Polar Coordinates and Conic Sections

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Abstract

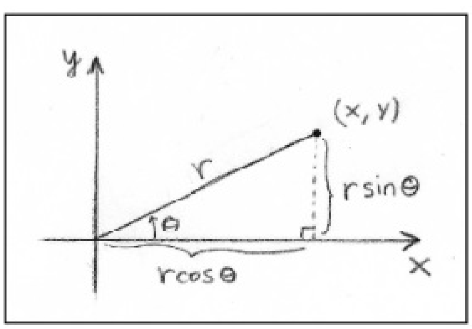
We study the algebra and calculus of polar coordinates. We find some curves are most naturally described with polar coordinates. In particular, problems involving sectors bounded by circular arcs are nicely described by polar coordinates. The calculation of geometric quantities in polar coordinates typically involve a significant twist from the more direct Cartesian coordinate approach. We review the concept of parametric curves once more, this is a review of a previous article, but I left it here since it might be helpful to read a different motivational discussion of why we parametrize curves. Next conic sections are described both via their geometric definition and by the more convenient algebraic characterization which we use in calculus applications. We also indicate a number of natural parametrizations of conic sections. We also explaining how conic sections can be formulated in polar coordinates. If we combine the parametric and polar concepts then we may study polar-parametric parametrizations of curves. Finally, I should mention, the discussion of polar coordinates is standard material, but you may naturally be curious about the existence and potential use of other coordinate systems. Time permitting, I may discuss rotated coordinate systems or why hyperbolic coordinates are troublesome.

1 Polar Coordinates

Cartesian coordinates x, y are related to polar coordinates r, θ by

$$\boxed{x = r\cos\theta, \qquad y = r\sin\theta.}$$

These formulas follow naturally from basic trigonometry:



Notice $x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$. Furthermore, if $x \neq 0$ we also find $\frac{y}{x} = \frac{r\sin\theta}{r\sin\theta}$. We should remember:

$$\boxed{r^2 = x^2 + y^2 \qquad \& \qquad \frac{y}{x} = \tan\theta.}$$

In graphing polar equations we typically allow $r < 0$ with the understanding that geometrically this indicates the point in question is on the diametrically opposite side. Notice

$$\begin{aligned}\cos(\theta+\pi) &= \cos\theta\cos\pi - \sin\theta\sin\pi = -\cos\theta\\ \sin(\theta+\pi) &= \sin\theta\cos\pi + \cos\theta\sin\pi = -\sin\theta.\end{aligned}$$

Consequently,

$$x = r\cos(\theta+\pi) = -r\cos\theta \qquad \& \qquad y = r\sin(\theta+\pi) = -r\sin\theta.$$

Apparently we can identify a given point P either with polar coordinates $(r, \theta+\pi)$ or polar coordinates $(-r, \theta)$. Polar coordinates are also degenerate in the choice of θ for fixed r since:

$$x = r\cos(\theta+2\pi n) = r\cos\theta \qquad \& \qquad y = r\sin(\theta+2\pi n) = r\sin\theta.$$

for any $n \in \mathbb{Z}$. We are free to shift θ by any multiple of 2π-radians. In terms of degree-measure, we can shift the angle by any multiple of 360^o. Let me pause to give an account of how to calculate the polar angle in different cases: we assume that $r = \sqrt{x^2+y^2} > 0$ in what follows[1]:

(1.) If $x > 0$ then $\theta = \tan^{-1}(y/x) \in (-\pi/2, \pi/2)$

(2.) If $x = 0$ and $y > 0$ then $\theta = \pi/2$

(3.) If $x = 0$ and $y < 0$ then $\theta = -\pi/2$ or

(4.) If $x < 0$ then $\theta = \tan^{-1}(y/x) + \pi \in (\pi/2, 3\pi/2)$

All of this said, I often simply calculate the angle in a triangle in the quadrant where the point is found and simply add or substract radians as geometric common sense indicates. If you are given instructions to find $\theta \in (-\pi, \pi]$ then you would need to adjust the answers given above, in particular case (4.) splits into two cases.

Example 1.1. *Let $P = (3,3)$ then $r = \sqrt{18}$ and $\theta = \tan^{-1}(3/3) = \pi/4$.*

Example 1.2. *Let $Q = (1,-1)$ then $r = \sqrt{2}$ and $\theta = \tan^{-1}(-1/1) + \pi = 3\pi/4$.*

Example 1.3. *Let $R = (-2,-2)$ then $r = \sqrt{8}$ and $\theta = \tan^{-1}(-2/-2) + \pi = 5\pi/4$.*

Example 1.4. *Let $S = (4,-4)$ then $r = \sqrt{32}$ and $\theta = \tan^{-1}(-4/4) = -\pi/4$.*

Notice P, Q, R, S in the preceding examples are respectively in Quadrants I, II, III and IV. And now for some $r < 0$ examples. Ok, actually, the same examples as above, just with the silly choice to write $r = -\sqrt{x^2+y^2}$:

Example 1.5. *Let $P = (3,3)$ if we set $r = -\sqrt{18}$ then $\theta = 5\pi/4$.*

Example 1.6. *Let $Q = (1,-1)$ if we use $r = -\sqrt{2}$ then $\theta = 3\pi/4$.*

[1] the polar angle at the origin where $r = 0$ is undefined

Example 1.7. *Let* $R = (-2, -2)$ *if we set* $r = -\sqrt{8}$ *and* $\theta = \pi/4$.

Example 1.8. *Let* $S = (4, -4)$ *if we use* $r = -\sqrt{32}$ *and* $\theta = \tan^{-1}(-4/4) = 3\pi/4$.

Next, I turn to the problem of converting equations to and from polar to Cartesian coordinates. In short, apply the boxed equations at the start of this section. In Cartesian coordinates, it is often customary to solve for $y = f(x)$ as to identify the curve as the graph of a function. The analog for polar coordinates it to solve for $r = f(\theta)$ where possible. If $r = f(\theta)$ describes C then we call C a **polar graph**.

Example 1.9. *Express* $x^2 + y^2 = 9$ *as a polar graph. So,* $r^2 = 9$ *and thus* $r = 3$. *This curve is the circle of radius* 3 *centered at the origin.*

Technically, we could also write $r = -3$ for the preceding example, but let us agree to avoid using $r < 0$ in as much as is possible.

Example 1.10. *Consider* $r = 2\cos\theta$. *Find the form of this curve in Cartesian coordinates. Observe* $r = 2\cos\theta$ *implies* $r^2 = 2r\cos\theta$ *which yields* $x^2 + y^2 = 2x$. *Hence* $x^2 - 2x + y^2 = 0$ *and completing the square reveals*

$$(x-1)^2 + y^2 = 1$$

this is a circle centered at $(1, 0)$ *with raduis* $R = 1$.

Example 1.11. *Consider the circle of radius* R *centered at* (h, k) *has equation*

$$(x-h)^2 + (y-k)^2 = R^2.$$

The form of this equation in polar coordinates is:

$$(r\cos\theta - h)^2 + (r\sin\theta - k)^2 = R^2.$$

Multiplying out the squares above and using $\cos^2\theta + \sin^2\theta = 1$ *gives:*

$$r^2 - 2r(h\cos\theta + k\sin\theta) + h^2 + k^2 = R^2$$

Interesting, the equation above is quadratic in r, *let us solve it by completing the square:*

$$(r - (h\cos\theta + k\sin\theta))^2 - (h\cos\theta + k\sin\theta)^2 = R^2 - h^2 - k^2$$

thus

$$(r - (h\cos\theta + k\sin\theta))^2 - h^2\cos^2\theta - 2hk\sin\theta\cos\theta - k^2\sin^2\theta = R^2 - h^2 - k^2$$

and

$$(r - (h\cos\theta + k\sin\theta))^2 = R^2 - h^2(1-\cos^2\theta) - k^2(1-\sin^2\theta) + 2hk\sin\theta\cos\theta$$

which becomes

$$(r - (h\cos\theta + k\sin\theta))^2 = R^2 - (h^2\sin^2\theta - 2hk\sin\theta\cos\theta + k^2\cos^2\theta)$$

or

$$(r - (h\cos\theta + k\sin\theta))^2 = R^2 - (h\sin\theta - k\cos\theta)^2$$

curious:

$$r = h\cos\theta + k\sin\theta \pm \sqrt{R^2 - (h\sin\theta - k\cos\theta)^2}.$$

Well, that was fun. Let's try this formula out on something we already did: $(x-1)^2 + y^2 = 1$ *has* $h = 1, k = 0$ *and* $R = 1$ *thus:*

$$r = \cos\theta \pm \sqrt{1 - \sin^2\theta} = 2\cos\theta.$$

The example above is not intended as result to memorize. However, the calculational principles have broad application.

