Chapter 5 : The Isospin Group

Are the proton and neutron the same particle?

(Table 5.1)

	Mass $m_o c^2$ [MeV]	Mass difference [MeV]	Spin	Lifetime [s]	Magnetic Moment [μ_N]
P	983.213	1.294	$\frac{1}{2}$	"stable"	2.793
N	939.507		$\frac{1}{2}$	918 ± 14	-1.913

very nearly in some respects. The mass difference can be understood in part by the different E & M interaction... hmm... Anyway with respect to the strong interaction the P & N look basically the same and have "equal" mass. Thus we can view P & N as different states of the same particle.

The wave function of the nucleons depends on

r — spatial position
t — time
s — spin
τ — isospin

Where by definition $\tau = 1 \Rightarrow$ proton and $\tau = -1 \Rightarrow$ nuetron hence

$$\psi_p = \psi(r, t, s, \tau = 1)$$
$$\psi_N = \psi(r, t, s, \tau = -1)$$

Equivalently we can replace the above with 2-component collumn vector

$$\psi = \begin{pmatrix} u_1(r,t,s) \\ u_2(r,t,s) \end{pmatrix} \quad \text{where} \quad \psi_p = \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_n = \begin{pmatrix} 0 \\ u_2 \end{pmatrix}$$

Where u_1 is for the proton and u_2 is for the nuetron, meaning $|u_1|^2$ is probability density of proton and likewise $|u_2|^2$ is prob. dens. for nuetron.

Introduce a 2×2 matrix operator

$$\hat{\tau}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then this operator's eigenvalues are isospin and of course ψ_p & ψ_n are eigenvectors

$$\hat{\tau}_3 \psi_p = 1 \cdot \psi_p$$
$$\hat{\tau}_3 \psi_n = -1 \cdot \psi_n$$

Lets use χ to denote a particular state $\chi = \chi(r,t,s)$ then

$$\chi_p = \begin{pmatrix} \chi(r,t,s) \\ 0 \end{pmatrix} \quad \underline{\text{AND}} \quad \chi_n = \begin{pmatrix} 0 \\ \chi(r,t,s) \end{pmatrix}$$

Consider the states $\chi_p = \begin{pmatrix} \chi(r,t,s) \\ 0 \end{pmatrix}$ and $\chi_n = \begin{pmatrix} 0 \\ \chi(r,t,s) \end{pmatrix}$ they are related by the raising & lowering operators $\hat{\tau}_+$ and $\hat{\tau}_-$

$$\hat{\tau}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \underline{\text{and}} \quad \hat{\tau}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Notice these change the isospin by ± 1, and limit τ to ± 1.

$$\hat{\tau}_+ \chi_p = 0 \qquad \hat{\tau}_- \chi_p = \chi_n$$
$$\hat{\tau}_+ \chi_n = \chi_p \qquad \hat{\tau}_- \chi_n = 0$$

These operators $\hat{\tau}_\pm$ are singular and nonhermitian are related to the pauli matrices as usual:

$$\boxed{\begin{aligned} \tau_1 &= \hat{\tau}_+ + \hat{\tau}_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \hat{\tau}_2 &= -i(\hat{\tau}_+ - \hat{\tau}_-) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \hat{\tau}_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}}$$

where $\hat{\tau}_i \hat{\tau}_j = i\epsilon_{ijk}\hat{\tau}_k + \delta_{ij} \cdot \mathbb{1}_{2\times 2}$ descibe the alegbra of the Pauli-matrices.

Now notice how the Pauli matrices act on χ_p and χ_n

$$\hat{\tau}_1 \chi_p = \chi_n \qquad \hat{\tau}_1 \chi_n = \chi_p$$
$$\hat{\tau}_2 \chi_p = i\chi_n \qquad \hat{\tau}_2 \chi_n = -i\chi_p$$
$$\hat{\tau}_3 \chi_p = \chi_p \qquad \hat{\tau}_3 \chi_n = -\chi_n$$

<u>Remark</u>: $\{\mathbb{1}, \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3\}$ span the space of 2x2 hermitian matrices. Which means that a self-adjoint operator which acts on 2 deg. of freedom of the Nucleon system with wavefunction ψ can be represented by some linear superposition of $\mathbb{1}$ and $\hat{\tau}_i$.

As it stands we can calculate $[\tau_i, \tau_j] = \tau_i\tau_j - \tau_j\tau_i = 2i\epsilon_{ijk}\hat{\tau}_k$ but we prefer for reasons to become clear soon that $[\hat{T}_i, \hat{T}_j] = i\epsilon_{ijk}\hat{T}_k$ clearly this can be accomplished thru defining (just like spin-operators $\hat{S}_k = \frac{1}{2}\hat{\sigma}_k$)

$$\boxed{\hat{T}_k \equiv \frac{1}{2}\hat{\tau}_k}$$

Isospin Group

"We can show that the two components (u_1, u_2) of the nucleon state ψ build up an elementary spinor in 3-dim'l isospin space". Recall that we defined $\hat{T}_i$ which satisfy $[\hat{T}_i, \hat{T}_j] = i\epsilon_{ijk}\hat{T}_k$ where $\hat{T}_i^{\dagger} = \hat{T}_i$ hence $\text{span}\{\hat{T}_i\}_{i=1}^{3}$ is a Lie algebra. ~~Hence~~ Now we can exponentiate this Lie algebra to find the corresponding Lie group of unitary operators:

$$\boxed{\begin{aligned}\hat{U}_{\text{ISOSPIN}}(\varepsilon_1, \varepsilon_2, \varepsilon_3) &= \exp\left(-i\varepsilon_\nu \hat{T}_\nu\right) \\ &= \exp\left(-\tfrac{i}{2}\left(\varepsilon_1\hat{\tau}_1 + \varepsilon_2\hat{\tau}_2 + \varepsilon_3\hat{\tau}_3\right)\right) \\ &= \mathbb{1}\cos(\varepsilon/2) - i\,\hat{n}\cdot\hat{\tau}\sin(\varepsilon/2)\end{aligned}}$$

See pg. 16 for algebra

Where $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \varepsilon\hat{n}$ as customary. This $\hat{U}_{IS}(\varepsilon)$ is a rotation in abstract isospace, in particular it is a rotation of angle ε about the $\hat{n}$ axis in isospace.

$$\begin{aligned}U_{IS}^{\dagger}(\varepsilon) &= \left(\mathbb{1}\cos(\varepsilon/2) - i\hat{n}\cdot\hat{\tau}\sin(\varepsilon/2)\right)^{\dagger} \\ &= \mathbb{1}\cos(\varepsilon/2) + i\hat{n}\cdot\hat{\tau}\sin(\varepsilon/2) \\ &= U_{IS}^{-1}(\varepsilon)\end{aligned}$$

which is clear from multiply the above with $U_{IS}(\varepsilon)$

$$U_{IS}(\varepsilon)U_{IS}^{-1}(\varepsilon) = \mathbb{1}\cos^2(\varepsilon/2) + (\hat{n}\cdot\hat{\tau})^2\sin^2(\varepsilon/2) = \mathbb{1}.$$

because $(\hat{n}\cdot\hat{\tau})^2 = \mathbb{1}$ from 16.

As we proved more generally before hermitian operators give unitary groups and vica-versa. More than just this note

$$\begin{aligned}\det\left(\hat{U}_{IS}(\varepsilon)\right) &= \det\left(\exp(-i\varepsilon_\nu\hat{T}_\nu)\right) \\ &= \exp\left(-i\varepsilon_\nu \,\text{Trace}(\hat{T}_\nu)\right) \\ &= \exp(0) \\ &= 1\end{aligned}$$

the $\hat{T}_\nu$ matrices are traceless.

You can also see 136 of Greiner for a more pedestrian proof of $\det\hat{U} = 1$.

- The group of isospin operators contains 2x2 unitary, determinant 1 matrices which are of course <u>SU(2)</u>

Isospin Space and Group

The iso spinor ψ transforms under an "isospin rotation" according to

$$\psi \longmapsto \psi'(r,t,s) = \begin{pmatrix} u_1'(r,t,s) \\ u_2'(r,t,s) \end{pmatrix} = \exp(-i\varepsilon_\nu \hat{T}_\nu)\, \psi(r,t,s)$$

$$\psi' = \exp(-i\varepsilon_\nu \hat{T}_\nu) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$$

The $\hat{T}_i$ are the "isospin operators" or the operators of "isobaric spin".
The nucleon states can be classified by the 3rd-component of the isospin:

$$\hat{T}_3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

The eigenstates $\frac{1}{2}$ and $\frac{-1}{2}$ denote the proton & nuetron respectively (the eigenstates with eigenvalues $\pm\frac{1}{2}$ really).
The charge operator is the operator with charge as it's eigenvalues,

$$\boxed{\hat{Q} = e\left(\hat{T}_3 + \frac{1}{2}\mathbb{1}\right) = \frac{1}{2}e(\hat{\tau}_3 + 1)}$$

Which is consistent with the known charges of P and N,

$$\begin{aligned} \hat{Q}\psi_p &= \frac{1}{2}e(\hat{\tau}_3 + 1)\psi_p \\ &= \frac{1}{2}e(\psi_p + \psi_p) \qquad : \text{since } \hat{\tau}_3\psi_p = \psi_p \\ &= e\psi_p \end{aligned}$$

$$\begin{aligned} \hat{Q}\psi_n &= \frac{1}{2}e(\tau_3 + 1)\psi_n \\ &= \frac{1}{2}e(-\psi_n + \psi_n) \qquad : \text{since } \hat{\tau}_3\psi_n = -\psi_n \\ &= 0 \end{aligned}$$

The charge of ψ_p (PROTON) is e while the charge of ψ_n (NUETRON) is zero.

$\boxed{\text{Exercise 5.1 : Addition Law for Infinitesimal Transformations}}$

pgs. 137–138 show how to calculate $\Phi(\theta, \delta\theta)$ where

$$\exp(i\Phi\cdot\hat{\tau}/2) = \exp(i\delta\theta\cdot\hat{\tau}/2)\exp(i\theta\cdot\hat{\tau}/2)$$

we attempt the calculation a different way, working at 1st order in the infinitesimal $\delta\theta$ just like GREINER,

Define $A = i\delta\theta\cdot\hat{\tau}/2$ and $B = i\theta\cdot\hat{\tau}/2$ consider that the Baker-Cam.-Haus. relation connects e^Ae^B to $e^{A+B+\cdots}$

$$e^Ae^B \cong e^{A+B+\frac{1}{2}[A,B]} \cong \exp(i\Phi\cdot\hat{\tau}/2)$$

(no higher commutator terms be only one A for 1st order.)

