}}\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\frac{\vec{J}(t)\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

unomitting the $\vec{r}\,'$ dependence.

$$= \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}\,',t)\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$\boxed{\text{P80}}$ Problem 10.17

Fortunately, I did this already, see LECTURE 35

$\boxed{\text{P81}}$ Problem 10.20

Show that $\dfrac{\partial t_r}{\partial t} = \dfrac{\mathscr{r} c}{\vec{\mathscr{r}} \cdot \vec{u}}$

Here, we defined $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$ where $\vec{\mathscr{r}} = \vec{r} - \vec{w}(t_r)$

and $\vec{v} = \dfrac{d\vec{w}}{dt_r}$ and $\|\vec{r} - \vec{w}(t_r)\| = \mathscr{r} = c(t - t_r)$ *

Square *,

$$c^2(t - t_r)^2 = \mathscr{r}^2 = \vec{\mathscr{r}} \cdot \vec{\mathscr{r}}$$

$$2c^2(t - t_r)\left[1 + \frac{\partial t_r}{\partial t}\right] = 2\vec{\mathscr{r}} \cdot \frac{\partial \vec{\mathscr{r}}}{\partial t} = 2\vec{\mathscr{r}} \cdot \frac{\partial}{\partial t}\left[\vec{r} - \vec{w}(t_r)\right]$$

$$c^2(t - t_r)\left(1 - \frac{\partial t_r}{\partial t}\right) = \vec{\mathscr{r}} \cdot \left(-\frac{d\vec{w}}{dt_r}\frac{\partial t_r}{\partial t}\right) = -\vec{\mathscr{r}} \cdot \vec{v}\,\frac{\partial t_r}{\partial t}$$

$$c\mathscr{r}\left(1 - \frac{\partial t_r}{\partial t}\right) = -\vec{\mathscr{r}} \cdot \vec{v}\,\frac{\partial t_r}{\partial t}$$

$$\Rightarrow c\mathscr{r} = \left(c\mathscr{r} - \vec{\mathscr{r}} \cdot \vec{v}\right)\frac{\partial t_r}{\partial t}$$

$$\Rightarrow c\mathscr{r} = \left(c\hat{\mathscr{r}} \cdot \vec{\mathscr{r}} - \vec{\mathscr{r}} \cdot \vec{v}\right)\frac{\partial t_r}{\partial t}$$

$$\Rightarrow c\mathscr{r} = \left(c\hat{\mathscr{r}} - \vec{v}\right) \cdot \vec{\mathscr{r}}\,\frac{\partial t_r}{\partial t}$$

$$\Rightarrow c\mathscr{r} = \vec{u} \cdot \vec{\mathscr{r}}\,\frac{\partial t_r}{\partial t}$$

$$\therefore \boxed{\frac{\partial t_r}{\partial t} = \frac{\mathscr{r} c}{\vec{\mathscr{r}} \cdot \vec{u}}}$$

P82 Problem 10.21

(a.) Suppose a point charge q is constrained to move on x-axis. Show the fields at points on the axis to the right of the charge given by

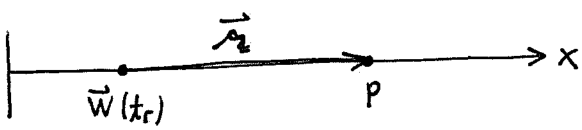
$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{x}, \quad \vec{B} = 0$$

(do not assume v is constant)

What are the fields on the axis to the left of the charge?

(b.) In Ex. 10.4 we noted $\vec{E}$ is reduced in forward/backward by factor of $(1 - v^2/c^2)$, in the case of constant velocity. This might appear to contradict your result in (a.). Resolve this "paradox"

(a.) $$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$



$\vec{v} = v\hat{x}$

$\vec{a} = a\hat{x}$

$\hat{\mathscr{r}} = \hat{x}$ for P to right of $\vec{w}(t_r)$

$\vec{u} = c\hat{\mathscr{r}} - \vec{v} = (c-v)\hat{x}$

$\vec{u}\times\vec{a} = ((c-v)\hat{x})\times(a\hat{x}) = 0$

$\vec{\mathscr{r}}\cdot\vec{u} = \mathscr{r}\hat{x}\cdot(c-v)\hat{x} = \mathscr{r}(c-v)$

$$\boxed{\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\mathscr{r}(c-v))^3}\left[(c^2-v^2)(c-v)\hat{x}\right] = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{x}}$$

Then $\vec{B} = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E} = 0 \quad \therefore \quad \boxed{\vec{B} = 0}$

On the left, $\hat{\mathscr{r}} = -\hat{x}$ and $\vec{u} = c\hat{\mathscr{r}} - \vec{v} = -(c+v)\hat{x}$ and $\vec{\mathscr{r}}\cdot\vec{u} = \mathscr{r}(c+v)$ consequently,

$$\vec{E} = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2-v^2)(c+v)\hat{x}$$

$$\Rightarrow \boxed{\vec{E} = \frac{-q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{x}}$$ and again $\boxed{\vec{B} = 0}$

(b.) I'm not sure what the resolution is here.