Example 1.12. *Consider $x^2 - y^2 = 1$. The polar form of this hyperbola is found by substituting $x = r\cos\theta$ and $y = r\sin\theta$ into the equation:*

$$(r\cos\theta)^2 - (r\sin\theta)^2 = 1 \quad \Rightarrow \quad r = \frac{\pm 1}{\sqrt{\cos^2\theta - \sin^2\theta}} = \frac{\pm 1}{\sqrt{\cos 2\theta}}$$

If we replace θ with $\theta - \pi/4$ then this amounts to rotating the given graph by $\pi/4$. Let us examine how such a rotation plays out in going from the preceding example to the example given next:

Example 1.13. *Consider $r = \frac{\pm 1}{\sqrt{\cos(2(\theta-\pi/4))}} = \frac{\pm 1}{\sqrt{\cos(2\theta-\pi/2)}}$. Notice,*

$$\cos\left(2\theta - \frac{\pi}{2}\right) = \sin(2\theta) = \frac{1}{2}\sin\theta\cos\theta.$$

Squaring the given equation gives:

$$r^2 = \frac{1}{\cos(2(\theta - \pi/4))} = \frac{2}{\sin\theta\cos\theta}.$$

Thus $r^2\sin\theta\cos\theta = 2$. Recall $r\sin\theta = y$ and $r\cos\theta = x$ thus we find $yx = 2$. Note $y = \frac{2}{x}$ is a hyperbola.

Example 1.14. *Consider $y = mx + b$ where m, b are constants. To find the polar equation of the given line we substitute $x = r\cos\theta$ and $y = r\sin\theta$ to obtain:*

$$r\sin\theta = mr\cos\theta + b$$

If $b \neq 0$ then we can express a line as a polar graph:

$$r = \frac{b}{m\cos\theta + \sin\theta}.$$

If $b = 0$ then the equation of a line in polar coordinates is given by a pair of rays. For example, $y = x$ is given by $\theta = \pi/4$ and $\theta = 5\pi/4$.

Notice $\theta = \theta_o$ for a given constant θ_o describes a **ray** based at the origin.

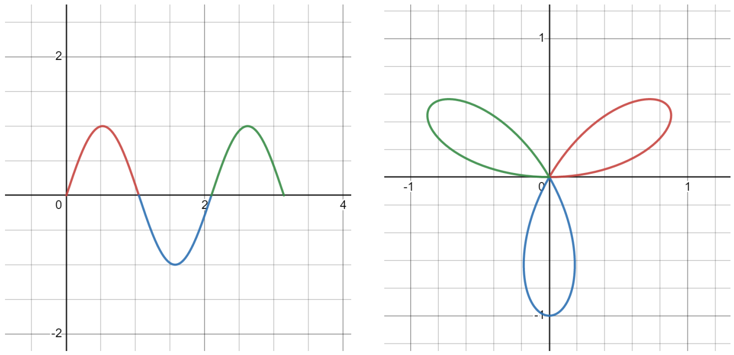
1.1 Plotting Polar Graphs

Given $r = f(\theta)$ we can graph the curve in the xy-plane by the following technique:

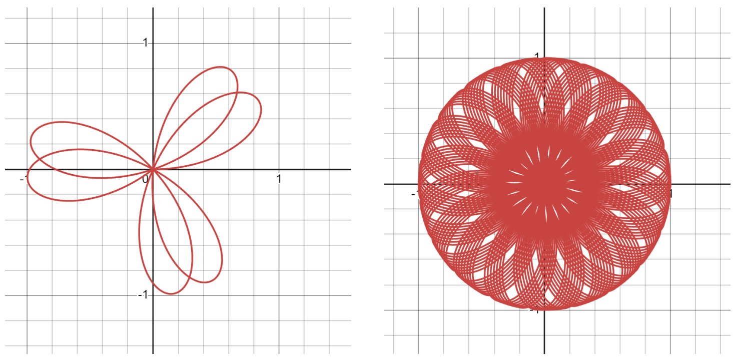
- **(i.)** graph $r = f(\theta)$ in the θr-plane,
- **(ii.)** plot the curve in the xy-plane using reference rays corresponding to important features in the θr-graph.

I will attempt to convey the concept outlined above in these notes, but I think this is better communicated in lecture where we draw the pictures step-by-step.

Example 1.15. *Color indicates which part of the $r = \sin(3\theta)$ graph corresponds to each petal. Each node of the graph of the sine function corresponds to a petal. It turns out that studying $0 \leq \theta \leq \pi$ suffices to cover the whole graph. If we extend θ further then we duplicate what is already shown.*

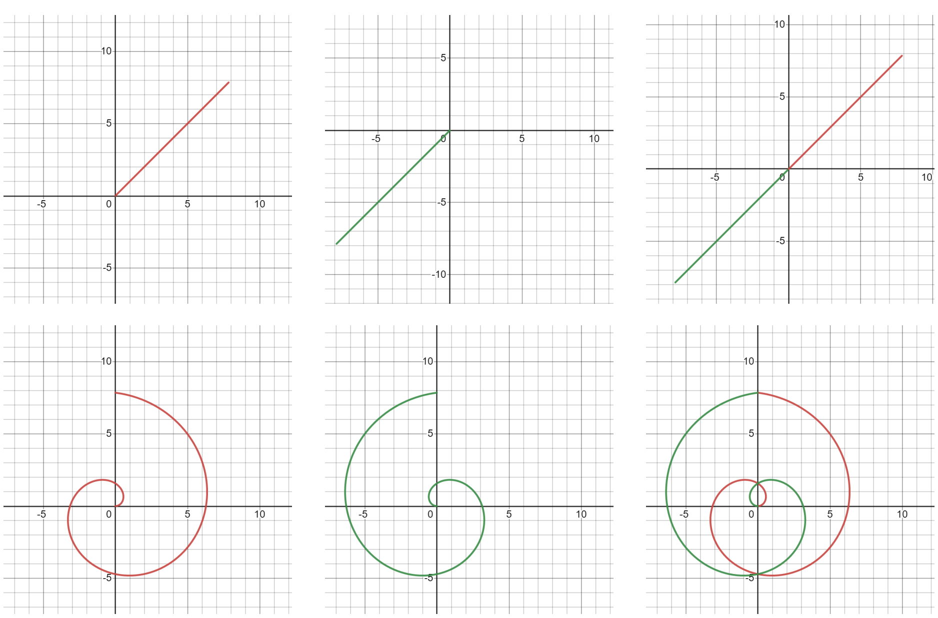


Example 1.16. *Consider $r = \sin(e\theta)$ for $0 \leq \theta \leq 6\pi/e$ and then for $0 \leq \theta \leq 150\pi/e$.*

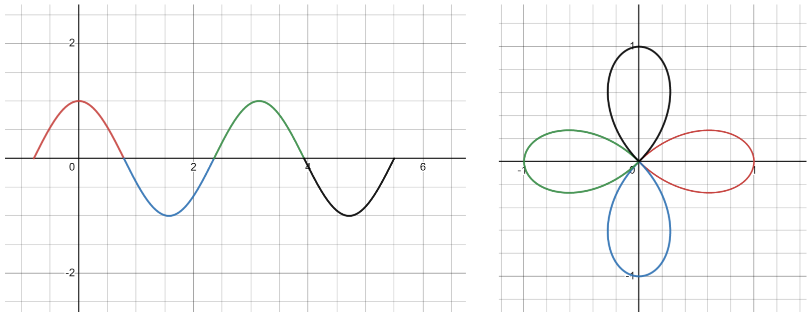


If I extended the domain of θ further the graph of this equation is dense on the unit-disk. The polar graph never returns to the same point in the same fashion. You can contrast this with $r = \sin(3\theta)$ which closes back on itself after once we extend θ over π-radians.