Comparing we find the formula for Φ,

$$\begin{aligned}
i\Phi\cdot\frac{\hat{\tau}}{2} &= A + B + \tfrac{1}{2}[A,B] \\
&= i(\delta\theta+\theta)\cdot\hat{\tau}/2 + \frac{i^2}{8}[\delta\theta\cdot\hat{\tau}, \theta\cdot\hat{\tau}] \\
&= \frac{i}{2}(\delta\theta+\theta)\cdot\hat{\tau} - \frac{1}{8}\Big((\delta\theta\cdot\hat{\tau})(\theta\cdot\hat{\tau}) - (\theta\cdot\hat{\tau})(\delta\theta\cdot\hat{\tau})\Big) \\
&= \frac{i}{2}(\delta\theta+\theta)\cdot\hat{\tau} - \frac{1}{8}\Big(\delta\theta^m\theta^n\hat{\tau}_m\hat{\tau}_n - \theta^n\delta\theta^m\hat{\tau}_n\hat{\tau}_m\Big) \\
&= \frac{i}{2}(\delta\theta+\theta)\cdot\hat{\tau} - \frac{1}{8}\delta\theta^m\theta^n\Big(\hat{\tau}_m\hat{\tau}_n - \hat{\tau}_n\hat{\tau}_m\Big) \\
&= \frac{i}{2}(\delta\theta+\theta)\cdot\hat{\tau} - \frac{1}{8}\delta\theta^m\theta^n\Big(2i\epsilon_{mnk}\hat{\tau}_k\Big) \\
&= \frac{i}{2}\Big[(\delta\theta+\theta)\cdot\hat{\tau} - \frac{1}{2}\underbrace{\epsilon_{mnk}\delta\theta^m\theta^n}_{(\delta\theta\times\theta)_k}\hat{\tau}_k\Big] \\
&= \frac{i}{2}\Big[\delta\theta+\theta - \frac{1}{2}\delta\theta\times\theta\Big]\cdot\hat{\tau}
\end{aligned}$$

Comparing we find $\boxed{\Phi = \delta\theta + \theta - \frac{1}{2}\delta\theta\times\theta}$

Hmm... $[A,[A,B]]$ and $[B,[B,A]]$ also matter because $[B,[B,A]]$ is 1st order as well... curious, anyway if you want the correct sol^{n} its in GREINER in some detail where he finds

$$\Phi = \theta + \delta\theta\left(\frac{\theta}{2}\cot\left(\frac{\theta}{2}\right) - \theta(\delta\theta\cdot\theta)\theta^{-2}\left(1-\frac{\theta}{2}\cot\left(\frac{\theta}{2}\right)\right)\right) - \frac{1}{2}(\delta\theta\times\theta)$$

obviously my sol^{n} while overlapping is incomplete w/o including higher commutator terms in BCH formula.

§ 5.1 : Isospin Operators for a Multi-Nucleon System

Consider a system of A protons and nuetrons (collectively called nucleons) the isospin operator for the n^{th} nucleon in the system,

$$\boxed{\hat{T}_i(n) = \frac{1}{2}\hat{\tau}_i(n) \qquad n = 1, 2, \dots, A}$$

the nucleons are independent, each has its own isospin space meaning

$$[\hat{T}_i(n), \hat{T}_i(m)] = 0 \qquad \text{for} \quad n \neq m.$$

The net-isospin of the system is simply

$$\hat{T}_i = \sum_{n=1}^{A} \hat{T}_i(n) = \frac{1}{2}\sum_{n=1}^{A} \hat{\tau}_i(n)$$

Calculate then the following, noting $[T_i(m), T_j(n)] = \delta_{mn}(i\epsilon_{ijk}\hat{T}_k)$

$$[\hat{T}_i, \hat{T}_j] = \sum_{n=1}^{A}\sum_{m=1}^{A}[\hat{T}_i(m), \hat{T}_j(n)]$$

$$= \sum_{n=1}^{A}\sum_{m=1}^{A}\delta_{mn}(i\epsilon_{ijk}\hat{T}_k)$$

$$= \sum_{n=1}^{A} i\epsilon_{ijk}\hat{T}_k(n)$$

$$= i\epsilon_{ijk}\hat{T}_k$$

(Some sign question here
Eq^n 5.93 $\Rightarrow [T_i(n), T_j(n)] = i\epsilon_{ijk}T_k$
but 5.206 $\Rightarrow [T_i, T_j] = -i\epsilon_{ijk}T_k$.

Likewise we can calculate the charge by summing over the constituent nucleons charges

$$\boxed{\hat{Q} = \sum_{n=1}^{A}\hat{Q}(n) = e\sum_{n=1}^{A}\frac{1}{2}(\hat{\tau}_3(n) + 1) = e\left(\hat{T}_3 + \frac{1}{2}A\right)}$$

"A" is also called the "mass number". Notice a nucleus is characterized by A and $Z = Q/e$ (# of protons).

"Isobars" are nuclei with the same mass number A but different "Isospin" (eigenvalue of $\hat{T}_3$) hence the more descriptive label "Isobaric spin"

Isobaric Spin & Multinucleon Systems

We were just discussing that there are many nuclei which have the same # of nucleons (and hence mass), these are called ISOBARS. The eigenvalues of $\hat{T}_3$ classify the states of a particular ISOBAR multiplet. Since $\hat{T}_3$ shares the same algebraic structure as $\hat{J}_3$ we can build the states just as in Chapter 2 of Greiner via the ladder operators and so on... This is because the groups are isomorphous (is this the same as isomorphic?)

Thus the multiplet can be labeled according to the total isospin and the 3rd component (like $|lm\rangle$)

$$\boxed{|T\,T_3\rangle}$$

Where as usual for angular momentum-type groups,

$$\boxed{\begin{array}{ll} \hat{T}^2|T\,T_3\rangle = T(T+1)|T\,T_3\rangle & T=0,\tfrac{1}{2},1,\tfrac{3}{2},\dots \\ \hat{T}_3|T\,T_3\rangle = T_3|T\,T_3\rangle & T \geq T_3 \geq -T \end{array}}$$

For each particular value of isospin T we get $(2T+1)$-states (Just like angular momentum l has $2l+1$ states). Particularly

$$T=\tfrac{1}{2} \longrightarrow |\tfrac{1}{2}\,\tfrac{1}{2}\rangle \equiv \tilde{\chi}_{\frac{1}{2}\frac{1}{2}} \equiv |p\rangle$$
$$\phantom{T=\tfrac{1}{2}} \longrightarrow |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle \equiv \tilde{\chi}_{\frac{1}{2}\frac{-1}{2}} \equiv |n\rangle$$

Then $T=1$ gives the ~~[illegible]~~ π-mesons and Σ baryons.

EXAMPLE 5.2 : THE DEUTERON

The DEUTERON is made of one proton and one nuetron. The wave function has a space-part $R_{n\ell}(r)\, Y_{\ell m_\ell}(\theta,\varphi)$ [decribing the relative motion of the two nucleons], a spin part $\chi_{s m_s}$ and the isospin part $|T T_3\rangle$ thus

$$\boxed{\psi_{deuteron} = R_{n\ell}\left[Y_{\ell m_\ell}\times \chi_{s m_s}\right]^{[j]} |T T_3\rangle}$$

Where $[\;]^{[j]}$ indicates the coupling of the angular and spin momentum to makeup the total ~~momentum~~ angular momentum.

$$[Y_{\ell m_\ell}\times\chi_{s m_s}]^{[j]} = \sum_{m_\ell m_s}(\ell s j\,|\,m_\ell m_s m)\, Y_{\ell m_\ell}\,\chi_{s m_s}(1,2)$$

$$\chi_{s m_s}(1,2) = \sum_{m_1 m_2}\left(\tfrac{1}{2}\tfrac{1}{2} s\,|\,m_1 m_2 m_s\right)\chi_{\frac{1}{2}m_1}(1)\,\chi_{\frac{1}{2}m_2}(2)$$

And the state $|TT_3\rangle$ is also a composite of the single nucleon states,

$$\boxed{|TT_3\rangle = \sum_{t_1 t_2}\left(\tfrac{1}{2}\tfrac{1}{2} T\,|\,t_1 t_2 T_3\right)\tilde{\chi}_{\frac{1}{2}t_1}(1)\,\tilde{\chi}_{\frac{1}{2}t_2}(2)}$$

Notice the $\sim$ indicates they are isospinors as opposed to the spinors $\chi_{\frac{1}{2}m_1}$. Also recall that $(\frac{1}{2}\frac{1}{2}s\,|m_1 m_2 m_s)$, $(\ell s j\,|m_\ell m_s m)$ and $(\frac{1}{2}\frac{1}{2}T\,|t_1 t_2 T_3)$ are "Clebsch - Gordon" coefficients. The angular, intrinsic and isospin all have the same group structure hence the same coefficients. This follows from the isomorphism of the algebras of $SO(3)$ and $SU(2)$ of the rotation and isospin groups respectively. Why? It is just that $\underbrace{[\hat{J}_i,\hat{J}_j] = i\epsilon_{ijk}\hat{J}_k}_{\text{angular momentum}}$ AND $\underbrace{[\hat{T}_i,\hat{T}_j] = i\epsilon_{ijk}\hat{T}_k}_{\text{isospin}}$

Recall $Q = \sum_{n=1}^{A}\frac{1}{2}e(T_3(n)+1) = e\,(T_3 + \frac{1}{2}A) = Q_p + Q_n$

So then we see that $e = e\,(T_3 + \frac{1}{2}(2))$ for deuteron thus

$$T_3 = 0$$

But for a proton-nuetron system clearly $T=0$ and $T=1$ are possible. The singlet is $T=0$, it is the "ground state" of the deuteron. When $T=1$ only $T_3=0$ is a "deuteron" because $T_3=\pm 1$ correspond to pp or nn the di-proton or dinuetron.
All the $T=1$ states are unstable.