$\boxed{P83}$ Problem 10.22

For a point charge moving at constant velocity, calculate the flux $\oint \vec{E}\cdot d\vec{a}$ (using Eq$^{\underline{n}}$ 10.75) over the surface of a sphere centered at the present location of the charge.

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\epsilon_0}\frac{1 - v^2/c^2}{\left(1 - \frac{v^2\sin^2\theta}{c^2}\right)^{3/2}}\frac{\hat{R}}{R^2} \quad \left(\vec{R} = \vec{r} - \vec{v}t\right)$$

The present location of the charge is $t\vec{v}$, $S_{R_0} = \{\vec{r} \mid \|\vec{r} - \vec{v}t\| = R_0\}$

$$\oint_{S_{R_0}} \vec{E}\cdot d\vec{a} = \frac{q(1 - v^2/c^2)}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{(\hat{R}/R_0^2)\cdot(R_0^2\sin\theta\, d\theta\, d\phi)\hat{R}}{\left(1 - \frac{v^2\sin^2\theta}{c^2}\right)^{3/2}}$$

$$= \frac{q(1 - v^2/c^2)}{2\epsilon_0}\int_0^{\pi}\frac{\sin\theta\, d\theta}{\left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}$$

$\sin^2\theta = 1 - \cos^2\theta$

$-\sin^2\theta = \cos^2\theta - 1$

$$= \frac{q(1 - v^2/c^2)}{2\epsilon_0}\int_0^{\pi}\frac{\sin\theta\, d\theta}{\left(1 - \frac{v^2}{c^2}(1 - \cos^2\theta)\right)^{3/2}}$$

$$= \frac{q(1 - v^2/c^2)}{2\epsilon_0}\int_{-1}^{1}\frac{du}{\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}u^2\right)^{3/2}}$$

$$= \frac{q(1 - v^2/c^2)}{2\epsilon_0}\frac{1}{(v^2/c^2)^{3/2}}\int_{-1}^{1}\frac{du}{\left(\frac{c^2}{v^2} - 1 + u^2\right)^{3/2}}$$

integration

$$= \frac{q(1 - v^2/c^2)}{2\epsilon_0}\cdot\frac{c^3}{v^3}\cdot\left(\frac{v}{c}\right)^3\left(\frac{2}{1 - v^2/c^2}\right)$$

$$= \boxed{\frac{q}{\epsilon_0}}.$$

P84 Problem 10.26

SOURCES

ρ, $\vec{J}$

$$\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

($\sim$ indicates evaluation at retarded time in my formulas)

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\tilde{\rho}\,d\tau}{\mathscr{r}}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\tilde{\vec{J}}}{\mathscr{r}}\,d\tau$$

$$\rho = -\epsilon_0\left(\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}\right)$$

$$\vec{J} = \frac{-1}{\mu_0}\left(\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\right)$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int\left(\frac{\tilde{\rho}\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\dot{\tilde{\rho}}\hat{\mathscr{r}}}{\mathscr{r}c} - \frac{\dot{\tilde{\vec{J}}}}{\mathscr{r}c^2}\right)d\tau$$

$$\vec{B} = \frac{\mu_0}{4\pi}\int\left(\frac{\tilde{\vec{J}}\times\hat{\mathscr{r}}}{\mathscr{r}^2} + \frac{\dot{\tilde{\vec{J}}}\times\hat{\mathscr{r}}}{\mathscr{r}c}\right)d\tau$$

Jefimenko's Eq^{ns}.

$$\rho = \epsilon_0\nabla\cdot\vec{E}$$

$$\vec{J} = \frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

POTENTIALS

V, $\vec{A}$

$$\nabla\cdot\vec{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$$

$$\vec{B} = \nabla\times\vec{A}$$

$$\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$$

FIELDS

$\vec{E}$, $\vec{B}$

$$\nabla\cdot\vec{B} = 0,\quad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$\boxed{\text{P85}}$ *Problem 11.1*