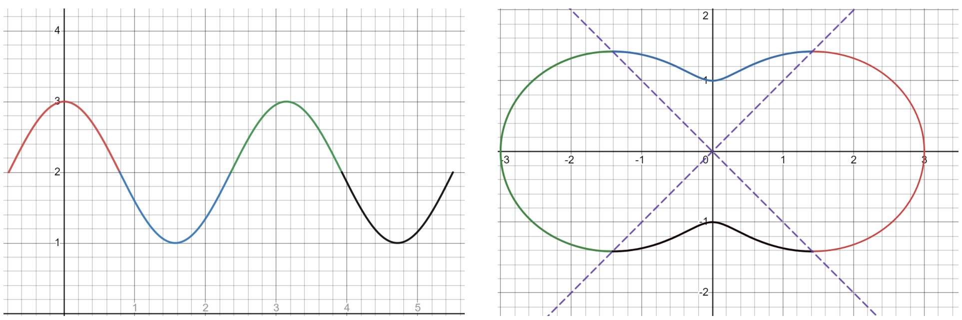
Example 1.17. *The graph of $r = \theta$ for $-5\pi/2 \leq \theta \leq 5\pi/2$ is pictured below. I broke the negative angle and postive angle cases into separate graphs in the interest of appreciating how negative r is graphed in contrast to postive r:*



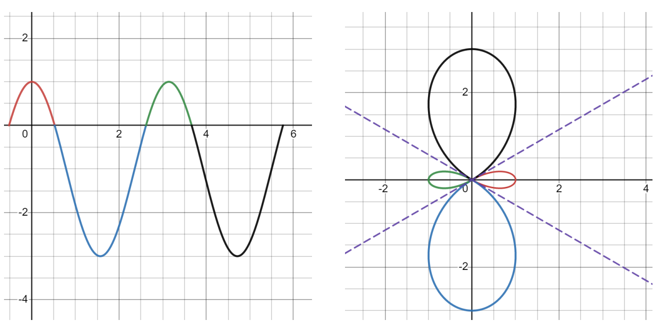
Example 1.18. *Let's graph $r = \cos(2\theta)$. Once more I use color coding to indicate the part of the graph in the xy-plane which corresponds to the graph of $r = \cos(2\theta)$ in the θr-plane:*



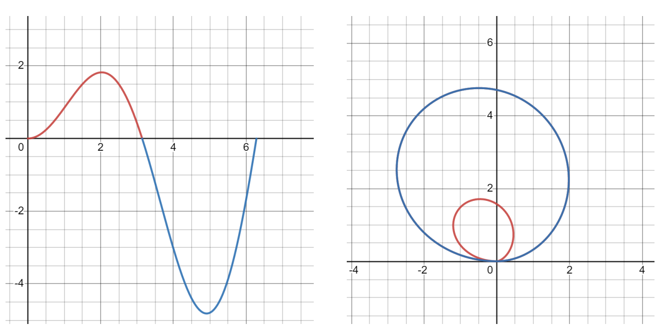
Example 1.19. *Let's graph $r = 2 + \cos(2\theta)$. Once more I use color coding to indicate the part of the graph in the xy-plane which corresponds to the graph of $r = \cos(2\theta)$ in the θr-plane. I also put $\theta = \pm\pi/4, \pm 3\pi/4$ as dotted-purple lines to help follow the cases.*



Example 1.20. *Let's graph $r = -1 + 2\cos(2\theta)$. Once more I use color coding to indicate the part of the graph in the xy-plane which corresponds to the graph of $r = -1 + 2\cos(2\theta)$ in the θr-plane. I also included $\theta = \pm\pi/6$ and $\theta = 5\pi/6$ and $\theta = 7\pi/6$ as dotted-purple lines to help track the geometric boundaries of each of the 4 petals seen in the graph below:*



Example 1.21. *Let's graph $r = \theta\sin(\theta)$. Once more I use color coding to indicate the part of the graph in the xy-plane which corresponds to the graph of $r = \theta\sin(\theta)$ in the θr-plane:*



2 Calculus in Polar Coordinates

Our goal in this section is to explain how formulas of calculus we know in Cartesian terms must be modified when our curve or region is described using polar coordinates.

2.1 arclength

If $x = r\cos\theta$ and $y = r\sin\theta$ where r, θ are both functions of time t then you are asked to show in a homework that

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

If we set $t = \theta$ then we derive the following special formula for a polar graph $r = r(\theta)$,

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 2.1. *Calculate the length of the polar graph* $r = R$ *where* $0 \leq \theta \leq 2\pi$. *Notice* $\frac{dr}{d\theta} = 0$ *and* $ds = Rd\theta$ *thus* $s = \int_0^{2\pi} Rd\theta = 2\pi R$. *This is not surprising.*

Example 2.2. *Calculate the length of the spiral* $r = \theta$ *where* $0 \leq \theta \leq 4\pi$. *Calculate,*

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \sqrt{\theta^2 + 1}\, d\theta$$

If we make a $\sinh t = \theta$ *substitution then* $\theta^2 + 1 = \sinh^2 t + 1 = \cosh^2 t$ *and* $d\theta = \cosh t\, dt$. *Note* $\sinh(0) = 0$ *and* $\sinh(\sinh^{-1}(4\pi)) = 4\pi$ *thus:*

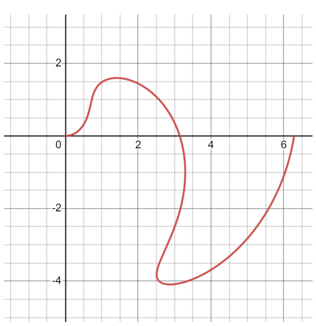
$$s = \int_0^{4\pi} \sqrt{\theta^2 + 1}d\theta = \int_0^{\sinh^{-1}(4\pi)} \cosh^2(t)\, dt = \frac{1}{2}\int_0^{\sinh^{-1}(4\pi)} (1 + \cosh(2t))\, dt$$

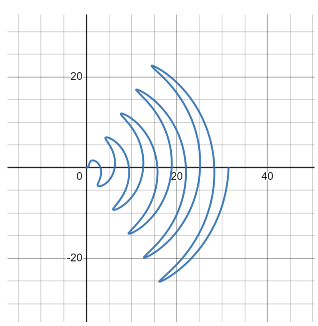
Thus $s = \frac{1}{2}\sinh^{-1}(4\pi) + \frac{1}{4}\sinh(2\sinh^{-1}(4\pi)) \approxeq 80.82$. *The second term could be further simplified using* $\sinh 2t = 2\cosh t \sinh t$.

Example 2.3. *A* **polar parametric** *curve is defined by giving both* r *and* θ *as functions of a parameter. For instance,* $r = t$ *and* $\theta = \sin t$ *for* $0 \leq t \leq 2\pi$ *defines such a curve. To calculate its arclength we use the following approach:*

$$ds = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}\, dt = \sqrt{1 + t^2\cos^2 t}\, dt$$

Numerical integration yields $s = \int_0^{2\pi} \sqrt{1 + t^2\cos^2 t}dt \approxeq 15.0131$. *The graph of this can be plotted as follows. I used the direct parametrization* $\vec{r}(t) = \langle t\cos(\sin t), t\sin(\sin t)\rangle$ *to make the plot in Desmos: (for the blue graph I used* $0 \leq t \leq 10\pi$*)*





I hope the examples above suffice to illustrate how we may calculate arclength for curves described with polar coordinates.

2.2 tangent lines

Next, we turn to the problem of finding the slope of a tangent line. If we suppose $r = r(t)$ and $\theta = \theta(t)$ describe a polar parametrized curve then $x = r\cos\theta$ and $y = r\sin\theta$ indicate the parametrization of the curve in Cartesian coordinates is simply:

$$\vec{r}(t) = \langle x(t), y(t)\rangle = \langle r(t)\cos\theta(t), r(t)\sin\theta(t)\rangle$$

where I have made the t-dependence explicit as to emphasize why calculating the velocity necessarily requires product-rules:

$$\vec{v} = \frac{d\vec{r}}{dt} = \left\langle \cos\theta\,\frac{dr}{dt} - r\sin\theta\,\frac{d\theta}{dt},\ \sin\theta\,\frac{dr}{dt} + r\cos\theta\,\frac{d\theta}{dt}\right\rangle$$

The formula above can be used to find the direction of the tangent line at a given point on a polar parametrized curve. Furthermore, since, $\frac{dx}{dt} = \cos\theta\,\frac{dr}{dt} - r\sin\theta\,\frac{d\theta}{dt}$ whereas $\frac{dy}{dt} = \sin\theta\,\frac{dr}{dt} + r\cos\theta\,\frac{d\theta}{dt}$ we could calculate the slope of the tangent line to a polar parametrized curve for such t that $\frac{dx}{dt} \neq 0$ via

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin\theta\,\frac{dr}{dt} + r\cos\theta\,\frac{d\theta}{dt}}{\cos\theta\,\frac{dr}{dt} - r\sin\theta\,\frac{d\theta}{dt}}.$$

Example 2.4. *Polar graphs use $t = \theta$ in which case:*

$$\vec{v} = \left\langle \cos\theta\,\frac{dr}{d\theta} - r\sin\theta,\ \sin\theta\,\frac{dr}{d\theta} + r\cos\theta\right\rangle \qquad \& \qquad \frac{dy}{dx} = \frac{\sin\theta\,\frac{dr}{d\theta} + r\cos\theta}{\cos\theta\,\frac{dr}{d\theta} - r\sin\theta}.$$