EXAMPLE 5.2 THE DEUTERON CONTINUED:

The singlet isospin wave function is

$$|T=0, T_3=0\rangle = \sum_{t_1} \left(\tfrac{1}{2}\tfrac{1}{2}0 \middle| t_1 \, -t_1 \, 0\right) \tilde{\chi}_{\frac{1}{2}t_1}(1)\, \tilde{\chi}_{\frac{1}{2}-t_1}(2)$$

$$= \underbrace{\left(\tfrac{1}{2}\tfrac{1}{2}0 \middle| \tfrac{1}{2}\tfrac{-1}{2}0\right)}_{1/\sqrt{2}} \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}-\frac{1}{2}}(2) + \underbrace{\left(\tfrac{1}{2}\tfrac{1}{2}0 \middle| \tfrac{-1}{2}\tfrac{1}{2}0\right)}_{-1/\sqrt{2}} \tilde{\chi}_{\frac{1}{2}-\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(2)$$

$$\equiv \frac{1}{\sqrt{2}}\left[\tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{-1}{2}}(2) + \tilde{\chi}_{\frac{1}{2}\frac{-1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(2)\right]$$

This is antisymmetric w.r.t. exchange $p \longleftrightarrow n$ (or $1 \longleftrightarrow 2$) whereas we can compute the isospin wavefunction of the triplet state,

$$|T=1, T_3=0\rangle = \sum_{t_1=} \left(\tfrac{1}{2}\tfrac{1}{2}1 \middle| t_1 \, -t_1 \, 0\right) \tilde{\chi}_{\frac{1}{2}t_1}(1)\, \tilde{\chi}_{\frac{1}{2}-t_1}(2)$$

$$= \underbrace{\left(\tfrac{1}{2}\tfrac{1}{2}1 \middle| \tfrac{1}{2}\tfrac{-1}{2}0\right)}_{1/\sqrt{2}} \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}-\frac{1}{2}}(2) + \underbrace{\left(\tfrac{1}{2}\tfrac{1}{2}1 \middle| \tfrac{-1}{2}\tfrac{1}{2}0\right)}_{1/\sqrt{2}} \tilde{\chi}_{\frac{1}{2}-\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(2)$$

$$\equiv \frac{1}{\sqrt{2}}\left[\tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{-1}{2}}(2) + \tilde{\chi}_{\frac{1}{2}\frac{-1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(2)\right]$$

Which is symmetric w.r.t. $p \longleftrightarrow n$ (or $1 \longleftrightarrow 2$). The Pauli-principal says the total-wave function Ψ_{dueteron} should be antisymmetric w.r.t. particle exchange. Thus, the isospin-singlet must occur with symmetric spin state $s=1$. Likewise the symmetric iso-triplet state must be paired with the spin-singlet state $s=0$.

Although these states are not deuteron states they do belong to $T=1$ so we mention them

$$|T=1, T_3=1\rangle = \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}\frac{1}{2}}(2) \qquad \text{(two protons)}$$

$$|T=1, T_3=-1\rangle = \tilde{\chi}_{\frac{1}{2}-\frac{1}{2}}(1)\, \tilde{\chi}_{\frac{1}{2}-\frac{1}{2}}(2) \qquad \text{(two neutrons)}$$

these are part of the diproton and dineutron. On next page we'll discuss some physical reasons for $T=1$ being unstable.

Example 5.2 the Deuteron Continued Again

Why are $T=1$ states unstable? The states of the $T=1$ for isospin are symmetric. "Since the nucleon-nucleon force in the spin 1 channel is attractive at short range, the wave function in configuration space must have quantum # $l=0$, that is it must be completely symmetric. That is the reason a bound state can only be constructed with $T=0$."

The Hamiltonian of the strong interaction is isospin invariant $\Rightarrow$ $[H_{strong}, \hat{T}^2] = 0$. Thus for different T we have different energies. If $\hat{H}(T) = f(r)\mathbb{1} + g(r)\hat{T}^2$

$$\hat{H}(T=0) = f(r)$$
$$\hat{H}(T=1) = f(r) + 2g(r)$$

The fact that $T=0$ are stable $\Rightarrow$ $f(r)$ is attractive with a "potential pocket" whereas the observation $T=1$ is unstable suggests that $f(r)+2g(r)$ is not.

Remark: It would be nice to have a more basic answer to the question above. Maybe Greiner's "Microscopic Theory of the Nucleus" 2nd Ed. Nuclear Theory Vol. 3. explains better.

§5.3: CHARGE INDEPENDANCE OF NUCLEAR FORCES

Lets see, we started with the idea that P and N are essentially equivalent to the strong force. It seems reasonable that we should be able to show the strong force is charge independent as a consequence of isospin invariance of $\hat{H}_{strong}$.

Isospin invariance of strong interactions can be compactly phrased in terms of the simple eq^n,

$$\boxed{[\hat{H}_{strong}, \hat{T}] = 0}$$

Which is equivalent to the Hamiltonian commuting with the group,

$$\boxed{[\hat{H}_{strong}, \hat{U}_{IS}(\varepsilon)] = 0}$$

Since $\hat{H}_{strong}$ is not known (! ?) we consider the $\hat{S}$-matrix

$$\boxed{\hat{S} = \exp\left(\frac{i}{\hbar}\hat{H}t\right)}$$

From which it is easily seen isospin invariance becomes

$$\boxed{[\hat{S}_{strong}, \hat{T}] = 0 \quad \text{OR} \quad [\hat{S}_{strong}, \hat{U}_{IS}(\varepsilon)] = 0}$$

From previous discussion we know $|T, T_3\rangle$'s states, ("the iso-multiplet") are energetically degenerate. Consider,

$$e^{-i\pi\hat{T}_2}|\tfrac{1}{2}\tfrac{1}{2}\rangle = \left(\cancel{\cos(\tfrac{\pi}{2})}^{\,0} - i\,\hat{\tau}_2 \sin(\tfrac{\pi}{2})\right)|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$= -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv |\tfrac{1}{2}\tfrac{-1}{2}\rangle$$

$$e^{-i\pi\hat{T}_2}|\tfrac{1}{2}\tfrac{-1}{2}\rangle = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv -|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

In other words we state

$$\boxed{e^{-i\pi\hat{T}_2}|p\rangle = |n\rangle} \quad \text{AND} \quad \boxed{e^{-i\pi T_2}|n\rangle = -|p\rangle}$$

CHARGE INDEPENDANCE OF STRONG FORCE

From our calculations on the last pg. we can easily see that for two-nucleon states we similarly find

$$\boxed{\exp\left[-i\pi\left(\hat{T}_z(1)+\hat{T}_z(2)\right)\right]|P(1)P(2)\rangle = |n(1)n(2)\rangle}$$

$$\boxed{\exp\left[-i\pi\left(\hat{T}_z(1)+\hat{T}_z(2)\right)\right]|n(1)n(2)\rangle = |P(1)P(2)\rangle}$$

Lets conclude this rather sneaky argument,

$$\boxed{\langle P(1)P(2)|\hat{H}_{strong}|P(1)P(2)\rangle} =$$

$$= \langle \exp\left[-i\pi\left(T_z(1)+T_z(2)\right)\right]n(1)n(2)|\hat{H}_{strong} \times$$

$$\times|\exp\left[-i\pi\left(T_z(1)+T_z(2)\right)\right]n(1)n(2)\rangle$$

$$= \langle nn|\exp\left[i\pi\left(T_z(1)+T_z(2)\right)\right]\hat{H}_{strong}\exp\left[-i\pi\left(T_z(1)+T_z(2)\right)\right]|nn\rangle$$

$$= \boxed{\langle nn|H_{strong}|nn\rangle}$$

Where we have used the fact that $\hat{U}_{IS}$ transformations deform the intial state to a "rotated" state of the same energy. And we have also used $H^{-}U = UH$. End result: the interaction of 2 protons and 2 nuetrons is the same, the strong interaction is independent of the charge.

Example 5.4 : The Pion Triplet

Experiment has revealed the following particles

Pion	Mass m_0c^2	Mass Difference [MeV]	Charge	Lifetime [s]	Spin	Magnetic Mom.
π^+	139.59	4.59	e	$2.55 \pm 0.03 \times 10^{-8}$	0	0
π^0	135.00	0	0	0.83×10^{-16}	0	0
π^-	139.59	4.59	$-e$	$2.55 \pm 0.03 \times 10^{-8}$	0	0

The masses are nearly equal. Because the strong interaction determines the dominant part of the mass $\Rightarrow$ the particles are equivalent to the strong-interaction. The small Δm of 4.59 MeV may be thought of as arising from E&M interactions. This is not unreasonable, consider that the Coulomb energy of a homogeneously charged sphere with radius $r_0 = \hbar / m_\pi c$ (the Compton wavelength of the pion)

$$|E_{\text{coulomb}}| = \frac{3}{5}\frac{e^2}{r_0} = \frac{3}{5}\frac{e^2}{\hbar c} m_\pi c^2 = \frac{3}{5}\cdot\frac{1}{137}\cdot 139\,\text{MeV} \cong \frac{3}{5}\,\text{MeV}.$$

Ok, its complicated but $\frac{3}{5} \neq 4.59$ by any stretch. Sufficient to say that isospin is an <u>approximate symmetry</u> here

So we identify the following isospin triplet,

$$\boxed{\begin{aligned} |T=1, T_3=1\rangle &= -|\pi^+\rangle \\ |T=1, T_3=0\rangle &= |\pi^0\rangle \\ |T=1, T_3=-1\rangle &= +|\pi^-\rangle \end{aligned}}$$

the phases chosen above follow for physical reasons, see next page.

As usual, due to the isomorphism to angular momentum,

$$\boxed{\begin{array}{l} \hat{T}_\pm |T T_3\rangle = \sqrt{T(T+1) - T_3(T_3 \pm 1)}\ |T\ T_3 \pm 1\rangle \\ \hat{T}_0 |T T_3\rangle = \hat{T}_3 |T T_3\rangle = T_3 |T T_3\rangle \end{array}}$$

Where the above flow from the relations,

$$\boxed{[\hat{T}_3, \hat{T}_\pm] = \pm \hat{T}_\pm \quad \& \quad [\hat{T}_+, \hat{T}_-] = 2\hat{T}_3}$$

Which come from $\hat{T}_\pm = \hat{T}_1 \pm i\hat{T}_2$ and $\hat{T}_0 = \hat{T}_3$. Just like ang. mom.

