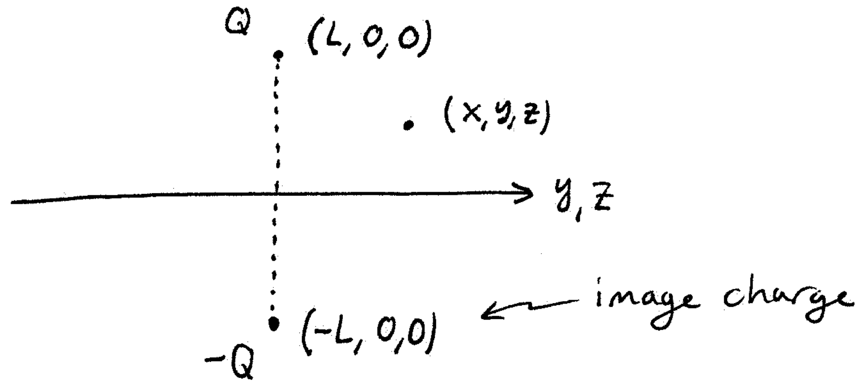
Printed Name: ____________________.

PHYSICS 331	TEST 1: ELECTROSTATICS (150PTS+10PTS)

Please work the problems in the white space provided and clearly box your solutions. You are allowed a page of notes, front and back is fine. Enjoy!

Problem 1 (20pts) A charge Q is at the point $(L, 0, 0)$ above the yz-plane where an hugely huge conductor extends to ludicrous distances. Find the charge density the conducting plane.



$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-L)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+L)^2+y^2+z^2}}\right)$$

The potential above has $V(0,y,z) = 0$ for all y, z and thus describes the electrostatics for the given charge and conductor where $x > 0$. Since $\vec{E} = 0$ within the conducting plane we obtain:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial x}\bigg|_{x=0}$$ since x is the normal direction to yz-plane

$$= \frac{-Q\varepsilon_0}{4\pi\varepsilon_0}\left[\frac{-2(x-L)}{2\left((x-L)^2+y^2+z^2\right)^{3/2}} + \frac{2(x+L)}{2\left((x+L)^2+y^2+z^2\right)^{3/2}}\right]\bigg|_{x=0}$$

$$= \frac{-Q}{4\pi}\left[\frac{2L+L}{(L^2+y^2+z^2)^{3/2}}\right]$$

$$= \boxed{\frac{-QL}{2\pi(L^2+y^2+z^2)^{3/2}}}$$

Problem 2 (20pts) You're given $\rho(\vec{r}) = Q\,\delta^3\left(\vec{r} - (d/2)\,\hat{x}\right) - Q\,\delta^3\left(\vec{r} + (d/2)\,\hat{x}\right)$ where $Q, d > 0$. Calculate the resulting electric field and potential and sketch the field lines.

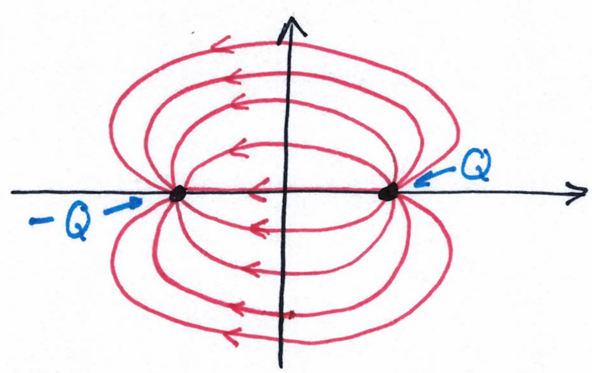
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \left[Q\,\delta^3\left(\vec{r}\,' - \left(\tfrac{d}{2}\right)\hat{x}\right) - Q\,\delta^3\left(\vec{r}\,' + \left(\tfrac{d}{2}\right)\hat{x}\right)\right]\frac{(\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3}\,d\tau'$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{\vec{r} - \left(\frac{d}{2}\right)\hat{x}}{\|\vec{r} - \left(\frac{d}{2}\right)\hat{x}\|^3} - \frac{\vec{r} + \left(\frac{d}{2}\right)\hat{x}}{\|\vec{r} + \left(\frac{d}{2}\right)\hat{x}\|^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\langle x - \frac{d}{2}, y, z\rangle}{\left((x - d/2)^2 + y^2 + z^2\right)^{3/2}} - \frac{\langle x + d/2, y, z\rangle}{\left((x + d/2)^2 + y^2 + z^2\right)^{3/2}}\right]$$

$$V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x - d/2)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x + d/2)^2 + y^2 + z^2}}\right]$$



