Printed Name: ______________________.

PHYSICS 331	TEST 2: MAGNETOSTATICS & ELECTRODYNAMICS (150PTS+10PTS)

Please work the problems in the white space provided and clearly box your solutions. You are allowed a page of notes, front and back is fine. Enjoy!

Problem 1 (20pts) Suppose we define $\vec{A}$ for a given steady current density $\vec{J}$. In this problem we wish to justify the choice of Coulomb Gauge $\nabla \bullet \vec{A} = 0$. You are given the existence of a vector field $\vec{A}_o$ for which $\vec{B} = \nabla \times \vec{A}_o$. Moreover, $\vec{A}_o = 0$ for $r >> 0$.

(a.) let λ be scalar function, show $\vec{A} = \vec{A}_o + \nabla\lambda$ is a vector potential for $\vec{B}$

(b.) show $\nabla \bullet \vec{A} = 0$ implies λ solves a Poisson equation where $\nabla \bullet \vec{A}_o$ is playing the role of charge density. By analogy with the electrostatics theorem:

If charge density is ρ then the potential is $V(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}d\tau'$

write down the integral formula for λ

(c.) given $\vec{A}$ satisfies the Coulomb Gauge, show that $\nabla^2\vec{A} = -\mu_o\vec{J}$.

(d.) explain why $\vec{A}(\vec{r}) = \dfrac{\mu_o}{4\pi}\displaystyle\int \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}d\tau'$ where $\mathscr{r} = ||\vec{r} - \vec{r}\,'||$

(a.) Suppose $\nabla \times \vec{A}_o = \vec{B}$. Let $\vec{A} = \vec{A}_o + \nabla\lambda$ and consider,

$$\nabla \times \vec{A} = \nabla \times (\vec{A}_o + \nabla\lambda) = \nabla \times \vec{A}_o + \underbrace{\nabla \times \nabla\lambda}_{0} = \vec{B}.$$

Thus $\vec{A} = \vec{A}_o + \nabla\lambda$ is a vector potential for $\vec{B}$.

(b.) $\nabla \cdot \vec{A} = 0 \Rightarrow \nabla \cdot \vec{A}_o + \nabla \cdot \nabla\lambda = 0 \Rightarrow \underline{\nabla^2\lambda = -\nabla \cdot \vec{A}_o}$.

$\nabla^2 V = -\rho/\epsilon_o \Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}d\tau'$ is like

$\nabla^2\lambda = -\nabla \cdot \vec{A}_o$ with $\rho = \epsilon_o \nabla \cdot \vec{A}_o$ so we find by substitution,

$$\boxed{\lambda(\vec{r}) = \frac{1}{4\pi}\int \frac{(\nabla \cdot \vec{A}_o)(\vec{r}\,')}{\mathscr{r}}d\tau'}$$

(c.) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$

If $\nabla \cdot \vec{A} = 0$ and $\nabla \times \vec{A} = \vec{B}$ where $\overbrace{\nabla \times \vec{B} = \mu_o\vec{J}}^{\text{steady current}}$

$\therefore \mu_o\vec{J} = -\nabla^2\vec{A} \rightarrow \boxed{\nabla^2\vec{A} = -\mu_o\vec{J}}$

(d.) $\underbrace{\nabla^2 A_i = -\mu_o J_i}_{\text{think } \frac{\rho}{\epsilon_o} = \mu_o J_i} \rightarrow A_i = \frac{\mu_o}{4\pi}\int \frac{J_i(\vec{r}\,')}{\mathscr{r}}d\tau'$ for $i = 1, 2, 3$

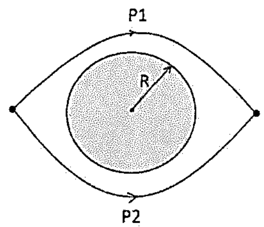
$$\therefore \boxed{\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}\,')d\tau'}{\mathscr{r}}}$$

Problem 2 (30pts) In quantum mechanics the phase of a charge q moving along a path with vector potential $\vec{A}$ is given a phase shift $\varphi = \frac{q}{\hbar}\int_P \vec{A}\bullet d\vec{l}$.

(a.) Consider a very long solenoid of radius R centered about the z-axis. A current I flows through the solenoid which has n turns per unit length. Find the vector potential for the magnetic field for $0 \le s \le R$.