So let's elaborate on the isospin multiplet here,

$\hat{T}_+ \|11\rangle = 0$	$\hat{T}_- \|11\rangle = \sqrt{2}\|10\rangle$	$\hat{T}_3 \|11\rangle = \|11\rangle$
$\hat{T}_+ \|10\rangle = \sqrt{2}\|11\rangle$	$\hat{T}_- \|10\rangle = \sqrt{2}\|1{-1}\rangle$	$\hat{T}_3 \|10\rangle = 0$
$\hat{T}_+ \|1{-1}\rangle = \sqrt{2}\|10\rangle$	$\hat{T}_- \|1{-1}\rangle = 0$	$\hat{T}_3 \|1{-1}\rangle = -\|1{-1}\rangle$

Using $|11\rangle = -|\pi^+\rangle$ and $|10\rangle = |\pi^0\rangle$ and $|1{-1}\rangle = |\pi^-\rangle$ the equations above can be written,

$\hat{T}_+(\sqrt{2}\|\pi^+\rangle) = 0$	$[T_+, T_+] = 0$
$\hat{T}_+(\|\pi^0\rangle) = -(\sqrt{2}\|\pi^+\rangle)$	$[T_+, T_3] = -T_+$
$\hat{T}_+(\sqrt{2}\|\pi^-\rangle) = 2\|\pi^0\rangle$	$[T_+, T_-] = 2T_3$
$\hat{T}_-(\sqrt{2}\|\pi^+\rangle) = -2(\|\pi^0\rangle)$	$[T_-, T_+] = -2T_3$
$\hat{T}_-(\|\pi^0\rangle) = (\sqrt{2}\|\pi^-\rangle)$	$[T_-, T_3] = T_-$
$\hat{T}_-(\sqrt{2}\|\pi^-\rangle) = 0$	$[T_-, T_-] = 0$
$\hat{T}_3(\sqrt{2}\|\pi^+\rangle) = (\sqrt{2}\|\pi^+\rangle)$	$[T_3, T_+] = T_+$
$\hat{T}_3(\|\pi^0\rangle) = 0$	$[T_3, T_3] = 0$
$\hat{T}_3(\sqrt{2}\|\pi^-\rangle) = -(\sqrt{2}\|\pi^-\rangle)$	$[T_3, T_-] = -T_-$

The choice of signs on 52 was made insure the correspondances

$$\hat{T}_+ \longleftrightarrow \sqrt{2}|\pi^+\rangle, \quad \hat{T}_- \longleftrightarrow \sqrt{2}|\pi^-\rangle, \quad \hat{T}_3 \longleftrightarrow |\pi^0\rangle$$

Pion Triplet Continued

Compactly we noticed that

$$\hat{T}_\mu |\pi^\nu\rangle \longleftrightarrow [\hat{T}_\mu , \hat{T}_\nu] \qquad \text{for } \mu, \nu = +, -, 0$$

This is why we said $\hat{T}_3 = \hat{T}_0$ for this formula. This shows there is isomorphism between transformations on $|\pi^\nu\rangle$ and the generators $\hat{T}_\mu$. We say that the transformation of the isospin group from spherical to cartesian

$$\begin{pmatrix} \hat{T}_1 \\ \hat{T}_2 \\ \hat{T}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(T_+ + T_-) \\ \frac{1}{2i}(T_+ - T_-) \\ \hat{T}_0 \end{pmatrix} \approx \begin{pmatrix} |\pi_1\rangle \\ |\pi_2\rangle \\ |\pi_3\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(|\pi^+\rangle + |\pi^-\rangle) \\ \frac{1}{i\sqrt{2}}(|\pi^+\rangle - |\pi^-\rangle) \\ |\pi^0\rangle \end{pmatrix}$$

$$[\hat{T}_i , \hat{T}_j] = i\epsilon_{ijk}\hat{T}_k \qquad \approx \qquad \hat{T}_i |\pi_j\rangle = i\epsilon_{ijk}\hat{T}_k$$

The $\frac{1}{\sqrt{2}}$ comes from the fact that $\langle \pi_i | \pi_j \rangle = \delta_{ij}$

In Ex. 5.4 we made the correspondance

$$\hat{T}_\mu |\pi^\nu\rangle \Longleftrightarrow [\hat{T}_\mu, \hat{T}_\nu]$$

the RHS of the above amounts to what is known as the adjoint representation (aka regular) of a Lie Algebra.

$$\hat{T}_\mu \longmapsto ad_{\hat{T}_\mu} \quad \text{where } ad_{T_\mu}(\hat{T}_\nu) = [T_\mu, T_\nu]$$

So what is a representation?

Def^n/ A representation is a mapping $\hat{D}$ from the Lie Algebra to some set of linear operators of a vector space.

$$\hat{L}_i \longmapsto \hat{D}(\hat{L}_i)$$

Where this mapping must be a Lie Algebra homomorphism,

① $\hat{D}(\alpha \hat{L}_i + \beta \hat{L}_j) = \alpha \hat{D}(L_i) + \beta \hat{D}(\hat{L}_j)$

② $\hat{D}([\hat{L}_i, \hat{L}_j]) = [\hat{D}(\hat{L}_i), \hat{D}(\hat{L}_j)]$

Notice that the vector space on which the operator $\hat{D}(\hat{L}_i)$ acts is not unique, we can have many rep. of a particular Lie Algebra.

Lets expound these ideas in a basis, suppose the representation vector space V has basis $\{|\varphi_k\rangle\}$ then

$$\hat{L}_i |\phi_j\rangle = D(\hat{L}_i)_{kj} |\phi_k\rangle$$

$$[D(\hat{L}_i)]_{kj} = \langle \phi_k | \hat{L}_i | \phi_j \rangle$$

It seems to confuse $\hat{L}_i$ with the representation of $\hat{L}_i$. The text did this before when discussing Casimirs

Matrix representations naturally satisfy

$$\hat{D}(\hat{L}_i)\hat{D}(\hat{L}_j) = \hat{D}(\hat{L}_i \hat{L}_j)$$

Because,

$$[\hat{D}(L_i)\hat{D}(L_j)]_{mn} = [\hat{D}(L_i)]_{mk}[\hat{D}(L_j)]_{kn}$$

$$= [\hat{D}(L_i L_j)]_{mn}$$

I think this follows from ②. In the sloppy notation

$$\langle n|\hat{L}_i|m\rangle\langle m|\hat{L}_j|k\rangle = \langle n|\hat{L}_i \hat{L}_j|k\rangle$$

Whatever, I don't understand the intent of pg. 147, the def^n of representation is the crucial thing.

ADJOINT REPRESENTATION

Represent the Lie algebra $\langle L_i \rangle$ on the vector-space of operators $\hat{L}_i$, that is span$(|L_i\rangle|_{i=1}^{n}$ the dimension of the vectorspace = dimension of the Lie Algebra. Anyway a typical vector in the rep.'s vector space is

$$|L_i\rangle$$

Now the text defines the adjoint rep. according to

$$\boxed{\begin{array}{l} \hat{L}_i |L_j\rangle \equiv C_{ijk} |L_k\rangle \\ \langle L_i | L_j \rangle \equiv \delta_{ij} \quad (\text{to make life easy.}) \\ \qquad\qquad\qquad\qquad \text{we give this inner product to } V. \end{array}}$$

We also need that $[\hat{L}_i, \hat{L}_j] = C_{ijk} \hat{L}_k$ still holds.

- I find this notation obscures what really is going on, its bad to use $\hat{L}_i$ and $\hat{L}_i$ for both the basic algebra generator and it's rep. in the adjoint.

The remedy is simple. Replace the definitions above with the following.

$$\boxed{ad(\hat{L}_i)|L_j\rangle \equiv C_{ijk}|L_k\rangle}$$

Which if we use $\hat{L}_i = |L_i\rangle$ can be more naturally phrased

$$\boxed{ad(\hat{L}_i)(\hat{L}_j) = [L_i, L_j]} \qquad (\text{same statement})$$

Notice the matrix of the representation is $n \times n$ and is given by the structure constants.

$$[D(\hat{L}_i)]_{kj} = C_{ijk}$$

in other words the structure constants form a matrix representation of the generators

- Next we'll prove the adjoint rep. is truly a representation.

ADJOINT REPRESENTATION IS A REPRESENTATION.

First note that linearity of $ad(\hat{L}_i)$ follows directly from the bilinearity of $[\,,\,]$ as $ad(\hat{L}_i)(\hat{L}_j) = [\hat{L}_i, \hat{L}_j]$.
All that remains is to show it is a Lie Algebra homomorphism, we use the explicit basis $|L_i\rangle = \hat{L}_i$ as it's natural,

$$ad([L_i, L_j])(L_k) = [[L_i, L_j], L_k]$$
$$= [L_i, [L_j, L_k]] - [L_j, [L_i, L_k]]$$

Where the Jacobi Identity below was used above

$$[[L_i, L_j], L_k] + [[L_j, L_k], L_i] + [[L_k, L_i], L_j] = 0$$
$$\Rightarrow [[L_i, L_j], L_k] = [L_i, [L_j, L_k]] + [L_j, [L_k, L_i]]$$

(And antisymmetry of bracket). Hence using definition of adjoint rep

$$ad([L_i, L_j])(L_k) = ad(L_i)(ad(L_j)(L_k)) - ad(L_j)(ad(L_i)(L_k))$$
$$= (ad(L_i)\,ad(L_j) - ad(L_j)\,ad(L_i))(L_k)$$
$$= [ad(L_i), ad(L_j)](L_k)$$

Thus $ad([L_i, L_j]) = [ad(L_i), ad(L_j)]$, the adjoint mapping is a Lie algebra homomorphism.

- THIS CONSTRUCTION MAKES SENSE FOR <u>ANY</u> LIE ALGEBRA. MORE THAN JUST THIS $Ker(ad) = \{0\}$ if the algebra is semi-simple, that is this becomes a Lie Algebra isomorphism which is useful.

- THE ARGUMENT IN GREINER IS THE SAME, JUST DIFFERENT NOTATION, again $\hat{L}_i \neq \hat{L}_i$ is troubling.

EXERCISE 5.5 : NORMALIZATION OF THE GROUP GENERATORS

Problem: Show that the generators $\hat{L}_i$ of a unitary matrix group, $U^{-1} = U^{\dagger}$ for each $U \in G$, can be chosen in such a way that the relation $\mathrm{Tr}(\hat{L}_i \hat{L}_j) = \frac{1}{2}\delta_{ij}$ is valid. Further, show that the resulting structure constants are pure imaginary and totally antisymmetric in this case.

Sol^n: Unitary groups have $\hat{L}_i^{\dagger} = \hat{L}_i$ (hermitian generators). Consider then:

$$\gamma_{ik} \equiv \mathrm{Tr}(\hat{L}_i \hat{L}_k)$$

Cleary by cyclicity $\gamma_{ik} = \gamma_{ki}$. Further conjugate γ_{ik}

$$\begin{aligned}
(\gamma_{ik})^* &= \mathrm{Tr}(\hat{L}_i^* \hat{L}_k^*) \\
&= \mathrm{Tr}\left((\hat{L}_i^* \hat{L}_k^*)^T\right) \\
&= \mathrm{Tr}\left((\hat{L}_i^*)^T (\hat{L}_k^*)^T\right) \\
&= \mathrm{Tr}[\hat{L}_i^{\dagger} \hat{L}_k^{\dagger}] \\
&= \mathrm{Tr}[\hat{L}_i \hat{L}_k] \\
&= \gamma_{ik} \implies \gamma_{ik}
\end{aligned}$$

can be written as a real symmetric matrix, thus it can be diagonalized by some orthogonal transformation

That is particularly for the orthogonal transformation R,

$$\lambda_i \delta_{il} = R_{ij} \gamma_{jk} R^{-1}_{kl}$$

(no sum)

$\therefore$ λ_i are the eigenvalues of γ_{jk} clearly & $R^{-1}_{kl} = (R_{kl})^T = R_{lk}$ it's orthogonal.