For $r = \theta^2$ we find:

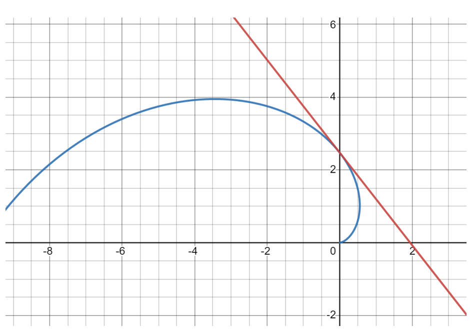
$$\frac{dy}{dx} = \frac{2\theta\sin\theta + \theta^2\cos\theta}{2\theta\cos\theta - \theta^2\sin\theta}$$

The tangent line to this curve when $\theta = \pi/2$ has

$$\frac{dy}{dx} = \frac{\pi\sin(\pi/2) + (\pi/2)^2\cos(\pi/2)}{\pi\cos(\pi/2) - (\pi/2)^2\sin(\pi/2)} = \frac{\pi}{-(\pi/2)^2} = -\frac{4}{\pi}.$$

The point on the curve which corresponds to $\theta = \pi/2$ has $x = (\pi/2)^2\cos(\pi/2) = 0$ and $y = (\pi/2)^2\sin(\pi/2) = \pi^2/4$. Thus the equation of the tangent line is:

$$y = \frac{\pi^2}{4} - \frac{4}{\pi}x$$



2.3 area

In polar coordinates if a region R can be characterized by the inequalities $r_{in}(\theta) \leq r \leq r_{out}(\theta)$ for $\theta_1 \leq \theta \leq \theta_2$ then the area of R can be calculated by the following integral:

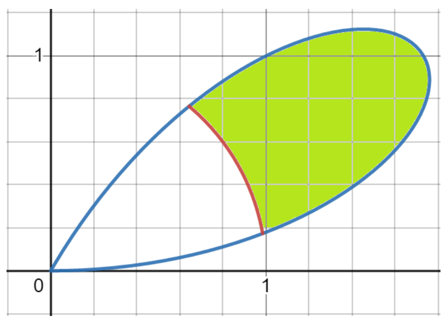
$$\text{area}(R) = \int_{\theta_1}^{\theta_2} \frac{1}{2}(r_{in}^2 - r_{out}^2)d\theta.$$

This integral follows naturally from the formula for the area of a sector which sweeps $\triangle\theta$-radians with inner radius r_{in} and outside radius r_{out}; $\triangle A = \frac{1}{2}(r_{out}^2 - r_{in}^2)\triangle\theta$. For a region as described above we can divide the area into a fan of n equal-angle sectors and as $n \to \infty$ we find $\triangle\theta \to 0$ and the summation can be identified as the Riemann sum of the integral given above. I'll probably draw a supporting picture in lecture.

Example 2.5. *Find area of R which is bounded by $r = \theta^2$ and the x-axis where $0 \leq \theta \leq \pi$. Calculate:*

$$\textit{area}(R) = \int_0^{\pi} \frac{1}{2}\theta^2 \, d\theta = \frac{\pi^3}{6}.$$

Example 2.6. *Find area bounded outside $r = 1$ and inside $r = 2\sin(3\theta)$ in Quadrant I.*



Notice the polar graphs intersect where $1 = 2\sin(3\theta)$ which suggest $\sin(3\theta) = \frac{1}{2}$ which has solutions $3\theta = \frac{\pi}{6}$ and $3\theta = \frac{5\pi}{6}$ yielding $\theta = \frac{\pi}{18}$ and $\theta = \frac{5\pi}{18}$. These are the only solutions with $0 < \theta < \pi/2$ as is required for Quadrant I. Calculate:

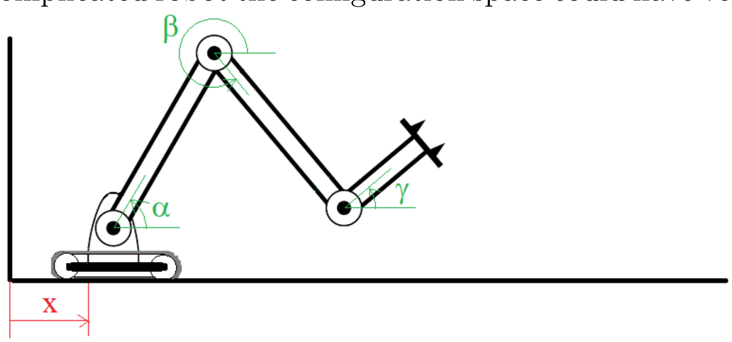
$$\begin{aligned}
\textit{area} &= \int_{\pi/18}^{5\pi/18} \frac{1}{2}\left(4\sin^2(3\theta) - 1\right) d\theta \\
&= \int_{\pi/18}^{5\pi/18} \left(2\sin^2(3\theta) - \frac{1}{2}\right) d\theta \\
&= \int_{\pi/18}^{5\pi/18} \left(\frac{1}{2} - \cos(6\theta)\right) d\theta \\
&= \left(\frac{\theta}{2} - \frac{1}{6}\sin(6\theta)\right)\Bigg|_{\pi/18}^{5\pi/18} \\
&= \left(\frac{5\pi}{36} - \frac{1}{6}\sin\left(\frac{5\pi}{3}\right)\right) - \left(\frac{\pi}{36} - \frac{1}{6}\sin\left(\frac{\pi}{3}\right)\right) \\
&= \frac{4\pi}{36} + \frac{1}{6}\frac{\sqrt{3}}{2} + \frac{1}{6}\frac{\sqrt{3}}{2} \\
&= \frac{\pi}{9} + \frac{\sqrt{3}}{6}
\end{aligned}$$

I will likely work additional examples in lecture.

3 Parametric Equations for Curves

What is dimension ? We live in a world of three spatial dimensions. If we fix an origin then from that reference point we can measure length, width and height to which we assign $(x, y, z) \in \mathbb{R}^3$. Three dimensional space is three-dimensional because it takes three independent real numbers to fix a point in space. Similarly, the xy-plane, or this sheet of paper (or screen) is two-dimensional as a point in the page is uniquely fixed by a pair of real parameters. The number of independent parameters of a space tells us its dimension.

It's hard to imagine more than three spatial dimensions[2]. However, you can face more than three dimensions in real world mechanical problems. The robot arm pictured below is controlled by motors which allow the track to move to position x and the arm to articulate according to the angles α, β, γ. The *configuration space* for this system is four dimensional. THINK: for a more complicated robot the configuration space could have very high dimension.



Now that we have some idea about dimension. Let's turn our attention to curves:

A curve is a one-dimensional object

In other words, each point on a curve can be described by a single real **parameter**. Another word that is helpful is **intrinsic**; a curve has an **intrinsic** dimension of one.

Example 3.1. *One direction of a highway can use mile-markers as a parameter. If you tell someone which mile-marker you are at then they know where you are on the highway.*

We make no assumption that this choice of parameter is **unique**. We can talk about a curve in the plane, or a curve in three-dimensional space, or even a curve in spacetime[3]. However, for the remainder of this talk I will focus our attention on curves in the xy-plane.

Definition 3.2. *A* **path** *in* $\mathbb{R}^2$ *is a pair of functions* $g_1 : I \to \mathbb{R}$ *and* $g_2 : I \to \mathbb{R}$ *where* $I \subseteq \mathbb{R}$ *is the* **domain** *of the path. The set* $\{(g_1(t), g_2(t)) \mid t \in I\}$ *is called the* **trace** *of the path. If a curve* C *in* $\mathbb{R}^2$ *is the trace of a path then that path is called a* **parametrization** *of the curve. We also say that* $x = g_1(t)$ *and* $y = g_2(t)$ *are* **parametric equations** *for* C*.* [4]

We have two main goals in the remainder of this section:

(i.) for a given path, decide if it is the parametrization of a *known* curve

(ii.) for a *known* curve, find a suitable parametrization.

[2]although, time as the fourth dimension is a basic idea of modern relativistic physics

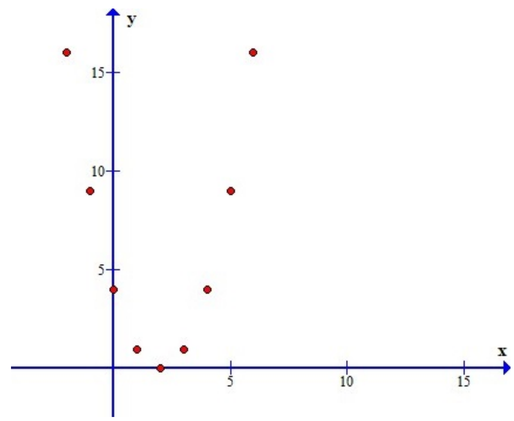
[3]also known as a worldline

[4]I should caution, there is not universal agreement on terminology. Often it is assumed that g_1, g_2 are continuous, and worse yet the term path and curve are often used interchangably.