(b.) Find the vector potential outside the solenoid where $s > R$.

(c.) Let P_1 and P_2 be two coterminal paths around the solenoid; that is suppose P_1, P_2 are paths for which $P_1 \cup (-P_2)$ then forms a loop enclosing around the solenoid. Let φ_j be the phase shift along P_j. Find explicitly how the difference in phase shift $\varphi_2 - \varphi_1$ depends on the current I.



(a.) we know the solenoid field $\vec{B} = \begin{cases} \mu_0 n I \hat{z} & \text{for } 0 \le s < R \\ 0 & \text{for } s > R \end{cases}$

Then, for $s < R$, $\vec{B} = \nabla \times \vec{A} = \frac{1}{s}\left[\frac{\partial}{\partial s}[sA_\phi] - \frac{\partial A_s}{\partial \phi}\right]\hat{z} = \mu_0 n I \hat{z}$

and if we set $A_s = 0$ and $A_z = 0$ we need to solve,

$$\frac{\partial}{\partial s}[sA_\phi] = \mu_0 n I s \Rightarrow sA_\phi = \frac{1}{2}\mu_0 n I s^2$$

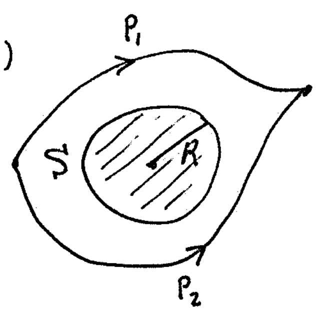
$$\Rightarrow A_\phi = \frac{1}{2}\mu_0 n I s$$

$$\therefore \boxed{\vec{A} = \frac{1}{2}\mu_0 n I s \hat{\phi}}$$

(b.) by continuity of $\vec{A}$,

$$\boxed{\vec{A} = \frac{1}{2}\mu_0 n I R \hat{\phi} \text{ for } s \ge R}$$

(c.)



$\partial S = P_2 \cup (-P_1)$

$$\int_{\partial S} \vec{A}\cdot d\vec{l} = \int_S (\nabla \times \vec{A})\cdot d\vec{a}$$

$$= \int_S \vec{B}\cdot d\vec{a}$$

$$= \int_{S \le R} \mu_0 n I \, da$$

$$= \pi R^2 \mu_0 n I = \int_{P_2} \vec{A}\cdot d\vec{l} + \int_{-P_1} \vec{A}\cdot d\vec{l}$$

$$\therefore \boxed{\varphi_2 - \varphi_1 = \frac{q}{\hbar}\int_{P_2} \vec{A}\cdot d\vec{l} - \frac{q}{\hbar}\int_{P_1} \vec{A}\cdot d\vec{l} = \frac{q}{\hbar}\pi R^2 \mu_0 n I}$$

Problem 3 (20pts) A very long circular cylinder of radius R carries a magnetization $\vec{M} = ks^2\hat{\phi}$ where k is constant. Find the magnetic field for points inside and outside $s = R$ as follows:

(a.) solve via the $\vec{H}$ arguments

(b.) solve via analysis of Ampere's Law on the bound currents

(a.) since free current is zero we deduce $\vec{H} = 0$ everywhere

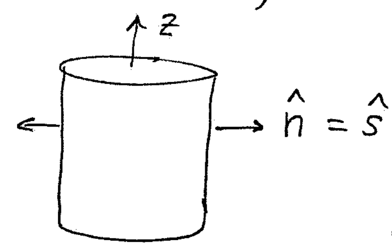
However, $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ thus for $s < R$ where $\vec{M} = ks^2\hat{\phi}$

we find $\frac{1}{\mu_0}\vec{B} - ks^2\hat{\phi} = 0 \therefore \boxed{\vec{B} = \mu_0 k s^2 \hat{\phi}}$ $(0 \leq s < R)$

Likewise, for $s > R$, we have $\vec{M} = 0 \therefore \frac{1}{\mu_0}\vec{B} = 0 \Rightarrow \boxed{\vec{B} = 0}$ $(s > R)$

(b.) $\vec{J}_b = \nabla \times \vec{M} = -\frac{\partial}{\partial z}\left[ks^2\right]\hat{s} + \frac{1}{s}\frac{\partial}{\partial s}\left[s \cdot ks^2\right]\hat{z}$ (other terms zero)

$\underline{\vec{J}_b = 3ks\,\hat{z}}$.