Check that the retarded potential of an oscillating dipole satisfy the Lorentz gauge condition

$$V(r,\theta,t) = \frac{P_0 \cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega}{c}\sin\left[\omega(t - r/c)\right] + \frac{1}{r}\cos\left[\omega(t-r/c)\right]\right\}$$

$$\vec{A}(r,\theta,t) = \frac{-\mu_0 P_0 \omega}{4\pi r}\sin\left[\omega(t - r/c)\right]\hat{z}\ , \quad \hat{z} = (\cos\theta)\hat{r} - (\sin\theta)\hat{\theta}$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 A_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, A_\theta\right] \qquad \left(\lambda = \frac{-P_0\mu_0\omega}{4\pi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[\lambda r\cos\theta\sin\left[\omega(t-r/c)\right]\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[-\frac{\lambda\sin^2\theta}{r}\sin\left[\omega(t-r/c)\right]\right]$$

$$= \frac{\lambda\cos\theta\sin\left[\omega(t-r/c)\right]}{r^2} + \frac{\lambda r}{r^2}\cos\theta\cos\left(\omega(t-r/c)\right)\left(\frac{-\omega}{c}\right)$$

$$-\frac{\lambda\sin\left[\omega(t-r/c)\right]}{r^2\sin\theta}\left(2\sin\theta\cos\theta\right)$$

$$= \frac{-\lambda\cos\theta\sin\left[\omega(t-r/c)\right]}{r^2} - \frac{\lambda\omega}{c}\frac{\cos\theta}{r}\cos\left[\omega(t-r/c)\right] \quad *$$

Likewise,

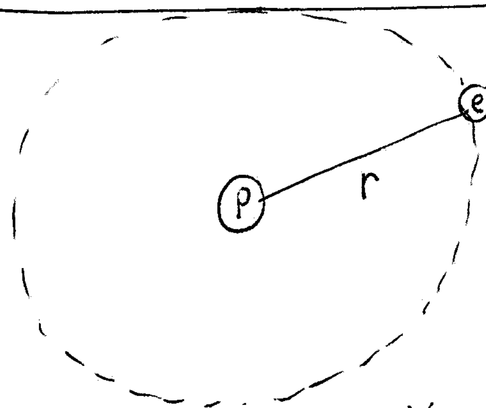
$$\frac{\partial V}{\partial t} = \frac{P_0\cos\theta}{4\pi\epsilon_0 r}\left\{-\frac{\omega^2}{c}\cos\left[\omega(t-r/c)\right] - \frac{\omega}{r}\sin\left[\omega(t-r/c)\right]\right\}$$

$$\mu_0\epsilon_0\frac{\partial V}{\partial t} = \underbrace{\frac{-\mu_0 P_0\omega}{4\pi}}_{\lambda}\left(\frac{\cos\theta}{r^2}\sin\left[\omega(t-r/c)\right] + \frac{\omega}{rc}\cos\theta\cos\left[\omega(t-r/c)\right]\right)$$

$$\therefore \quad \nabla\cdot\vec{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t} \quad //$$

P86 PROBLEM 11.14

Bohr's hydrogen model has electrons orbiting in circle $R = 5\times10^{-11}$ m via Coulomb attraction of proton. Show $v \ll c$ for most of the orbit so we can use Larmor formula and calculate lifespan of Bohr's atom from classical perspective



$$\frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} = \frac{mv^2}{r}$$

$$\Rightarrow \quad v = \sqrt{\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr}}$$

$$\frac{v}{c} = \left[\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(5\times10^{-11})}\right]^{1/2}\frac{1}{3\times10^8}$$

Then $\frac{v}{c} \approx 0.0075$ hence motion not too relativistic.

Larmor formula gives $P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2$ since $a = \frac{v^2}{r}$

thus $P = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2 = -\frac{dU}{dt}$, $U = \underbrace{U_{kinetic}}_{mechanical} + \underbrace{U_{potential}}_{E\&M}$

(for electron orbiting)

$$U = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0}\frac{q^2}{r} = \frac{-1}{8\pi\epsilon_0}\frac{q^2}{r}$$

$$\therefore \quad -\frac{dU}{dt} = \frac{-1}{8\pi\epsilon_0}\frac{q^2}{r^2}\frac{dr}{dt} = \frac{\mu_0 q^2}{6\pi c}\left(\frac{1}{4\pi\epsilon_0}\frac{q^2}{mr^2}\right)^2$$

$$\frac{dr}{dt} = \frac{-\mu_0(8\pi\epsilon_0)}{6\pi c}r^2\frac{1}{(4\pi\epsilon_0)^2}\frac{q^4}{m^2r^4}$$

$$\frac{dr}{dt} = \frac{-1}{3c}\left(\frac{q^2}{2\pi\epsilon_0 mc}\right)^2\frac{1}{r^2} \qquad \left(\frac{1}{c^2} = \mu_0\epsilon_0\right)$$

$$\int dt = -3c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2\int_{r_0}^{0} r^2 dr \hookrightarrow \underline{t = c\left(\frac{2\pi\epsilon_0 mc}{q^2}\right)^2 r_0^3}.$$

$$\Rightarrow \boxed{t \approx 1.3\times10^{-11}\,s}$$