Normalizing Generators : 5.5

Define (indices differ from GREINER slightly)

$$\boxed{\hat{F}_j \equiv R_{ji}\hat{L}_i}$$

Where R_{ji} is the orthogonal matrix with $\lambda_i \delta_{il} = R_{ij}\gamma_{jk}R^{-1}_{kl}$
notice that $\hat{F}_j^{\dagger} = \hat{F}_j$ since R is a real matrix. The inverse transformation is easily calculated,

$$(R^{-1})_{mj}\hat{F}_j = (R^{-1})_{mj}R_{ji}\hat{L}_i = \delta_{mi}\hat{L}_i = \hat{L}_m$$

And $(R^{-1})_{mj} = R_{jm}$ because $R^T = R^{-1} \Leftrightarrow RR^T = I$ that is to say R is orthogonal. Hence

$$\boxed{\hat{L}_m = R_{jm}\hat{F}_j}$$

Consider then

$$\begin{aligned}
Tr(\hat{F}_i\hat{F}_l) &= Tr(R_{ij}\hat{L}_j R_{lk}\hat{L}_k) \\
&= R_{ij}R_{lk}\,Tr(\hat{L}_j\hat{L}_k) \\
&= R_{ij}R_{lk}\gamma_{jk} \\
&= R_{ij}\gamma_{jk}(R^{-1})_{kl} \\
&= \lambda_i\delta_{il} \leftarrow \text{(no sum)}
\end{aligned}$$

Further we can deduce that $\lambda_i > 0$, let $l = i$ then

$$\begin{aligned}
Tr(\hat{F}_l\hat{F}_l) &= (\hat{F}_l)_{\alpha\beta}(\hat{F})_{\alpha\beta} \\
&= (\hat{F}_l)_{\alpha\beta}(\hat{F}_l^{\dagger})_{\alpha\beta} \\
&= \sum_{\alpha\beta} \|(\hat{F}_l)_{\alpha\beta}\|^2 > 0 \quad \text{since } F \neq 0.
\end{aligned}$$

- Notice $(\hat{F}_l)_{\alpha\beta}$ is the matrix component constructed not truly from $\hat{L}_i$ rather from the adjoint!

Ex 5.5 continued

We just proved that $\hat{F}_j = R_{ji}\hat{L}_i \Rightarrow Tr(\hat{F}_j\hat{F}_k) = \lambda_j \delta_{jk}$ and that $\lambda_j > 0 \quad \forall j \in \{1, 2, \ldots, \dim(L)\}$. So now construct a third set of generators, (no sum)

$$\boxed{\hat{T}_i = \frac{1}{\sqrt{2\lambda_i}}\hat{F}_i}$$

Clearly these have $Tr(\hat{T}_i\hat{T}_j) = \frac{1}{2\sqrt{\lambda_i\lambda_j}} Tr(F_iF_j) = \frac{1}{2}\delta_{ij}$ which was the desired outcome. It remains to show the structure constants for $\hat{T}_i$ are pure imaginary.

$$[\hat{T}_i, \hat{T}_j] = C_{ijk}\hat{T}_k$$

$$\begin{aligned}
\Rightarrow Tr\{[\hat{T}_i, \hat{T}_j]\hat{T}_\ell\} &= C_{ijk} Tr(\hat{T}_k\hat{T}_\ell) \\
&= \tfrac{1}{2} C_{ijk}\delta_{k\ell} \\
&= \tfrac{1}{2} C_{ij\ell}
\end{aligned}$$

Using cyclicity of trace,

$$\begin{aligned}
Tr\{[T_i, T_j]T_\ell\} &= Tr\{T_iT_jT_\ell - T_jT_iT_\ell\} \\
&= Tr\{T_jT_\ell T_i - T_\ell T_j T_i\} \\
&= Tr\{[T_j, T_\ell]T_i\} \; \star \\
&= \tfrac{1}{2} C_{j\ell i}
\end{aligned}$$

Comparing we find $C_{ij\ell} = C_{j\ell i}$

Notice $C_{ijk} = -C_{jik} = -C_{kji} = C_{jki}$

$= C_{jki}$ using the above

$= -C_{kji}$ ... any switch of two indices is antisymmetric.

Ex. 5.5 Continued, show C_{ijk} are imag. for $[T,T]=cT$.

By definition we have

$$\hat{T}_i \hat{T}_j - \hat{T}_j \hat{T}_i = C_{ijk} \hat{T}_k$$

Now use hermiticity,

$$(C_{ijk})^* \hat{T}_k^{\dagger} = (T_i T_j - T_j T_i)^{\dagger}$$
$$= T_j^{\dagger} T_i^{\dagger} - T_i^{\dagger} T_j^{\dagger}$$

$$\therefore (C_{ijk})^* \hat{T}_k = T_j T_i - T_i T_j = -C_{ijk} \hat{T}_k$$

Hence $\boxed{(C_{ijk})^* = -C_{ijk}}$ thus the C_{ijk} are pure imaginary, we can write for real f_{ijk} that

$$\boxed{C_{ijk} \equiv i f_{ijk}}$$

Now the Killing form can be expressed with C or f

$$\boxed{g_{ij} \equiv C_{ikl} C_{jlk} = -f_{ikl} f_{jlk} = f_{ikl} f_{jkl}}$$

Interesting, now assume $\det(g) = 0 \Rightarrow \exists$ an eigenvector μ

$$\sum_j g_{ij} \mu_j = 0$$

hence also,

$$0 = \mu_i g_{ij} \mu_j = \mu_i f_{ikl} f_{jkl} \mu_j = \sum_{kl} \Big(\sum_i \mu_i f_{ikl} \Big)^2 \geq 0$$

which is only zero if

$$\sum_i \mu_i f_{ikl} = 0$$

Which would say that $[\mu_j \hat{T}_j, \hat{T}_k] = i \sum_j \mu_j f_{jkl} \hat{T}_l = 0$

Yielding that the linear combination $\mu_j \hat{T}_j$ commutes with the whole algebra. Thus it is a generators for an independent $U(1)$-subgroup. As we know a unitary matrix group is semisimple only in the case no such subgroup exists (Cartan's Criteria again)

§5.4: Transformation Law for Isospin Vectors

The adjoint rep. or regular representation acts on the basis $|\pi_j\rangle$ for the ISOSPIN ALGEBRA,

$$\boxed{\hat{T}_i |\pi_j\rangle = i\epsilon_{ijk} |\pi_k\rangle}$$

Where we recall that $[\hat{T}_i, \hat{T}_j] = i\epsilon_{ijk}\hat{T}_k$ so when we identify $ad_{T_i} \equiv ad(\hat{T}_i) \longleftrightarrow \hat{T}_i$ the above makes sense as,

$$ad(T_i)(T_j) \equiv [\hat{T}_i, \hat{T}_j] = i\epsilon_{ijk}\hat{T}_k \longleftrightarrow i\epsilon_{ijk}|\pi_k\rangle$$

The author is again capitalizing on the isomorphism of $L \cong ad(L)$, the Lie Algebra & it's adjoint are the same, in the sense they share the same algebra structure. BUT I argue that $\hat{T}_i$ and $ad(\hat{T}_i)$ are very different objects. grr...

Anyway, an infinitesimal isospin rotation in isospace is,

$$\begin{aligned}
|\pi_j'\rangle &= e^{-i\delta\varepsilon_i T_i}|\pi_j\rangle \\
&\cong (1 - i\delta\varepsilon_i T_i)|\pi_j\rangle \\
&= |\pi_j\rangle - i^2\delta\varepsilon_i \epsilon_{ijk}|\pi_k\rangle \\
&= |\pi_j\rangle + \delta\varepsilon_i \epsilon_{ijk}|\pi_k\rangle
\end{aligned}$$

More precisely,

$$|\pi_j'\rangle = |\pi_j\rangle - \epsilon_{jik}\delta\varepsilon_i|\pi_k\rangle + O(\delta\varepsilon^2)$$

This resembles an ordinary 3-d rotation, except here $\mathbb{R}^3$ is really "ISOSPACE" which is spanned by

$$\{|\pi_1\rangle, |\pi_2\rangle, |\pi_3\rangle\} \quad \text{with } \langle\pi_i|\pi_j\rangle = \delta_{ij}$$

An isovector then is simply $V = V_k|\pi_k\rangle$ where $V_k \in \mathbb{R}$. (they're real.)

§5.4: Isospin Vectors and How They Transform

A vector in isospace is $V = V_k |\pi_k\rangle$ consider then the rotated basis gives new components V_k' as,

$$V = V_k' |\pi_k'\rangle$$
$$= V_k |\pi_k\rangle \qquad \text{(passive rotation)}$$
$$= V_k \left(|\pi_k'\rangle + \epsilon_{kil}\, \delta\varepsilon_i\, |\pi_k'\rangle \right)$$

no primes on (62) (?)

$$\Rightarrow V_k' = V_k - \epsilon_{ki\alpha}\, \delta\varepsilon_i\, V_\alpha \qquad \text{(passive rotation, just rotated basis)}$$
$$V_i' = V_i + \epsilon_{ijk}\, \delta\varepsilon_j\, V_k \qquad \text{(active rotation, actually changes vector's direction)}$$

These match usual 3-d inf. rotations. More succinctly $\delta\varepsilon = \{\delta\varepsilon_i\}$ we write

$$V' = V + \delta\varepsilon \times V \equiv V + \delta V$$
$$\boxed{\delta V = V' - V = \delta\varepsilon \times V}$$

Now dot the above to find

$$(V')^2 = V^2 + 2V\cdot\delta V + (\delta V)^2$$

Hence calculate to 1st order,

$$\delta(V^2) = \cancel{2V\cdot\delta V}$$
$$\equiv (V')^2 - V^2$$
$$= 2\,\delta V\cdot V$$
$$= 2\,(\delta V)_i\, V_i$$
$$= 2\,\epsilon_{ijk}\, \delta\varepsilon_j\, V_k V_i \quad \leftarrow \text{vanishes because } V_k V_i \text{ is symmetric while } \epsilon_{ijk} \text{ is antisym. in } (ki).$$
$$= 0$$

That is isospin rot. preserves lengths of isovectors, surprise surprise.