The goals above beg an obvious question: *what curves do we know*? Before I answer that question let us examine a mystery curve. Let's see if we can cipher its true identity.

Example 3.3. *Consider the path given by* $\left\{ \begin{array}{c} x = t+2 \\ y = t^2 \end{array} \right\}$ *for* $-4 \leq t \leq 4$. *We can study the path by plotting some points:*

t	$x = t+2$	$y = t^2$
-4	-2	16
-3	-1	9
-2	0	4
-1	1	1
0	2	0
1	3	1
2	4	4
3	5	9
4	6	16



You probably can see this is a **parabola**. *We can see this by* **algebra***: eliminate* t *by noting* $t = x - 2$ *thus* $y = t^2 = (x-2)^2$. *Indeed,* $y = (x-2)^2$ *describes a parabola with vertex* $(2, 0)$.

We can think about graphs of functions of the Cartesian variable x[5].

Definition 3.4. *If* f *is a function of* x *then* $graph(f) = \{(x, f(x)) \mid x \in dom(f)\}$.

The graph $y = f(x)$ must pass the **vertical line test**; each vertical line in $\mathbb{R}^2$ intersects graph(f) at most once. Graphs are simple to parametrize.

Example 3.5. *The graph* $y = f(x)$ *is nicely parametrized by using* $x = t$. *Observe* $\left\{ \begin{array}{c} x = t \\ y = f(t) \end{array} \right\}$ *for* $t \in dom(f)$ *parametrizes* $graph(f)$.

Example 3.6. *If* $f(x) = x\cos(x)$ *then* $x = t$ *and* $y = t\cos(t)$ *for* $-\infty < t < \infty$ *parametrizes* $graph(f)$. *I've assumed that* f *is given the natural domain for the given the formula for* $f(x)$.

Line segments can be vertical so it may be impossible to look at a given line-segment as the graph of $f(x)$. In contrast, it is always possible to parametrize a line segment.

Example 3.7. *Let* $P = (p_1, p_2)$ *and* $Q = (q_1, q_2)$ *be a pair of points in the plane. Then*

$$x = p_1 + t(q_1 - p_1), \qquad y = p_2 + t(q_2 - p_2)$$

for $0 \leq t \leq 1$ *parametrizes the line-segment from* P *to* Q. *For instance, if* $P = (3, 5)$ *and* $Q = (3, 8)$ *then*

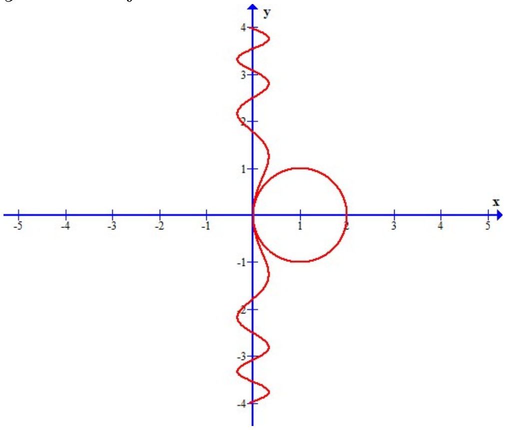
$$x = 3, \qquad \& \qquad y = 5 + 3t$$

for $0 \leq t \leq 1$ *parametrize the line segment from* P *to* Q.

[5]admittedly, we could also think about graph$(g) = \{(g(y), y) \mid y \in \text{dom}(g)\}$ if we were given a g as a function of y. That sort of graph would have to pass the horizontal line test. I decided to let you talk to me about this in office hours if you're interested

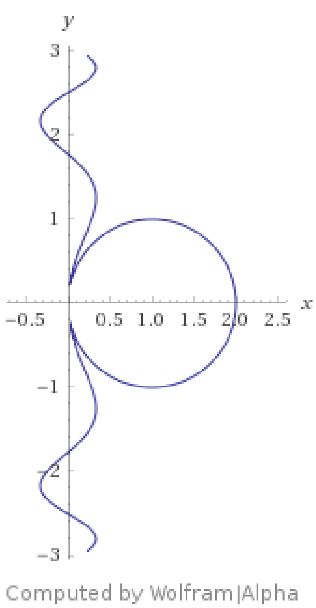
Intuitively, one of the advantages of the parametric viewpoint is it treats x and y on an equal footing. There is no vertical line test for parametrized curves. A parametrized curves allow much creativity.

Example 3.8. *We define* $x = \frac{1}{3}\sin(t^2)$ *and* $y = t$ *for* $-4 \leq t \leq 4$ *and* $x = 1 + \cos(t)$ *and* $y = \sin t$ *for* $4 \leq t \leq 4 + 2\pi$. *Here's a picture of this path:*



To be fair, solution sets to equations also allow nearly limitless possibility. I made an octopus of sorts if you use your imagination:

Example 3.9. *The solution set of the equation* $(3x - \sin(y^2))((x-1)^2 + y^2 - 1) = 0$ *as plotted by Wolfram Alpha is:*

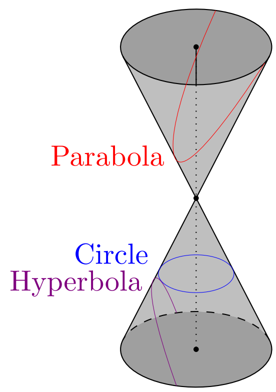


The idea of building curves is fairly simple; for parametric curves I merely splice things together when convenient. In contrast, to combine *solution sets* we can simply multiply the defining equations. To be clear, what follows is probably more important than my octo-construction shannigans on this page.

Remark 3.10. *Wolfram Alpha and other such software cannot be universally trusted for complicated graphs or analysis. It is a tool which must be used responsibily. One of our goals in calculus is gaining intuition for when machines are lying. The more you know the easier it is to see computer failure.*

4 Conic Sections

I created this picture with the Tikz package in LaTeX. Thanks to Mark Wibrow for posting code which I found at this website. The idea of the picture is that we create parabolas, circles and hyperbolas by taking particular slices of the double-cone. It is also possible to hit just the point where the cones meet to get a single point. Furthermore, if you take a vertical plane you can obtain a pair of lines as a cross-section of the cone. The point and line cases are known as degnerate conics.



A second-degree quadratic equation in x, y has the general form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

It turns out that the solution to the above equation is a conic section (possibly degenerate) and so it is convenient to provide the following **algebraic** definitions for a conic section:

Definition 4.1. *Given constants A, B, C, D, E, F the solution set of*

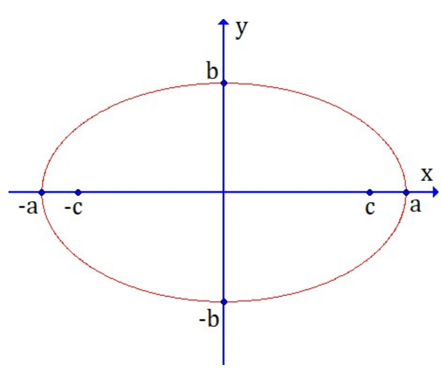
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

is a **conic section**. *We say the conic section is degenerate if it is a line or a point.*

In this section I focus on how we graph and parametrize an ellipse or a hyperbola.

4.1 Short Ellipse

Given $a \geq b$ and $c = \sqrt{a^2 - b^2}$, solutions of $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$ form an ellipse.



- ⋄ center of ellipse is at $(0, 0)$ and **foci** are at $(-c, 0)$ and $(c, 0)$

- $\diamond$ **major axis** connects the **major vertices** $(-a, 0)$ and $(a, 0)$ on the x-axis
- $\diamond$ **minor axis** connects the **minor vertices** $(0, -b)$ and $(0, b)$ on the y-axis

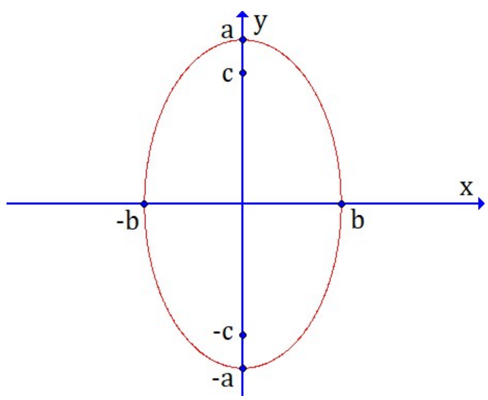
Concerning the Geometric Definition: ellipse is defined as the collection of points for which the sum of the distance from the focal points is a fixed constant. For this ellipse that means the point (x, y) is on the ellipse if

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

Algebra[6] reveals that the above is equivalent to the beautiful $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

4.2 Tall Ellipse

Given $a \geq b$ and $c = \sqrt{a^2 - b^2}$, the solution set of $\boxed{\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1}$ is an ellipse.