$\vec{K}_b = \vec{M} \times \hat{n} = \left(ks^2\hat{\phi} \times \hat{s}\right)\Big|_{s=R} = \underline{-kR^2\hat{z}}$

Consider amperian loop about z-axis of radius $s_0 < R$ then by cylindrical symmetry, $\vec{B} = B_\phi\hat{\phi}$ and,

$$\oint_{C_s} \vec{B} \cdot d\vec{l} = (2\pi s_0)B = \mu_0 I_{enc} = \mu_0 \int_0^{s_0} (3ks)(2\pi s\,ds) = 2\pi\mu_0 k s_0^3$$

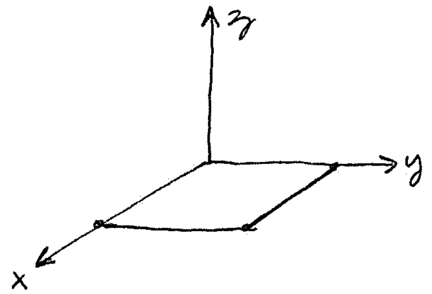
(setting $s_0 \to s$) $\therefore \boxed{\vec{B} = \mu_0 k s^2 \hat{\phi}}$ for $0 \leq s < R$

On the other hand, for $s > R$,

$$\vec{I}_{enc} = \underbrace{(2\pi R)(-kR^2\hat{z})}_{\text{from surface current}} + \underbrace{(2\pi k R^3\hat{z})}_{\text{volume bound current}} = 0$$

$\therefore \quad B(2\pi s) = \mu_0(0) \quad \Rightarrow \quad \boxed{\vec{B} = 0}$ for $s > R$.

Problem 4 (20pts) A square loop of side length a lies in the xy-plane in the first quadrant with one corner at $(0,0)$. In this region $\vec{B} = 3ky^2t^2\hat{z}$ where k is a constant. Calculate the emf in the loop.



$$\Phi_B = \int_0^a \int_0^a (3ky^2t^2\,\hat{z}) \cdot (\hat{z}\,dx\,dy)$$
$$= (3kt^2)\left(\int_0^a dx\right)\left(\int_0^a y^2\,dy\right)$$
$$= (3kt^2)(a)(a^3/3)$$
$$= kt^2a^4$$

$$\mathcal{E}mf = -\frac{d\Phi_B}{dt} = \boxed{-2kta^4}$$

Problem 5 (5pts) We introduced the stress energy tensor and we found the formula

$$\vec{F} = \int_{\partial V} T(d\vec{a},\ ) - \mu_o\epsilon_o \frac{d}{dt}\int_V \vec{S}\,d\tau.$$

Give a qualitative description of what this formula tells us.

Force on ∂V is due to the electromagnetic stress-energy tensor and the flow of momentum $\approx$ Here $\vec{F}$ accounts for the net-electromagnetic force on charges within a volume V.

Problem 6 (5pts) We defined $\vec{g} = \mu_o\epsilon_o\vec{S}$ and we showed

$$-\frac{\partial\vec{g}}{\partial t} + \sum_{i,j=1}^{3}(\partial_i T_{ij})\hat{x}_j = 0.$$

Qualitatively describe the significance of the above equation.

local conservation of field momentum in the absence of charge (LECTURE 26)

Problem 7 (20pts) Notice for part (b.) We showed in lecture that the magnetic field within a spinning charged spherical shell with uniform charge density σ at radius R with angular velocity $\vec{\omega}$ is given by $\vec{B} = \frac{2}{3}\mu_o \sigma R \vec{\omega}$.

(a.) Suppose a solid sphere of radius R has uniform magnetization $\vec{M} = M\hat{z}$ where $M > 0$. Calculate the bound current densities inside the sphere and on its surface

(b.) Calculate the magnetic field for the uniformly magnetized sphere in the case $r < R$ (inside the sphere). Your solution should not use $\vec{H}$.

(c.) Derive the magnetic field within the uniformly magnetized sphere via auxillary field arguments in the case $r < R$. (now use $\vec{H}$)

(d.) Find the magnetic field for the uniformly magnetized sphere in the case $r > R$.