§5.4 Continued

Another type of isovector is a vector of operators which act on some Hilbert space, say $|\varphi_\nu\rangle$. For example the isospin operators $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$. So we can consider how a iso-vector operator rotates

$$\delta\hat{V} = \hat{V}' - \hat{V} = \delta\hat{\varepsilon} \times \hat{V}$$

Explicitly

$$\delta\hat{V}_i = \hat{V}_i' - \hat{V}_i = \epsilon_{ijk}\,\delta\varepsilon_j\,\hat{V}_k$$

Yet we also know according to QM the operator transforms like a matrix namely:

$$\begin{aligned}
\hat{V}_i' &= \hat{U}_{IS}(\delta\varepsilon)\,\hat{V}_i\,\hat{U}_{IS}^{-1}(\delta\varepsilon) \\
&= e^{-i\delta\varepsilon_k \hat{T}_k}\,\hat{V}_i\,e^{i\delta\varepsilon_k \hat{T}_k} \qquad (1^{st}\text{ order}) \\
&= (1 - i\delta\varepsilon_k\hat{T}_k)\,\hat{V}_i\,(1 + i\delta\varepsilon_k\hat{T}_k) \qquad (1^{st}\text{ order}) \\
&= \hat{V}_i - i\delta\varepsilon_k \hat{T}_k\hat{V}_i + i\delta\varepsilon_k\hat{V}_i\hat{T}_k + \cdots \\
&= \hat{V}_i - i\delta\varepsilon_k[\hat{T}_k, \hat{V}_i]
\end{aligned}$$

Thus

$$\delta\hat{V}_i = \hat{V}_i' - \hat{V}_i = -i\delta\varepsilon_j[\hat{T}_j, \hat{T}_i]$$

Compare that with $\delta\hat{V}_i = \epsilon_{ijk}\,\delta\varepsilon_j\,\hat{V}_k$ to see that we have that $\hat{V}_i$ is a "iso-vector operator", meaning it obeys the relation here

$$\boxed{[\hat{T}_j, \hat{V}_i] = i\,\epsilon_{ijk}\hat{V}_k}$$

- To summarize SU(2) ISOSPIN gives rotations in 3-dim'l isospace with the standard basis $|\pi_j\rangle \approx$ regular rep.

§5.4 Concluding remarks

The abstract concept of isospace arose from the observed symmetry of the N-P doublet and the $\pi^{\pm}, \pi^{0}$ triplet. While they're is an isomorphism between $\mathbb{R}^3_{\text{ISOSPACE}}$ and $\mathbb{R}^3_{\text{physical}}$" space they are quite different. The rotational invariance of $\mathbb{R}^3_{\text{phy}}$. gives the rotation group and the associated angular momentum multiplets. Whereas ISOSPACE with it's isosymmetry rotations just gives formal symmetries which are physically realized in the symm. of the elementary particles (particularly the charge multiplets are isospin multiplets). Physically what is observed are the vectors $|\pi^{\pm}\rangle$ in the "spherical" rep. of isospin space

$$|\pi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\pi_1\rangle \pm i|\pi_2\rangle\right) \quad \& \quad |\pi_0\rangle$$

these are eigenstates of the charge operator while $|\pi_1\rangle, |\pi_2\rangle$ are not.

//

Example 5.6   G-PARITY OR ISOSPIN PARITY

Helps explain the decays of the ω and ρ mesons, we shall see... Define parity in iso space in the natural way

$$\boxed{\hat{G}|\pi_j\rangle \equiv -|\pi_j\rangle}$$

Clearly analogous to the usual parity $\hat{P}$ which has $\hat{P}|e_i\rangle = -|e_i\rangle$

G - Parity Continued

We defined G - Parity by $\hat{G}|\pi_j\rangle = -|\pi_j\rangle$
and claim it can be represented by $\exp(-i\pi\hat{T}_2)\hat{C}$ where $\hat{C}$ is the charge conjugation operator:

$$\boxed{\hat{C}|\pi^+\rangle = |\pi^-\rangle \qquad \hat{C}|\pi^-\rangle = |\pi^+\rangle \qquad \hat{C}|\pi^0\rangle = |\pi^0\rangle}$$

So in the ~~basis~~ cartesian basis what is $\hat{C}$ look like? Recall that we had $|\pi^\pm\rangle = \frac{1}{\sqrt{2}}(|\pi_1\rangle \pm i|\pi_2\rangle)$ and $|\pi^0\rangle$ is $|\pi^0\rangle$. Lets invert this for $|\pi_1\rangle$ and $|\pi_2\rangle$ by adding/subtracting $|\pi^+\rangle \pm |\pi_-\rangle$

$$|\pi^+\rangle + |\pi^-\rangle = \frac{1}{\sqrt{2}}\cdot 2|\pi_1\rangle \longrightarrow |\pi_1\rangle = \frac{1}{\sqrt{2}}(|\pi^+\rangle + |\pi^-\rangle)$$

$$|\pi^+\rangle - |\pi^-\rangle = \frac{1}{\sqrt{2}}\cdot 2i|\pi_2\rangle \longrightarrow |\pi_2\rangle = \frac{-i}{\sqrt{2}}(|\pi^+\rangle - |\pi^-\rangle)$$

Then calculate the action of $\hat{C}$ in the cartesian basis,

$$\begin{aligned}\hat{C}|\pi_1\rangle &= \frac{1}{\sqrt{2}}(\hat{C}|\pi^+\rangle + \hat{C}|\pi^-\rangle)\\ &= \frac{1}{\sqrt{2}}(|\pi^-\rangle + |\pi^+\rangle)\\ &= \boxed{|\pi_1\rangle = \hat{C}|\pi_1\rangle}\end{aligned}$$

$$\begin{aligned}\hat{C}|\pi_2\rangle &= \frac{-i}{\sqrt{2}}(\hat{C}|\pi^+\rangle - \hat{C}|\pi^-\rangle)\\ &= \frac{-i}{\sqrt{2}}(|\pi^-\rangle - |\pi^+\rangle)\\ &= \boxed{-|\pi_2\rangle = \hat{C}|\pi_2\rangle}\end{aligned}$$

$$\boxed{\hat{C}|\pi_0\rangle = |\pi_0\rangle}$$

Equivalently we can express that in $\{|\pi_1\rangle, |\pi_2\rangle, |\pi_0\rangle\}$

$$\hat{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

G-parity Continued

one should notice that $\hat{T}_+$ means different things depending on what it is applied to! Likewise for $\hat{T}_i$ from pg. 145 we note

$$\hat{T}_+ |\pi^+\rangle = 0 \quad , \quad \hat{T}_+ |\pi_0\rangle = -\sqrt{2}\,|\pi^+\rangle \quad , \cdots$$

From those and $|\pi_1\rangle = \frac{1}{\sqrt{2}}\left(|\pi^+\rangle + |\pi^-\rangle\right)$ and $|\pi_2\rangle = \frac{-i}{\sqrt{2}}\left(|\pi^+\rangle - |\pi^-\rangle\right)$ we can deduce the form of $\hat{T}_2$ in this basis, or just use the simpler method of recalling that

$$\boxed{\hat{T}_2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}$$

$$\hat{T}_i |\pi_j\rangle = i\epsilon_{ijk}|\pi_k\rangle$$

which clearly gives when $i=2$

$$\hat{T}_2 |\pi\rangle_j = i\,\epsilon_{2jk}\,|\pi_k\rangle$$

Now how to calculate the exponential of this $\hat{T}_2$ in the adjoint rep?

$$e^{-i\pi T_2} = \; ?$$

$$(\hat{T}_2)^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\hat{T}_2)^2 = i^2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(\hat{T}_2)^3 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \hat{T}_2$$

$$(\hat{T}_2)^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \hat{T}_2^{\,2}$$

$$e^{-i\pi T_2} = \sum_{n=\text{even}} \frac{(-i\pi T_2)^n}{n!} + \sum_{n\ \text{odd}} \frac{(-i\pi T_2)^n}{n!}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underbrace{\sum_{n\ \text{even}} \frac{(i)^n(\pi)^n}{n!}}_{1-\frac{\pi}{2}+\frac{\pi^2}{4!}+\cdots} + i\hat{T}_2 \underbrace{\sum_{n\ \text{odd}} \frac{(-i)^{n+1}\pi^n}{n!}}_{-\pi+\frac{\pi^3}{3!}+\cdots} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underbrace{1-\frac{\pi}{2}+\frac{\pi^2}{4!}+\cdots}_{\cos(\pi) = -1} \qquad \underbrace{-\pi+\frac{\pi^3}{3!}+\cdots}_{-\sin(\pi) = 0}$$

(because of $n=0$)

$$= \boxed{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}$$

G-parity continued

We found that $e^{-i\pi T_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ while $\hat{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ thus the product gives $\hat{G}$ supposedly $\exp(-i\pi T_2)\hat{C} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cancel{0} & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cancel{0} & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Hence $\hat{G} = e^{-i\pi\hat{T}_2}\hat{C} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ which is precisely the matrix form of $\hat{G}|\pi_j\rangle = -|\pi_j\rangle$.

//

Now that we have established some mathematics of G-parity what does it mean? If we assume that G-parity is an internal symmetry which represents a conserved quantity in particle interactions (just strong int. considered.)

Under this assumption G-parity must be conserved in particle interactions. If the intial state has positive G-parity then so must the final state. Notice that

- 2 pion system has positive G-parity
- 3 pion system has negative G-parity

Further it is observed that under strong interactions that

$$\omega \longrightarrow 3\pi \quad \text{BUT} \quad \omega \not\longrightarrow 2\pi$$

Then it follows that ω must also have negative G-parity.

$$\boxed{\hat{G}|\omega\rangle = -|\omega\rangle}$$

Likewise it is observed,

$$\rho \longrightarrow 2\pi \quad \text{BUT} \quad \rho \not\longrightarrow 3\pi$$

$$\therefore \boxed{\hat{G}|\rho\rangle = |\rho\rangle}$$

- Each meson gets some partkular G-parity. This helps us understand the interactions, why some happen and why others are forbidden.

While G-parity is a symmetry for mesons, the same is not true for BARYONS because charge conjugation takes us from one multiplet to another in this case. (More later.)