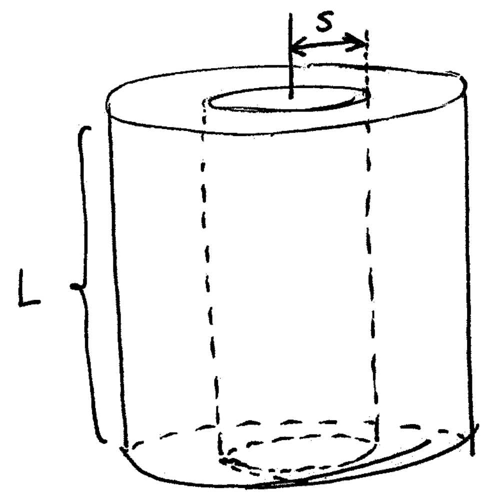
It's a physical dipole,

$\vec{p} = Qd\,\hat{x}$

Problem 3 (20pts) Consider the infinite cylindrical charge of radius R with uniform density ρ_o,

(a.) Find the electric field as a function of distance s from the central axis of the charge,

(b.) Find the potential for which $V = 0$ when $s = R$



For $0 \le s \le R$,

$$Q_{enc} = \int \rho_o \, d\tau = \rho_o(\pi s^2 L)$$

$$\frac{Q_{enc}}{\varepsilon_o} = \oint \vec{E} \cdot d\vec{a}$$

$$\frac{\rho_o \pi s^2 L}{\varepsilon_o} = E\,(2\pi s L)$$

$$E = \frac{\rho_o \pi s^2 L}{(2\pi s L)\,\varepsilon_o} = \frac{\rho_o s}{2\varepsilon_o}$$

$$\boxed{\vec{E} = \begin{cases} \left(\dfrac{\rho_o s}{2\varepsilon_o}\right)\hat{s} & \text{for } 0 \le s \le R \\ \left(\dfrac{\rho_o \pi R^2}{2\pi \varepsilon_o s}\right)\hat{s} & \text{for } s > R \end{cases}}$$

We noticed $Q_{enc} = \rho_o(\pi R^2 L)$ whenever $s \ge R$ to derive $s > R$ case.

(b.) $$V = \begin{cases} \dfrac{-\rho_o}{4\varepsilon_o} s^2 + C_1 & \text{for } 0 \le s \le R \\ -\dfrac{\rho_o R^2}{2\varepsilon_o} \ln(s/C_2) & \text{for } s \ge R \end{cases}$$

We set $C_2 = R$ then $V = \dfrac{-\rho_o R^2}{2\varepsilon_o} \ln(R/R) = 0$ for $s = R$.

Likewise $0 = \dfrac{-\rho_o R^2}{4\varepsilon_o} + C_1 \Rightarrow C_1 = \dfrac{\rho_o R^2}{4\varepsilon}$

$$\boxed{V = \begin{cases} \dfrac{\rho_o}{4\varepsilon_o}(R^2 - s^2) & \text{for } 0 \le s \le R \\ \dfrac{-\rho_o R^2}{2\varepsilon_o} \ln(s/R) & \text{for } s \ge R \end{cases}}$$

Problem 4 (20pts) Suppose the potential on a sphere is known to be $V = V_o + V_1 \cos\theta$ for $r = R$.

(a.) Find the potential inside the sphere,

(b.) Find the potential outside the sphere.

(a.) $$V_I(r,\theta) = \sum_{m=0}^{\infty} A_m r^m P_m(\cos\theta) \quad \text{for } 0 \le r \le R$$

$$V_{II}(r,\theta) = \sum_{m=0}^{\infty} \frac{B_m}{r^{m+1}} P_m(\cos\theta) \quad \text{for } r \ge R$$

From $V_I(R,\theta) = V_{II}(R,\theta)$ we find $A_m R^m = \dfrac{B_m}{R^{m+1}}$

thus $B_m = A_m R^{2m+1}$. We also have $V(R,\theta) = V_o + V_1 \cos\theta$

$$\sum_{m=0}^{\infty} A_m R^m P_m(\cos\theta) = V_o + V_1 \cos\theta$$
$$= V_o P_o(\cos\theta) + V_1 P_1(\cos\theta)$$

Thus, by LI of Legendre polynomials, $A_m = 0$ for $m \ne 0, 1$

and $A_o R^0 = V_o$ and $A_1 R^1 = V_1$. That

is, $\underline{A_o = V_o}$. and $\underline{A_1 = V_1/R}$.