- $\diamond$ **center** of ellipse is at $(0, 0)$, **focal points**[7] are at $(0, -c)$ and $(0, c)$
- $\diamond$ **major axis** connects the **major vertices** $(0, -a)$ and $(0, a)$ on the y-axis
- $\diamond$ **minor axis** connects the **minor vertices** $(-b, 0)$ to $(b, 0)$ on the x-axis

Concerning the Geometric Definition: ellipse is defined as the collection of points for which the sum of the distance from the focal points is a fixed constant. For this ellipse that means the point (x, y) is on the ellipse if

$$\sqrt{x^2 + (y+c)^2} + \sqrt{x^2 + (y-c)^2} = 2a.$$

Algebra[8] reveals that the above is equivalent to the beautiful equation $\dfrac{y^2}{a^2} + \dfrac{x^2}{b^2} = 1$.

---

[6] There is much more to learn than I share here. Please see the *Conic Sections: Visualization and Derivation* powerpoint if you are interested in the details of the geometric derivations I mention in this section

[7] fine, fine, **foci** if you insist

[8] yes, you could simply derive the results here by interchanging x and y in the previous subsection

4.3 Parametrization of Ellipse

The reason we insisted that $a > b$ in the previous two subsections was to simplify the discussion of focal points. Now that we are past that it is convenient to explain the form is of an ellipse centered at (h, k). Generally, if we have a particular curve centered at $(0, 0)$ then we may shift it to (h, k) by replacing x with $x - h$ and y with $y - k$ in the formulas which desribe the curve.

Definition 4.2. Ellipse centered at (h, k) paralell to the coordinate axes: *given $A, B > 0$ the ellipse centered at (h, k) is the solution set of*

$$\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2} = 1$$

If $A > B$ then the ellipse has a horizontal major axis. If $B > A$ then the ellipse has a vertical major axis. If $A = B$ then the ellipse is a **circle** *and we say $R = A$ is the* **radius** *of the circle.*

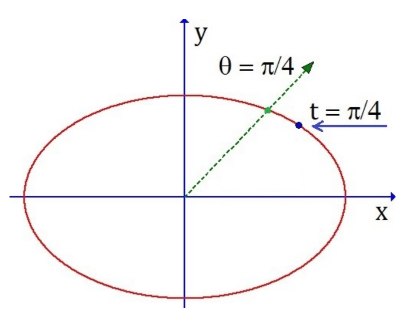
To parametrize the ellipse we need to find functions $g_1(t)$ and $g_2(t)$ such that when we set $x = g_1(t)$ and $y = g_2(t)$ we solve the equation which defines the ellipse. My intuition is that $\cos^2 t + \sin^2 t = 1$ gives us hope since it resembles the ellipse equation. In particular, the pattern I match is just what I indicate below:

$$\underbrace{\frac{(x-h)^2}{A^2}}_{\cos^2 t} + \underbrace{\frac{(y-k)^2}{B^2}}_{\sin^2 t} = 1$$

This suggests we set $\frac{x-h}{A} = \cos t$ and $\frac{y-k}{B} = \sin t$. Therefore, let us set:

$$\boxed{x = h + A\cos t \qquad \& \qquad y = k + B\sin t \qquad \text{for} \qquad 0 \leq t \leq 2\pi.}$$

You can check, if you plug the above parametric equations into the expression $\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2}$ then the result is 1. In other words, our formulas serve to parametrize the ellipse. In fact, they give a CCW[9] parameterization of the ellipse. Geometrically the parameter t is like the standard angle from polar coordinates. However, to be careful, it is only the standard angle in the special case that $A = B$. Otherwise, if look at the intersection of the ellipse with the ray $\theta = t_o$ you will not usually obtain the point mapped to by t_o. For example, the graph below illustrates my comment for $x^2/9 + y^2/25 = 1$ parametrized by $x = 3\cos t$ and $y = 5\sin t$. You can see at $t = \pi/4$ the blue dot is where the parametrization maps to whereas the green dot is at standard angle $\pi/4$.



[9] this is an abbreviation for Counter Clock Wise which refers to going opposite the standard motion of old-fashioned analog clocks. In contrast, CW means Clock Wise, as in the same way a clock goes.

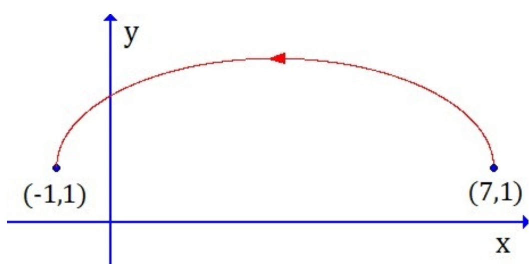
I should point out we can modify our formulas above as follows:

- ⋄ let $\left\{ \begin{array}{c} x = h + A\cos(\omega t) \\ y = k + B\sin(\omega t) \end{array} \right\}$ for $0 \leq t \leq 2\pi$ and appropriate ω.
- ⋄ set $\omega = 1$ for CCW motion, or $\omega = -1$ for CW motion to cover the ellipse. The domain $0 \leq t \leq 2\pi$ can be modified to cover less or more of the ellipse as needed.

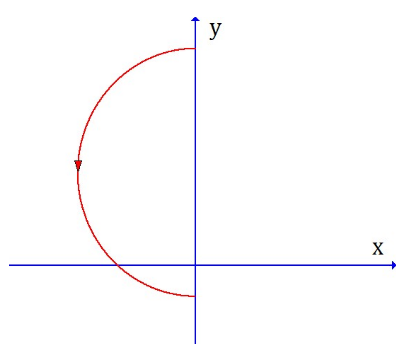
Example 4.3. *To CW parametrize* $\frac{x^2}{9} + \frac{y^2}{25} = 1$ *simply set* $x = 3\cos t$ *and* $y = -5\sin t$ *for* $0 \leq t \leq 2\pi$.

Example 4.4. *To CCW parametrize* $\frac{x^2}{25} + \frac{y^2}{9} = 1$ *simply set* $x = 5\cos t$ *and* $y = 3\sin t$ *for* $0 \leq t \leq 2\pi$.

Example 4.5. *Parametrize the top-half of* $x^2 - 6x + 4y^2 - 8y = 3$ *in the CCW sense. By algebra of completing the square,* $(x-3)^2 + 4(y-1)^2 = 3 + 9 + 4 = 16$. *Therefore,* $\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$. *hence we set* $x = 3 + 4\cos t$ *and* $y = 1 + 2\sin t$. *We select the top-half by limiting the parameter to* $0 \leq t \leq \pi$.

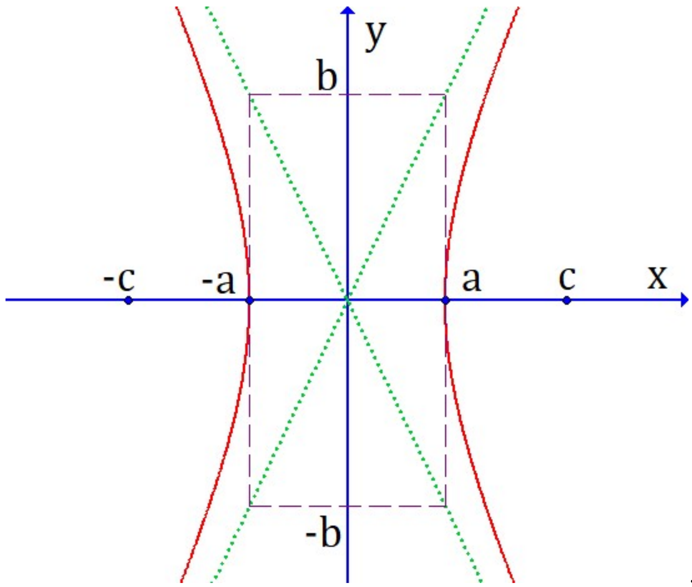


Example 4.6. *Parametrize the left half of* $x^2 + y^2 - 6y = 7$ *in a CCW direction. By algebra,* $x^2 + (y-3)^2 = 16$. *Identify the curve is a circle of radius* 4 *centered at* $(0,3)$. *The left half of the circle begins at* $(0,7)$ *and ends at* $(0,-1)$. *We parametrize this half-circle by* $x = 4\cos t$ *and* $y = 3 + 4\sin t$ *for* $\pi/2 \leq t \leq 3\pi/2$.