However for nucleon/antinucleon systems G-parity is still useful. The $N\bar{N}$ system for example is either in an iso-triplet state $|T=1, T_3=0\rangle$ or the singlet $|T=0, T_3=0\rangle$ the rotation about T_2 - axis is different for the above,

$$e^{-i\pi\hat{T}_2}|T=1,T_3=0\rangle = \exp\left[-i\frac{\pi}{2}(\tilde{\tau}_2(1)+\tilde{\tau}_2(2))\right]\frac{1}{\sqrt{2}}\left(|p\rangle_1|\bar{p}_2\rangle + |n\rangle_1|\bar{n}\rangle_2\right)$$

$$= \left(\exp\left(-\frac{i\pi}{2}\tilde{\tau}_2(1)\right), \exp\left(-\frac{i\pi}{2}\tilde{\tau}_2(2)\right)\right)\frac{1}{\sqrt{2}}\left(|p_1\rangle|\bar{p}_2\rangle + |n\rangle_1|\bar{n}\rangle_2\right)$$

$$= \left(-i\tilde{\tau}_2(1), -i\tilde{\tau}_2(2)\right)\left(\frac{1}{\sqrt{2}}|p_1\rangle|\bar{p}_2\rangle + |n\rangle_1|\bar{n}\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}\left(|n\rangle_1(-|\bar{n}\rangle_2) - |p\rangle_1(+|\bar{p}\rangle_2)\right)$$

$$= \frac{-1}{\sqrt{2}}\left(|n\rangle_1|\bar{n}\rangle_2 + |p\rangle_1|\bar{p}\rangle_2\right)$$

$$= -|T=1, T_3=0\rangle$$

using the result of 143
$e^{-i\pi\hat{T}_2}|p\rangle = |n\rangle$
$e^{-i\pi\hat{T}_2}|n\rangle = -|p\rangle$.

$\Rightarrow e^{-i\pi T_2}|\bar{p}\rangle = -|\bar{n}\rangle$
& $e^{-i\pi\hat{T}_2}|\bar{n}\rangle = |\bar{p}\rangle$

Similarly can see that

$$e^{-i\pi\hat{T}_2}|T=0,T_3=0\rangle = \left(\exp\left(\frac{-i\pi}{2}\tilde{\tau}_2(1)\right), \exp\left(\frac{-i\pi}{2}\tilde{\tau}_2(2)\right)\right)\frac{1}{\sqrt{2}}\left(|p\rangle_1|\bar{p}\rangle_2 - |n\rangle_1|\bar{n}\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}\left(|n\rangle_1(-|\bar{n}\rangle_2) + |p\rangle_1(|\bar{p}\rangle_2)\right)$$

$$= |T=0, T_3=0\rangle$$

Used that the generators for group operation in multiplet is same as that of the multiplet of the antiparticle, except the change of sign. Maybe we'll explain this better later on in SU(3) discussion.

G - PARITY CONTINUED

Positronium is the $e^+ e^-$ system and for total angular momentum J it can be shown (how?)

$$\hat{C} |e^+ e^-\rangle = (-1)^J |e^+ e^-\rangle$$

Well likewise for the $N\bar{N}$ system,

$$\hat{C} |N\bar{N}\rangle = (-1)^J |N\bar{N}\rangle$$

We know how $\exp(-i\pi T_2)$ acts on $|T=1, T_3=0\rangle$ and $|T=0, T_3=0\rangle$ and we also showed that G-parity $\hat{G} = \exp(-i\pi T_2)\hat{C}$ thus, looking back and thinking a bit,

$$\boxed{\hat{G}|N\bar{N}\rangle = (-1)^{J+T} |N\bar{N}\rangle}$$

It has been observed at CERN that

$$|N\bar{N}\,(J=0, T=0)\rangle \not\longrightarrow 3\pi \quad \text{(negative G-parity)}$$
$$|N\bar{N}\,(J=0, T=1)\rangle \not\longrightarrow 2\pi \quad \text{(positive G-parity)}$$

therefore we can see that

$|N\bar{N}, T=0\rangle$ has positive G-parity
$|N\bar{N}, T=1\rangle$ has neg. G-parity

Interestingly we can use the fact that G(nucleon) = antinucleon to connect the nucleon — nucleus to the antinucleon - antinucleus interaction. The effective antinucleon potential $V_{eff}(r)$ is the G-conjugate of the nucleon-nucleus potential. For example:

$$V_{eff}^{(N)}(r) = V^{\pi}(r) + V^{\rho}(r) + V^{\omega}(r) + \cdots$$

π, ρ, ω are all mesons.

As we have discussed the G-parity of π, ρ, ω can be inferred from there pionic decays. $G|\pi\rangle = -|\pi\rangle$, $G|\rho\rangle = |\rho\rangle$, $G|\omega\rangle = -|\omega\rangle$ hence

$$
\begin{aligned}
V_{eff}^{(\bar{N})}(r) &= \hat{G}\; V_{eff}^{(N)}(r)\; \hat{G} \\
&= G_\pi V^{\pi}(r) + G_\rho V^{\rho}(r) + G_\omega V^{\omega}(r) + \cdots \\
&= -V^{\pi}(r) + V^{\rho}(r) - V^{\omega}(r) + \cdots
\end{aligned}
$$

This has been observed! For example the ω-meson has a strongly repulsive interaction in short range of NN system while its very attractive in the $N\bar{N}$ system.

$\Rightarrow$ $N\bar{N}$ stable highly localized? (Ask Jorge)

Exercise 5.7 : Representations of Lie Algebras

A representation of a Lie Algebra $\mathcal{A}$ is an assignment of an $n\times n$ matrix $\hat{a}$ for each $\hat{A}\in\mathcal{A}$ such that the assignment $\varphi(\hat{A}) = \hat{a}$ is a Lie Algebra Homomorphism, meaning

1.) φ is linear

2.) $\varphi([\hat{A},\hat{B}]) = [\varphi(\hat{A}),\varphi(\hat{B})] \equiv [\hat{a},\hat{b}]$

The regular or adjoint representation for the Lie Algebra $\mathcal{A}$ generated by $\{\hat{T}_i\}_{i=1}^{n}$ is given by $([\hat{T}_i,\hat{T}_j] = C_{ijk}\hat{T}_k)$

$$\left(\varphi(\hat{T}_i)\right)_{jk} = C_{ijk} \quad \text{where } \varphi(\hat{T}_i)(\hat{T}_j) = \epsilon_{ijk}\hat{T}_k$$

In other perhaps clearer words,

$$(\hat{a}_i)_{jk} = C_{ijk}$$

And it can be shown that $[\hat{a}_i,\hat{a}_j] = C_{ijk}\hat{a}_k$ see 56 $\to$ 57.

—//—

The Lie Algebra of Angular momentum $\{\hat{L}_i\}$ satisfies the algebra $[\hat{L}_i,\hat{L}_j] = i\epsilon_{ijk}\hat{L}_k$, these are the orbital angular momentum operators. The regular or adjoint representation where $\hat{L}_1 \longleftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\hat{L}_2 \longleftrightarrow \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\hat{L}_3 \longleftrightarrow \begin{pmatrix}0\\0\\1\end{pmatrix}$ is our basis of operators equivalently denoted $|\hat{L}_i\rangle$ then the matrix representations of $\hat{L}_i$ are simply $(\hat{L}_i)_{jk} = i\varepsilon_{ikj}$

$$\boxed{\begin{aligned}
\hat{L}_1 &\longmapsto -i\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}\\
\hat{L}_2 &\longmapsto -i\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}\\
\hat{L}_3 &\longmapsto -i\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}
\end{aligned}}$$

- One should note these are the spin one matrices, from chapter one.
- Also we used $\hat{T}_2$ in the adjoint back on pg. 67, the confusion between $\hat{T}_2$ and $\hat{T}_2$'s adjoint has waisted my time already... not again.

§5.5 Experimental Test of Isospin Invariance

The hypothesis that ISOSPIN GROUP = Symmetry group of strong interaction. This hypothesis is consistent with several experimental observations we have discussed thus far,

1. Small mass difference in ISOSPIN MULTIPLETS. ($M_{proton} \approx M_{neutron}$ exc... $M_{\pi^\pm} \approx M_{\pi^0}$)
2. CHARGE INDEPENDANCE of strong force

Mathematically, we can summarize $\boxed{[\hat{H}_{strong}, \hat{T}_i] = 0.}$

Lie algebra of angular momentum ≈ Lie Algebra of isospin hence we can add isospins just like we add angular momenta. The whole machinery of Clebsch Gordon applies again. For two particles,

$$\boxed{|T\,T_3\rangle = \sum_{T_3(1)+T_3(2)=T_3} \underbrace{\big(T(1)\,T(2)\,T \,\big|\, T_3(1)\,T_3(2)\,T_3\big)}_{\text{Clebsch Gordon Coefficient}} |T(1)\,T_3(1)\rangle\,|T(2)\,T_3(2)\rangle}$$

Where T is the total isospin which must be between $T(1)+T(2)$ and $|T(1)-T(2)|$ where $T(i)$ is total isospin of the i^{th} particle.

- See table 5.2 for some explicit C.B. coefficients.

Consider the decay of a particle with isospin T into two other particles with $T(1)$ and $T(2)$. It is described by the matrix element,

$$\boxed{\begin{aligned} &\langle \underbrace{T(1)\,T_3(1);\,T(2)\,T_3(2)}_{\text{AFTER}} | \underbrace{\hat{S}}_{\text{PHYSICS}} | \underbrace{T\,T_3}_{\text{BEFORE}} \rangle = \\ &= \sum_{T'} \big(T(1)\,T(2)\,T' \,\big|\, T_3(1)\,T_3(2)\,T_3'\big) \langle T'\,T_3' | \hat{S} | T\,T_3\rangle \end{aligned}}$$

Where we have used that the C.B. sum above can be inverted to read

$$\boxed{|T(1)\,T_3(1)\rangle\,|T(2)\,T_3(2)\rangle = \sum_{T'=|T(1)-T(2)|}^{T(1)+T(2)} \big(T(1)\,T(2)\,T' \,\big|\, T_3(1)\,T_3(2)\,T_3'\big) |T'\,T_3'\rangle}$$

Where $T_3' = T_3(1) + T_3(2)$.

EXPERIMENTAL CONSEQUENCES OF ISOSPIN Continued

ISOSPIN INVARIANCE OF HAMILTONIAN MEANS THAT,

$$[\hat{T}_j, \hat{H}] = 0 \longrightarrow [\hat{T}_j, \hat{S}] = 0 \quad \text{for} \quad \hat{S} = \exp(-i\hat{H}t/\hbar)$$

From which it follows that the isospin-space portion of $\hat{H}$ is built from $\mathbb{1}$ and the casimir $\hat{T}^2$ of isospin. That is $\hat{H}$ must be an isospin scalar $\Rightarrow \hat{H} = \hat{H}(\hat{T}^2)$ thus $\hat{S} = \hat{S}(\hat{T}^2)$ and,

$$\boxed{\langle T' T_3' | \hat{S} | T T_3 \rangle = \delta_{TT'} \delta_{T_3 T_3'} \langle T \| \hat{S} \| T \rangle}$$

This is just the statement that multiplets $|T T_3\rangle$ do not couple to each other, the Hamiltonian is Block diagonal each block corresponding to a particular multiplet aka irred rep.

$$\boxed{\begin{aligned} \langle T \| \hat{S} \| T \rangle &\equiv \text{reduced matrix element} \\ &= f(T(1), T(2), T) \end{aligned}}$$ independent of T_3.