(a.) $\nabla \times \vec{M} = \boxed{\vec{J}_b = 0}$. (within and without the sphere, but to be careful $\nabla \times \vec{M}$ is infinite at $r = R$ since $\exists$ Θ - function in r, $\vec{M} = M\,\Theta(r-R)\,\hat{z}$)

$\vec{K}_b = \vec{M} \times \hat{n} = M\hat{z} \times \hat{r}$ where $\hat{r} = \cos\phi\sin\theta\,\hat{x} + \sin\phi\sin\theta\,\hat{y} + \cos\theta\,\hat{z}$

$$\vec{K}_b = M\left(\cos\phi\sin\theta\,\hat{y} - \sin\phi\sin\theta\,\hat{x}\right) = M\sin\theta\left(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right) = \boxed{(M\sin\theta)\hat{\phi}}$$

(b.) $\vec{K} = \sigma\vec{v} = \sigma\omega R\sin\theta\,\hat{\phi} \Rightarrow \vec{B} = \frac{2}{3}\mu_o\sigma R\vec{\omega}$ (where $\vec{\omega} = \omega\hat{z}$)

We have $\vec{K}_b = (M\sin\theta)\hat{\phi}$ so identify $M = \sigma\omega R$ to borrow from our given solution for the spinning charged sphere, we find effectively $\omega\sigma = \frac{M}{R}$ $\hookrightarrow$ $\vec{B} = \frac{2}{3}\mu_o R \cdot \frac{M}{R}\hat{z}$

$$\therefore \quad \boxed{\vec{B} = \frac{2}{3}\mu_o M\hat{z}}$$

(c.) $\vec{H} = 0$ for $r < R$ since no free current, hence, $\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M} = 0$

$$\Rightarrow \vec{B} = \mu_o\vec{M} =$$

$$\Rightarrow \boxed{\vec{B} = \mu_o M\hat{z}}$$

Cannot be correct! So, what is wrong this my analysis here?

(d.) From our analysis of spinning charged sphere, or from multipole concept, it is as if there is a single pure dipole with $\vec{m} = \frac{4}{3}\pi R^3\vec{M}$ at the origin. Hence,

$$\vec{B} = \left(\frac{\mu_o}{4\pi r^3}\right)\left(\frac{4\pi R^3 M}{3}\right)\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) = \boxed{\frac{\mu_o R^3 M}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

PROBLEM 7 continued

From (b.) we found $\vec{B} = \frac{2}{3}\mu_0 M \hat{z}$ whereas from (c) $\vec{B} = \mu_0 M \hat{z}$

Notice, the approach of (c.) applied to (d.) also suggests $\vec{B} = 0$ rather than $\vec{B} = \frac{\mu_0 R^3 M}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$. Obviously these results disagree. Which approach is incorrect?

Let's revisit (a.) and the calculation of $\vec{J_b} = \nabla\times\vec{M}$.

$$\vec{M} = M\,\Theta(r-R)\,\hat{z} \qquad (\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

Hence,

$$= \underbrace{M\,\Theta(r-R)\cos\theta}_{M_r}\,\hat{r} - \underbrace{M\,\Theta(r-R)\sin\theta}_{M_\theta}\,\hat{\theta}$$

$$\nabla\times\vec{M} = \frac{-1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\phi}[M_\theta]}\,\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\cancelto{0}{\frac{\partial M_r}{\partial\phi}} - \frac{\partial}{\partial r}\left[r\cancelto{0}{M_\phi}\right]\right]\hat{\theta}$$

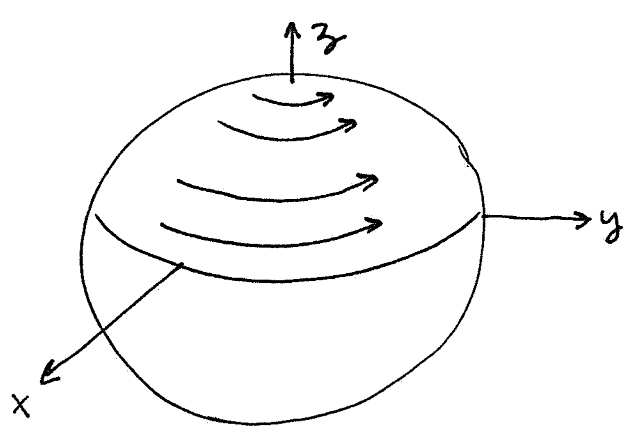
(oops didn't need to write this one)