Likewise $\underline{B_o = A_o R = V_o R}$. and $\underline{B_1 = A_1 R^3 = V_1 R^2}$.

$$\boxed{V(r,\theta) = \begin{cases} V_o + \dfrac{V_1 r}{R}\cos\theta & : \ 0 \le r \le R \\ \dfrac{V_o R}{r} + \dfrac{V_1 R^2}{r^2}\cos\theta & : \ r \ge R \end{cases}}$$

Problem 5 (20pts) Assume f and $\vec{A}$ are smooth and S is a simply connected surface.

(a.) Derive $\nabla \times (f\vec{A}) = (\nabla f) \times \vec{A} + f(\nabla \times \vec{A})$

(b.) Show that $\int_S f(\nabla \times \vec{A}) \cdot d\vec{a} = \int_S \left[\vec{A} \times (\nabla f)\right] \cdot d\vec{a} + \oint_{\partial S} f\vec{A} \cdot d\vec{l}$

(a.)
$$\nabla \times (f\vec{A}) = \sum_{i,j,k} \epsilon_{ijk}\, \partial_i (f A_j)\, \hat{x}_k$$

product rule

$$= \sum_{i,j,k} \epsilon_{ijk} \left[(\partial_i f) A_j + f\, \partial_i A_j\right] \hat{x}_k$$

$$= \sum_{i,j,k} \epsilon_{ijk} (\partial_i f) A_j \hat{x}_k + f \sum_{i,j,k} \epsilon_{ijk} (\partial_i A_j)\, \hat{x}_k$$

$$= \underline{(\nabla f) \times \vec{A} + f(\nabla \times \vec{A})} \;//$$

(b.)
$$\int_S f(\nabla \times \vec{A}) \cdot d\vec{a} = \int_S \left[\nabla \times (f\vec{A}) - (\nabla f) \times \vec{A}\right] \cdot d\vec{a}$$

$$= \int_S (\nabla \times (f\vec{A})) \cdot d\vec{a} - \int_S ((\nabla f) \times \vec{A}) \cdot d\vec{a}$$

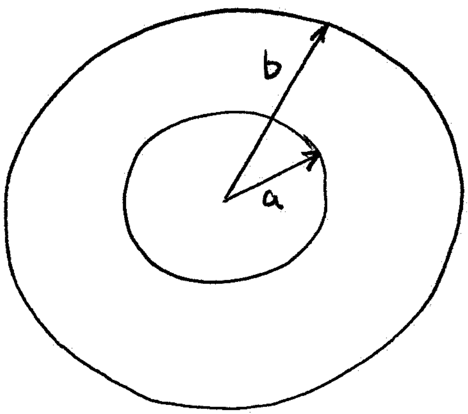
Stokes' Th$^{\text{m}}$

$$= \underline{\oint_{\partial S} f\vec{A} \cdot d\vec{\ell} + \int_S [\vec{A} \times (\nabla f)] \cdot d\vec{a}} \;//$$

Problem 6 (20pts) A spherical rubber shell has polarization $\vec{P} = kr^2\,\hat{r}$ for $a \le r \le b$. There is no free charge in the system.

(a.) Find the bound charge inside the rubber and find the bound surface charge on the inner and outside edge of the rubber shell,

(b.) Calculate the potential inside the sphere given that $V(\infty) = 0$. Surely much partial credit can be earned by calculating $\vec{D}$ and hence $\vec{E}$ in each region.



$$\rho_b = -\nabla\cdot\vec{P} = \frac{-1}{r^2}\frac{\partial}{\partial r}\left[r^2(kr^2)\right]$$

$$\rho_b = \frac{-1}{r^2}\frac{\partial}{\partial r}\left[kr^4\right] = \frac{-4kr^3}{r^2}$$

$$\boxed{\rho_b = -4kr}$$

$$\sigma_b(r=a) = \vec{P}\cdot\hat{n}_a = (ka^2\hat{r})\cdot(-\hat{r}) = -ka^2$$

$$\sigma_b(r=b) = \vec{P}\cdot\hat{n}_b = (kb^2\hat{r})\cdot(\hat{r}) = kb^2$$

} bound charge densities at $r=a$ and $r=b$ respective

(b.) $\vec{D} = 0$ since $\oint_S \vec{D}\cdot d\vec{a} = Q_f = 0$ for any S.

However, $\vec{D} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \underline{\vec{E} = 0 \text{ for } r > b \text{ \& } r < a}$.

Likewise, $\vec{D} = \varepsilon_0 E + kr^2\hat{r} \Rightarrow \underline{\vec{E} = -\frac{kr^2}{\varepsilon_0}\hat{r} \text{ for } a < r < b}$.

