EXAMPLES FOR PROBABILITY

- our focus is on when x denotes a <u>continuous</u> random variable.

– (the larger study of Probability involves more than calculus... we need discrete variables / counting / ... for elsewhere) –

Def^{n}/ f is a <u>probability density function</u> on $\mathbb{R}$ if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

- $P[a \leq x \leq b] = \int_a^b f(x)\,dx$ = probability x is found in $[a,b]$

$$\overline{x} = \mu = \int_{-\infty}^{\infty} x f(x)\,dx = \text{mean or average of } x$$

1.) If $f(x) = \begin{cases} 0 & : x < 0 \\ Ae^{-x} & : x \geq 0 \end{cases}$ then find A for which $f(x)$ forms a prob. density.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} Ae^{-x}\,dx = \lim_{b \to \infty}\left(\int_0^b Ae^{-x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-A\underbrace{e^{-b}}_{0} + Ae^{0}\right)$$

$$= A \quad \therefore \text{ set } \boxed{A = 1}$$

I should mention,

$f(x) = \begin{cases} 0 & : x < 0 \\ e^{-x} & : x \geq 0 \end{cases}$ is an example of an exponential distribution. We can calculate the mean,

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u}\,\underbrace{e^{-x}\,dx}_{dv}$$

$$= \lim_{b\to\infty}\left(-xe^{-x}\Big|_0^b + \int_0^b e^{-x}\,dx\right)$$

$$= \lim_{b\to\infty}\left(-be^{-b}\right) + 1 = \lim_{b\to\infty}\left(\frac{-b}{e^b}\right) + 1 \overset{\mathcal{L}}{=} \lim_{b\to\infty}\left(\frac{-1}{e^b}\right) + 1$$

$\therefore \boxed{\mu = 1}$

2.) $f(x) = \dfrac{A}{4+x^2}$ find A to make $f(x)$ a prob. density. Then calculate $P[0 \le x \le \pi]$.

$$\int \frac{dx}{4+x^2} = \int \frac{2\cancel{\sec^2\theta}\, d\theta}{4\cancel{\sec^2\theta}}$$

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta\, d\theta$$
$$4+x^2 = 4\sec^2\theta$$

$$= \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Consequently,

$$1 = \int_{-\infty}^{\infty} \frac{A\,dx}{4+x^2} = 2\int_0^{\infty} \frac{A\,dx}{4+x^2}$$

-(reasonable step <u>if</u> the resulting $\int_0^\infty$ does exist.)-

$$= 2A \lim_{b\to\infty}\left[\frac{1}{2}\underbrace{\tan^{-1}\left(\frac{b}{2}\right)}_{\frac{\pi}{2}} - \frac{1}{2}\underbrace{\tan^{-1}\left(\frac{0}{2}\right)}_{0}\right]$$

$$= A\cdot\frac{\pi}{2} \Rightarrow \boxed{A = \frac{2}{\pi}}$$

$$\boxed{f(x) = \frac{2/\pi}{4+x^2}}$$

$$P[0 \le x \le \pi] = \int_0^{\pi}\left(\frac{2/\pi}{4+x^2}\right)dx$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\tan^{-1}\left(\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{2}\right)$$

$$\cong \boxed{0.3195}$$

<u>Remark</u>: this application section need not be difficult... you just need to have improper integration mastered.