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rM_\theta) - \frac{\partial M_r}{\partial\theta}\right]\hat{\phi}$$

$$\nabla\times\vec{M} = \left(\frac{1}{r}\frac{\partial}{\partial r}\left[-Mr\,\Theta(r-R)\sin\theta\right] - \frac{\partial}{\partial\theta}\left[M\,\Theta(r-R)\cos\theta\right]\right)\hat{\phi}$$

$$= M\hat{\phi}\left(-\frac{1}{r}\left(\Theta(r-R) + r\frac{d}{dr}\left[\Theta(r-R)\right]\right)\sin\theta + (\sin\theta)\,\Theta(r-R)\right)$$

$$= M\hat{\phi}\left(\left(1-\frac{1}{r}\right)\sin\theta\,\Theta(r-R) - \sin\theta\,\delta(r-R)\right)$$



$$\underbrace{\vec{J_b} = M\left[\left(1-\frac{1}{r}\right)\Theta(r-R) - \delta(r-R)\right]\sin\theta}$$

I think this gives another surface current at $r = R$, how to calculate it?

PROBLEM 7 continued

We should consider the divergence of $\vec{M} = M\,\Theta(r-R)\,\hat{z}$ which gives $\vec{M} = \underbrace{M\,\Theta(r-R)\cos\theta}_{M_r}\,\hat{r} - \underbrace{M\,\Theta(r-R)\sin\theta}_{M_\theta}\,\hat{\theta}$

$$\nabla\cdot\vec{M} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 M_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, M_\theta\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[M r^2 \Theta(r-R)\cos\theta\right] - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[M\,\Theta(r-R)\sin^2\theta\right]$$

$$= \frac{M}{r^2}\left[2r\cos\theta\,\Theta(r-R) + r^2\cos\theta\,\delta(r-R)\right] + \swarrow$$

$$- \frac{M}{r\sin\theta}\,\Theta(r-R)\cdot 2\sin\theta\cos\theta$$

$$= M\left(\cos\theta\,\Theta(r-R)\left[\frac{2r}{r^2} - \frac{2}{r}\right] + \frac{r^2}{r^2}\cos\theta\,\delta(r-R)\right)$$

$$= \underline{M\cos\theta\,\delta(r-R)}$$

Ok, so (c.) was in error, even w/o free current $\not\Rightarrow$ $\vec{H} = 0$ when $\nexists$ symmetry to make it so. Let's find $\vec{H}$ from the correct solution to (b.) and (d.)

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$\underline{r < R}$ $\quad \vec{H}_{in} = \frac{2}{3}\vec{M} - \vec{M} = \frac{-1}{3}\vec{M}$

$\underline{r > R}$ $\quad \vec{H}_{out} = \frac{R^3 M}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) - 0.$

We have the situation that $\oint \vec{H}\cdot d\vec{l} = 0$ for all loops, however, this does not force $\vec{H} = 0$.

Problem 8 (20pts) We argued that the rate of work done on charges due to the electric and magnetic fields in some volume V is given by $\frac{dW}{dt} = \int_V \vec{E} \cdot \vec{J} d\tau$. Show that $\frac{dW}{dt} = -\frac{d}{dt}\int_V u\, d\tau - \int_{\partial S} \vec{S} \cdot d\vec{A}$ where u is the electromagnetic field energy density and $\vec{S}$ is the Poynting vector.

Suppose $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ then solving for $\vec{J}$ we obtain $\vec{J} = \frac{1}{\mu_0}(\nabla \times \vec{B}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. Therefore,

$$\underline{\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\nabla \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}} \; *$$

Recall,

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{B})$$

$$\therefore \quad \underline{\vec{E} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{B})} \; *$$

Substitute ** into * to obtain

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0}\left[\vec{B} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{B})\right] - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$$