4.4 Horizontal Hyperbolas

Given nonzero a, b, the solution set of $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ is a **horizontal hyperbola**.



- ◇ with **foci** on the x-axis at $(c, 0)$ and $(-c, 0)$
- ◇ **slant-asyptotes** at $y = \pm\frac{b}{a}x$
- ◇ **transverse** or **conjugate** y-axis which the hyperbola does not cross
- ◇ the box drawn using $y = \pm b$ and $x = \pm a$ allows us to graph the asymptotes with ease.

Concerning the Geometric Definition: hyperbola is defined as the collection of points for which the difference of the distances from a pair of focal points is a fixed constant. For this hyperbola that means the point (x, y) is on the **right branch** of the hyperbola if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a.$$

Similarly, (x, y) is on the **left branch** of the hyperbola if

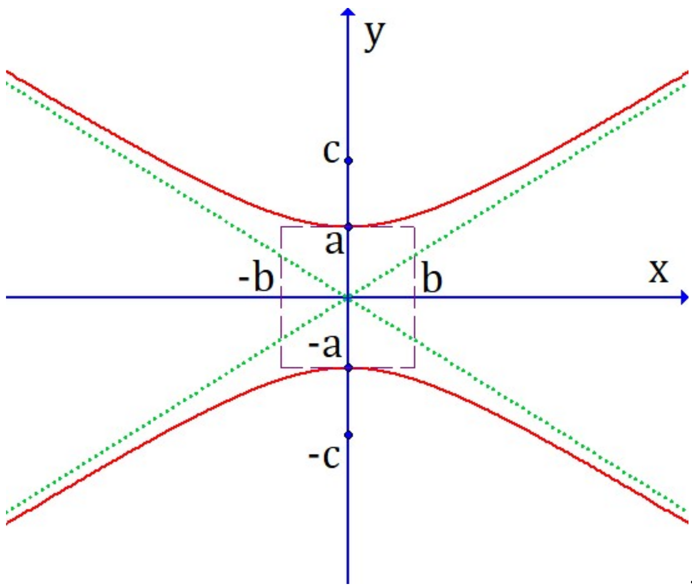
$$\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = 2a.$$

Isolating and eliminating the radicals in the right or left branch equations bring us to the beautiful standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ for the hyperbola where b is defined by $b = \sqrt{c^2 - a^2}$.

4.5 Vertically Opening Hyperbolas:

Given nonzero a, b the solution set of $\boxed{\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1}$ is a **vertical hyperbola**

- ◇ is a hyperbola with foci on the y-axis and asymptotes $y = \pm\frac{a}{b}x$



- ◇ with **foci** on the y-axis at $(0, c)$ and $(0, -c)$
- ◇ **slant-asyptotes** at $y = \pm\frac{a}{b}x$
- ◇ **transverse** or **conjugate** x-axis which the hyperbola does not cross
- ◇ the box drawn using $y = \pm b$ and $x = \pm a$ allows us to graph the asymptotes with ease.

Concerning the Geometric Definition: hyperbola is defined as the collection of points for which the difference of the distances from a pair of focal points is a fixed constant. For this hyperbola that means the point (x, y) is on the **upper branch** of the hyperbola if

$$\sqrt{x^2 + (y + c)^2} - \sqrt{x^2 + (y - c)^2} = 2a.$$

Similarly, (x, y) is on the **lower branch** of the hyperbola if

$$\sqrt{x^2 + (y - c)^2} - \sqrt{x^2 + (y + c)^2} = 2a.$$

Isolating and eliminating the radicals in the right or left branch equations bring us to the standard form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ for the hyperbola where b is defined by $b = \sqrt{c^2 - a^2}$.

4.6 Rotated Conic Sections

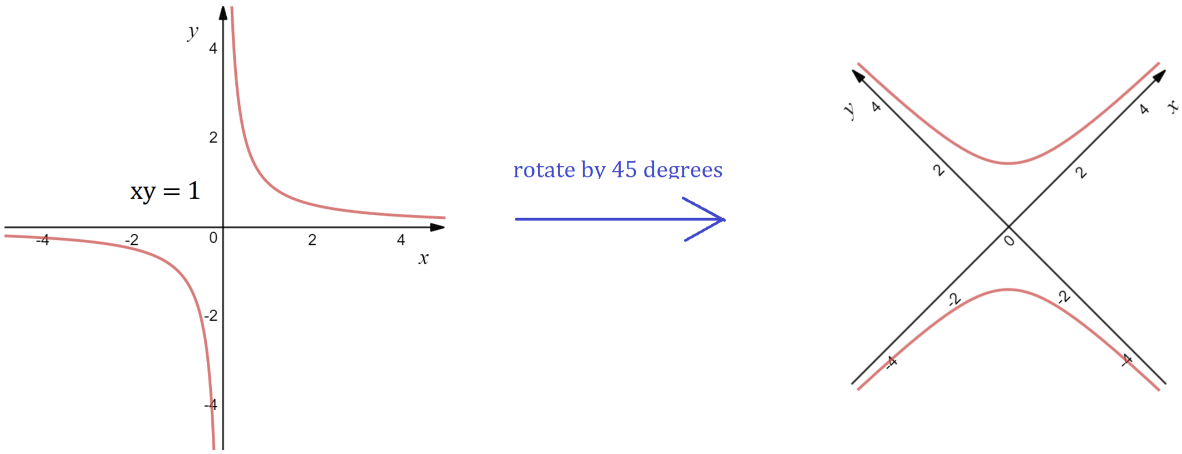
A second-degree quadratic equation in x, y has the general form:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The solution to the above equation is a conic section, possibly degenerate. We have seen this in some detail for examples where the Bxy term is missing. It turns out that it is always possible to remove the *cross-term* by rotating coordinates. That is, there exists a coordinate system for the plane such that the above equation is modified to

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

You can read more about this in Chapter 12 of Howard Anton's *Calculus*. He explains how to find the rotated coordinate system at the level of your current course. I will be content to share the following to illustrate the claim:



4.7 Parametrization of Hyperbolas

Let me begin by forging a general, algebraically convenient, definition:

Definition 4.7. HYPERBOLA CENTERED AT (h, k) PARALELL TO THE COORDINATE AXES: *given* $A, B > 0$ *the ellipse centered at* (h, k) *is the solution set of*

$$\frac{(x-h)^2}{A^2} - \frac{(y-k)^2}{B^2} = \varepsilon$$

for $\varepsilon = \pm 1$. *In particular, if* $\varepsilon = -1$ *then the hyperbola opens vertically . If* $\varepsilon = 1$ *then the ellipse opens horizontally. The lines* $y = k \pm \frac{B}{A}(x-h)$ *are* ***asymptotes*** *of the hyperbola.*

Focus on the $\varepsilon = 1$ **case.** To parametrize the hyperbola we need to find functions $g_1(t)$ and $g_2(t)$ such that when we set $x = g_1(t)$ and $y = g_2(t)$ we solve the equation which defines the horizontal hyperbola. For the ellipse, the fact that $\cos^2 t + \sin^2 t = 1$ served us well. Let us suppose there are hyperbolic analogs of sine and cosine called **hyperbolic sine** (sinh) and **hyperbolic cosine** (cosh). We would like for these functions to solve the identity

$$\cosh^2 t - \sinh^2 t = 1$$

since if they did it would be really really nice for the calculation below.

$$\underbrace{\frac{(x-h)^2}{A^2}}_{\cosh^2 t} - \underbrace{\frac{(y-k)^2}{B^2}}_{\sinh^2 t} = 1$$

This suggests we set $\frac{x-h}{A} = \cosh t$ and $\frac{y-k}{B} = \sinh t$. Therefore, let us set:

$$\boxed{x = h + A\cosh t \qquad \& \qquad y = k + B\sinh t \qquad \text{for} \qquad -\infty < t < \infty.}$$

We'll see that $\cosh t \geq 1$ in the next section hence the formula above has $x \geq h + A$. In orther words, the above merely parametrizes the **right- branch** of the hyperbola. To cover the **left-branch** we use

$$\boxed{x = h - A\cosh t \qquad \& \qquad y = k + B\sinh t \qquad \text{for} \qquad -\infty < t < \infty.}$$

If you can set aside disbelief just a little longer, we can cover the case $k = -1$ be almost the same arguments. For the vertical hyperbola the **upper- branch** is parametrized by:

$$\boxed{x = h + A\sinh t \qquad \& \qquad y = k + B\cosh t \qquad \text{for} \qquad -\infty < t < \infty.}$$

whereas the **lower-branch** is covered by

$$\boxed{x = h + A\sinh t \qquad \& \qquad y = k - B\cosh t \qquad \text{for} \qquad -\infty < t < \infty.}$$

Example 4.8. *Parametrize $x^2 + 2x - y^2 = 0$ for $x \geq 0$. Notice $(x+1)^2 - y^2 = 1$ thus $(0,0)$ is the vertex of the right-branch of this horizontal hyperbola. Therefore, set $x = -1 + \cosh t$ and $y = \sinh t$ for $t \in \mathbb{R}$.*

It remains to define cosh and sinh. See the next section for details.