Since $\vec{E} = 0$ outside the shell we need $V = 0$ for $r = b$

$$\vec{E} = \frac{-kr^2}{\varepsilon_0}\hat{r} = -\hat{r}\frac{\partial V}{\partial r} \hookrightarrow V(\vec{r}) = \frac{k}{3\varepsilon_0}r^3 + C_1$$

Then $V = \frac{kb^3}{3\varepsilon_0} + C_1 = 0 \therefore C_1 = \frac{-kb^3}{3\varepsilon_0}$

$$\therefore \underline{V(\vec{r}) = \frac{k}{3\varepsilon_0}(r^3 - b^3) \text{ for } a \le r \le b}.$$

$$\Rightarrow \boxed{V = \frac{k}{3\varepsilon_0}(a^3 - b^3)}$$ inside the shell.

Problem 7 (20pts) Coulomb's Law for the electric field due to a distribution ρ over volume $\mathcal{V}$ is given by

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_o} \int_{\mathcal{V}} \rho(\vec{r}\,') \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

Show that Coulomb's Law implies Gauss' Law in both differential and integral forms.

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \nabla \cdot \left(\rho(\vec{r}\,') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau' \qquad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \rho(\vec{r}\,')\, \nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'$$

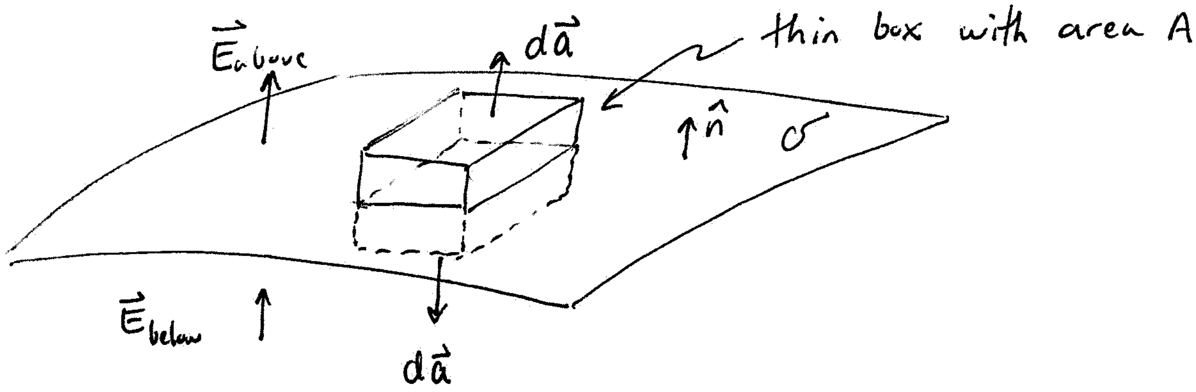
$$= \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \cdot 4\pi \delta^3(\vec{r} - \vec{r}\,')\, d\tau'$$

$$= \rho(\vec{r})/\varepsilon_0 \qquad \therefore \quad \boxed{\nabla \cdot \vec{E} = \rho/\varepsilon_0}$$

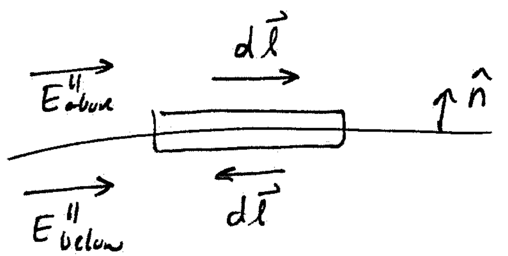
$$\frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_{\mathcal{V}} \rho\, d\tau = \int_{\mathcal{V}} (\nabla \cdot \vec{E})\, d\tau = \oint_{\partial\mathcal{V}} \vec{E} \cdot d\vec{a}$$

$$\boxed{\oint_{\partial\mathcal{V}} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}}$$

Problem 8 (20pts) Explain why $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\epsilon_o}\hat{n}$ where $\hat{n}$ points in the above direction of a surface with surface charge density σ.



$$\oint \vec{E} \cdot d\vec{a} = E^{\perp}_{above} A - E^{\perp}_{below} A = \frac{Q_{enc}}{\varepsilon_o} = \frac{\sigma A}{\varepsilon_o}$$



$$\underline{E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_o}} \quad (1)$$

$$0 = \oint_{loop} \vec{E} \cdot d\vec{l} = E^{\parallel}_{above}\, l - E^{\parallel}_{below}\, l = 0 \quad \therefore \quad \underline{E^{\parallel}_{above} = E^{\parallel}_{below}} \quad (2)$$

Combining (1) and (2) we deduce

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_o}\hat{n}}$$