Faraday's Law

$$= \frac{1}{\mu_0} \vec{B} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \nabla \cdot \left(\frac{1}{\mu_0} \vec{E} \times \vec{B}\right)$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left[\frac{1}{\mu_0} \vec{B} \cdot \vec{B} + \epsilon_0 \vec{E} \cdot \vec{E}\right] - \nabla \cdot \vec{S} \quad , \quad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$= -\frac{\partial}{\partial t}\Big[\underbrace{\tfrac{1}{2}\left(\epsilon_0 E^2 + \tfrac{1}{\mu_0} B^2\right)}_{u}\Big] - \nabla \cdot \vec{S}$$

Thus,

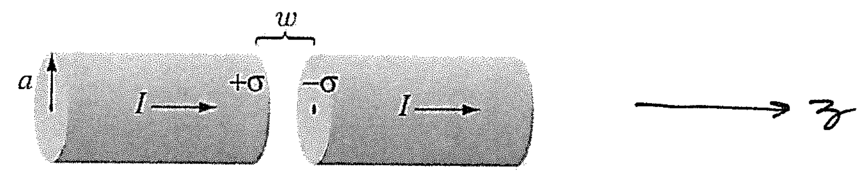
$$\frac{dW}{dt} = \int_V (\vec{E} \cdot \vec{J}) d\tau = \int_V \left(-\frac{\partial u}{\partial t} - \nabla \cdot \vec{S}\right) d\tau$$

$$= -\frac{d}{dt}\int_V u\, d\tau - \int_V (\nabla \cdot \vec{S})\, d\tau$$

Divergence Th$\stackrel{m}{.}$

$$\underline{= -\frac{d}{dt}\int_V u\, d\tau - \oint_{\partial V} \vec{S} \cdot d\vec{a}}\,.$$

Problem 9 (20pts) Consider the model of the charging capacitor pictured below.



(a.) Find $\vec{E}$ near the center of the gap assuming that the charging begins at $t = 0$

(b.) Find $\vec{B}$ due to the changing electric field near the center of the gap

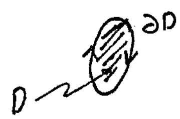
(c.) Find the electromagnetic field energy density u and Poynting vector $\vec{S}$ near the center of the gap of width w

(d.) Do u and $\vec{S}$ relate in the appropriate manner ?

(a.) $Q = It$ is charge on cap at time t thus $\sigma = \frac{Q}{A} = \frac{It}{\pi a^2}$

$$\boxed{\vec{E} = \frac{\sigma}{\epsilon_0}\hat{z} = \frac{It}{A\epsilon_0}\hat{z}}$$ near center of gap (can set $A = \pi a^2$ if wish)

(b.) $$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\mu_0 I}{A}\hat{z}$$



little loop in the gap near center

$$\int_D (\nabla \times \vec{B})\cdot d\vec{a} = \int_{\partial D} \vec{B}\cdot d\vec{l} = \int_D \frac{\mu_0 I}{A} da = \frac{\mu_0 I}{A}\pi s^2 = (2\pi s)B$$

$$\vec{B} = \frac{\mu_0 I \pi s^2}{(\pi a^2)(2\pi s)}\hat{\phi}$$

$$\boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

(c.) $$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$
$$= \frac{1}{\mu_0}\left(\frac{It}{\pi a^2 \epsilon_0}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)\hat{z}\times\hat{\phi}$$
$$= \frac{-I^2 t s}{2\pi^2 a^4 \epsilon_0}\hat{s}$$

$\hat{s}$, $\hat{\phi}$, $\hat{z}$

$$u = \frac{1}{2}\left[\epsilon_0\left(\frac{It}{\pi a^2 \epsilon_0}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \frac{I^2}{2\pi^2 a^4}\left(\frac{t^2}{\epsilon_0} + \mu_0 s^2\right)$$

$$u = \frac{I^2}{2\pi^2 a^4}\left(\frac{t^2}{\epsilon_0} + \mu_0 s^2\right)$$

Observe $\frac{\partial u}{\partial t} = \frac{I^2 t}{\pi^2 a^4 \epsilon_0}$ and $\nabla\cdot\vec{S} = \frac{1}{s}\frac{\partial}{\partial s}\left[s\left(\frac{-I^2 t s}{2\pi^2 a^4 \epsilon_0}\right)\right] = \frac{-I^2 t}{\pi^2 a^4 \epsilon_0}$

$$\therefore \quad \underline{\frac{\partial u}{\partial t} + \nabla\cdot\vec{S} = 0}.$$ (as to be expected in gap where no charge is)