5 Review of Hyperbolic Functions

Just in case you missed it. Here is a quick review of the basics of hyperbolic functions.

Let us begin with a rather bizarre step; if f has a domain for which $x \in \text{dom}(f)$ implies $-x \in \text{dom}(f)$ then we have the following identity by adding zero

$$f(x) = \frac{1}{2}\big(f(x) + f(-x)\big) + \frac{1}{2}\big(f(x) - f(-x)\big)$$

you can check that the function $\frac{1}{2}\big(f(x)+f(-x)\big)$ is even whereas the function $\frac{1}{2}\big(f(x)-f(-x)\big)$ is odd. In other words, the identity above shows that any function with a domain symmetric about the origin may be decomposed into the sum of an even and odd function. When we apply this general fact to the exponential function we find the hyperbolic functions.

Definition 5.1. *Let $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. We say $\cosh$ is the* **hyperbolic cosine** *whereas $\sinh$ is the* **hyperbolic sine**.

In many ways cosh and sinh behave just as do cos and sin. For example,

$$\cosh(0) = \frac{1}{2}(e^0 + e^{-0}) = 1$$

and

$$\sinh(0) = \frac{1}{2}(e^0 - e^{-0}) = 0.$$

We also note $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$. Therefore, by their definition and an algebra step, $e^x = \cosh x + \sinh x$ and $e^{-x} = \cosh(-x) + \sinh(-x) = \cosh x - \sinh x$. Therefore,

$$1 = e^x e^{-1} = (\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x.$$

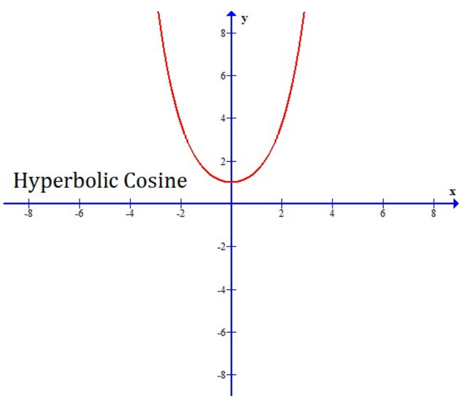
We have shown $\cosh^2 x - \sinh^2 x = 1$ for all real x. These new hyperbolic functions will work nicely to parametrize hyperbolas according to the arguments of the previous section.

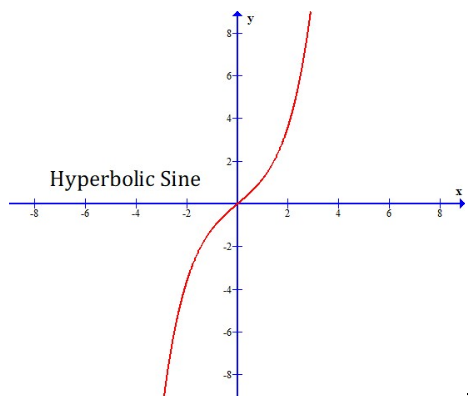
Let me share a few basic facts about the hyperbolic functions. I want you to know the theoretical minimum to understand how these functions behave.

Theorem 5.2.
$$\frac{d}{dx}\cosh x = \sinh x \qquad \& \qquad \frac{d}{dx}\sinh x = \cosh x$$

Proof: exercise for the reader ☺.

Furthermore, since $e^x > 0$ for all $x \in \mathbb{R}$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$ it is immediately clear that $\cosh x > 0$. Therefore, $\sinh x$ gives an increasing function everywhere. Since $\sinh(0) = 0$ and hyperbolic sine has no critical points we find $\sinh(x) < 0$ whenever $x < 0$ and $\sinh(x) > 0$ whenever $x > 0$. In contrast, since $\frac{d}{dx}\cosh x = \sinh x$, it follows hyperbolic cosine decreases for $x < 0$ and then increases when $x > 0$. I usually tell students hyperbolic sine is sort of like a cubic whereas hyperbolic cosine is at a glance a upward opening parabola with y-intercept 1. This is not really accurate since the hyperbolic functions have exponential growth or decay for both $x \to \infty$ and $x \to -\infty$. I'll summarize our findings with a picture:





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The next definition should not be too surprising:

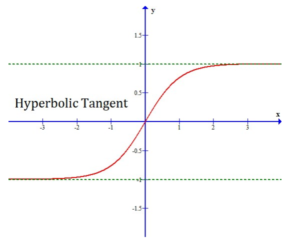
Definition 5.3. *Let* $\tanh(x) = \frac{\sinh x}{\cosh x}$. *We say* tanh *is the* **hyperbolic tangent**. *Similarly* $\text{sech}x = \frac{1}{\cosh x}$ *defines the* **hyperbolic secant**.

The hyperbolic tangent function has many interesting applications. One near and dear to my heart is **rapidity**. In special relativity we define the **rapidity** to be the hyperbolic angle ϕ such that $\tanh\phi = \beta = \frac{v}{c}$ where v is velocity of a particle and c is the speed of light. It turns out that while velocites do no add, rapidities do. For example, if I throw a baseball on a train with rapidity ϕ_1 and I throw that ball with a speed relative the train such that it has rapidity ϕ_2 in the train-frame then with respect to the frame in which the train tracks are stationary the ball has rapidity $\phi_1 + \phi_2$.

Experiments show again and again that velocities of material objects have $-c < v < c$ where c is the speed of light in empty space. Thus, experiments show $-1 < \frac{v}{c} < 1$ for material objects. The hyperbolic tangent gives us a mathematical object which complies with this cosmic speed limit. In particular, observe

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

and think about what happens in the limits $x \to \infty$ and $x \to -\infty$ to see $-1 < \tanh x < 1$. Of course, it was precisely this feature that caused physicists to define rapidity in terms of a hyperbolic tangent. For all ϕ we have $-1 < \tanh \phi = \beta = \frac{v}{c} < 1$.



Remark 5.4. *Hyperbolic functions commonly appear in many physics and engineering problems. For example, the voltage function on a rectangular plate where the sides are insulated but the base and top edges are held at different voltages. The solution is commonly given in terms of hyperbolic functions.*

A hanging chain takes the shape of the graph of hyperbolic cosine. Hyperbolic functions have a beautiful synergy with trigonometric functions and we have only begun to scratch the surface here. I could show you integrals that were difficult with trigonometric subsitution that become nearly trivial with a hyperbolic subsitution. Email me if you're interested, I'd be happy to show you where to look.

5.1 Playing with Euler's Formula

We discussed this in lecture earlier this semester, I leave it here as a digression.

What do you know? If you know that $e^{ix} = \cos x + i \sin x$ since

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \cdots \\ &= \left(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots\right) + i\left(x - \frac{1}{3}x^3 + \frac{1}{5!}x^5 + \cdots\right) \\ &= \cos x + i \sin x \end{aligned}$$

and you notice, since sine is odd $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i \sin x$ thus

$$1 = e^{ix}e^{-ix} = (\cos x + i \sin x)(\cos x - i \sin x) = \cos^2 x + \sin^2 x.$$

then you see we can derive the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$ via these rather crafty series arguments. It stands to reason then that our hyperbolic functions ought to appear as subseries for something like e^{ix}. Why not try e^x ?

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ &= \underbrace{1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots}_{\text{defines}\cosh x} + \underbrace{x + \frac{1}{3}x^3 + \frac{1}{5!}x^5 + \cdots}_{\text{defines}\sinh x} \\ &= \cosh x + \sinh x \end{aligned}$$

By construction cosh is even and sinh is odd hence $\cosh(-x) = \cosh x$ and $\sinh(-x) = -\sinh x$. Thus $e^{-x} = \cosh(-x) + \sinh(-x) = \cosh x - \sinh x$ and so we find:

$$1 = e^x e^{-x} = (\cosh x + \sinh x)(\cosh x - \sinh x) = \cosh^2 x - \sinh^2 x.$$

6 Conic Sections in Polar Coordinates

The conic section of eccentricity $e > 0$ with focus at the origin and directrix $x = d$ has polar equation:

$$r = \frac{ed}{1 + e\cos\theta}$$

The conic section from $e = 0$ is simply the circle $r = R$.