The reduced matrix element is characteristic of a multiplet with total isospin T, it is independent of states of a particular T_3. Hence we can simplify the expression of 72 for the s-matrix,

$$\boxed{\begin{aligned} &\langle T(1) T_3(1); T(2) T_3(2) | \hat{S} | T T_3 \rangle = \\ &= (T(1) T(2) T | T_3(1) T_3(2) T_3) \langle T(1) T(2) \| \hat{S} \| T \rangle \end{aligned}}$$

the sum over T' is killed by the $\delta_{TT'}$. (Special Case of Wigner-Eckart Thm which we'll explore next.) The probability of the decay is given by the square of the matrix element,

$$P_{|TT_3\rangle \to |T(1)T_3(1); T(2) T_3(2)\rangle} \propto |\langle T(1) T_3(1); T(2) T_3(2) | \hat{S} | T T_3 \rangle|^2$$

$$\propto |(T(1) T(2) T | T_3(1) T_3(2) T_3)|^2 |\langle T(1) T(2) \| \hat{S} \| T \rangle|^2$$

Thus for a particular intial state with T we can compare intensities of resultant decays, when taking the ratio the reduced matrix element will cancel (provided the decays compared are both in the $|T(1) T_3(2); T(2) T_3(2)\rangle$ final state, just with different T_3's). Physically this means we can observe the consequence of isospin w/o even knowing the complete Hamiltonian of the strong interaction.

EXAMPLE 5.8: THE WIGNER-ECKART THEOREM

We define that the $2k+1$ operators $\hat{T}_q^{(k)}$, $(q = -k, -k+1, \dots, k)$ form the components of an irreducible tensor operator of rank k if they satisfy:

$$[\hat{J}_\pm, \hat{T}_q^{(k)}] = \sqrt{k(k+1) - q(q\pm1)}\; T_{q\pm1}^{(k)}$$

$$[\hat{J}_0, \hat{T}_q^{(k)}] = q\,\hat{T}_q^{(k)}$$

Th^{m} (WIGNER-ECKART) A representation with $\hat{J}^2$, $\hat{J}_3$ with basis $|\tau jm\rangle$ has that a matrix element of an irreducible tensor operator $\hat{T}_q^{(k)}$ (as defined above) can be decomposed using the "reduced" matrix element and Clebsch-Gordon coeff,

$$\langle \tau' j' m' | \hat{T}_q^{(k)} | \tau j m\rangle = (j k j' | m q m') \underbrace{\langle \tau' j' \| T^{(k)} \| \tau j\rangle}_{\text{doesn't depend on } m, m' \text{ and } q}$$

the lable τ refers to quantum #'s from operators that do not commute with all $\hat{J}_q$

Lets see how to prove it, consider the $(2b+1)\times(2j+1)$ vectors ($b = q$ maybe?) yep

$$\hat{T}_q^{(k)} |\tau j m\rangle$$

Or a linear combination of the above,

$$|\tau J M\rangle \equiv \sum_{m,q} (j k J | m q M)\, \hat{T}_q^{(k)} |\tau j m\rangle$$

Using that $\hat{T}_q^{(k)}$ is a irred. tensor op. of rank k,

$$\hat{J}_\pm \hat{T}_q^{(k)} |\tau j m\rangle = [\hat{J}_\pm, \hat{T}_q^{(k)}] |\tau j m\rangle + \hat{T}_q^{(k)} \hat{J}_\pm |\tau j m\rangle$$

$$= \underbrace{\sqrt{k(k+1) - q(q\pm1)}\, \hat{T}_{q\pm1}^{(k)}}_{\text{using def}^{n}\text{ of irred. tensor op rank } k} |\tau j m\rangle + \underbrace{\sqrt{j(j+1) - m(m\pm1)}}_{\text{using that } \hat{J}_\pm \text{ is the familiar op from ang. mom. theory.}}\, \hat{T}_q^{(k)} |\tau j m\pm1\rangle$$

Now apply $J_\pm$ to the linear comb. state $|\tau J M\rangle$:

$$\hat{J}_\pm |\tau J M\rangle = \sum_{m,q} (j k J | m q M)\, \underbrace{\hat{J}_\pm \hat{T}_q^{(k)} |\tau j m\rangle}_{\text{using last few steps.}}$$

$$= \sum_{m,q} (j k J | m q M) \left(\sqrt{k(k+1) - q(q\pm1)}\, \hat{T}_{q\pm1}^{(k)} |\tau j m\rangle + \sqrt{j(j+1) - m(m\pm1)}\, \hat{T}_q^{(k)} |\tau j m\pm\right.$$

$$= \sum_{m,q} \sqrt{k(k+1) - q(q\pm1)}\, (j k J | m q M)\, \hat{T}_{q\pm1}^{(k)} |\tau j m\rangle$$

$$+ \sum_{m,q} \sqrt{j(j+1) - m(m\pm1)}\, (j k J | m q M)\, \hat{T}_q^{(k)} |\tau j m\pm1\rangle$$

Proof of Wigner - Eckart Th^m continued

Continuing where we finished on 74 switch $m \to m \mp 1$ and $q \to q \mp 1$ to combine the sums (converts $|\tau j m \pm 1\rangle \longrightarrow |\tau j m\rangle$ and $\hat{T}_{q\pm 1} \to \hat{T}_q$ thus,

$$\hat{J}_\pm |\tau J M\rangle = \sum_{m,q} \hat{T}_q^{(k)} |\tau j m\rangle \cdot \text{?}$$

$$\text{?} \cdot \left\{ \sqrt{k(k+1) - q(q\mp 1)}\, (j k J | m\, q\mp 1\, M) + \sqrt{j(j+1) - m(m \mp 1)}\, (j k J | m\mp 1\, q\, M) \right\}$$

From the nuts and bolts of Clebsch gordan as discussed in Chpt. 2,

$$\{\text{quote above}\} = \sqrt{J(J+1) - M(M\pm 1)}\, (j k J | m q M \pm 1)$$

there's a jump. (to fill in need to go back to chpt. 2 for an hour or three) Assuming this fact we find,

$$\hat{J}_\pm |\tau J M\rangle = \sqrt{J(J+1) - M(M\pm 1)} \sum_{m,q} (j k J | m q M\pm 1)\, \hat{T}_q^{(k)} |\tau j m\rangle$$

$$= \sqrt{J(J+1) - M(M\pm 1)}\, |\tau J M \pm 1\rangle$$

using def^n as on 74 for $|\tau J M\rangle$ with $M \to M \pm 1$.

Now apply $\hat{J}_0$ to our state,

$$\hat{J}_0 \hat{T}_q^{(k)} |\tau j m\rangle = [J_0, \hat{T}_q^{(k)}] |\tau j m\rangle + \hat{T}_q^{(k)} J_0 |\tau j m\rangle$$

$$= q\, \hat{T}_q^{(k)} |\tau j m\rangle + m\, \hat{T}_q^{(k)} |\tau j m\rangle$$

$$= (q + m)\, \hat{T}_q^{(k)} |\tau j m\rangle$$

Then try on the linear. comb. state $|\tau J M\rangle$

$$\hat{J}_0 |\tau J M\rangle = \sum_{m,q} (j k J | m q M)\, \hat{J}_0 \hat{T}_q^{(k)} |\tau j m\rangle$$

$$= \sum_{m,q} (j k J | m q M)(q + m)\, \hat{T}_q^{(k)} |\tau j m\rangle$$

$$= M |\tau J M\rangle$$ because unless $q + m = M$ the cleb. gord. coeff. goes to zero which leaves just M as the result.

What we see now is that the states $|\tau J M\rangle$ fufill ~~the~~ the angular momentum algebra. Hence they are (possibly unormalized) eigen functions of $\hat{J}^2$ and $\hat{J}_z$

Curious
Usually $\hat{J}_3$ is picked. hmm... what's the deal?

Proof of Wigner - Eckart Th^{m} continued

We just saw that $|\tau J m\rangle$ are eigenfunctions of $\hat{J}^2$ and $\hat{J}_z$ hence,

$$\langle \tau' J' M' | \tau J M \rangle = \delta_{JJ'} \delta_{MM'} \langle \tau' J M | \tau J M \rangle$$

Where the reduced matrix element $\langle \tau' J M | \tau J M \rangle$ doesn't depend on M, since,

$$\langle \tau' J M | \tau J M \rangle = \frac{1}{\sqrt{J(J+1) - M(M\mp 1)}} \langle \tau' J M | \hat{J}_\pm | \tau' J M \mp 1 \rangle$$

$$= \langle \tau' J M \mp 1 | \tau J M \mp 1 \rangle$$

(Where we just applied $\hat{J}_\pm$ to both sides of the above.)

Consequently we can write,

$$\boxed{\langle \tau' J' M' | \tau J M \rangle = \delta_{JJ'} \delta_{MM'} \langle \tau' J | \tau J \rangle}$$

Inverting an earlier expression from 74 using details of Clebsch Gordon ($\perp$)

$$\hat{T}^{(k)}_q |\tau j m\rangle = \sum_{jm} (j k J | m q M) |\tau J M\rangle$$

Evaluate then

$$\langle \tau' j' m' | \hat{T}^{(k)}_q | \tau j m \rangle = \sum_{jm} (j k J | m q M) \langle \tau' j' m' | \tau J M \rangle$$

$$= (j k j' | m q m') \langle \tau' j' | \tau J \rangle$$

where j' is the angular momentum arising from j and k. Thus

$$\boxed{\langle \tau' j' m' | \hat{T}^{(k)}_q | \tau j m \rangle = (j k j' | m q m') \langle \tau' j' \| \hat{T}^{(k)} \| \tau J \rangle}$$

Selection Rules

1 $\langle \tau' j' m' | \hat{T}^{(k)}_q | \tau j m \rangle \neq 0$ iff $q + m = M$ and j, k and j' fulfill Δ ineq.

- THE PHYSICS OF A GIVEN PROBLEM THEN %S INTO TWO PIECES,
  1.) THAT WHICH FOLLOWS FROM THE SYMMETRY, JWT INVOLVES CLEBSCH - GORD.
  2.) OTHER DETAILS (CONTAINED IN THE REDUCED MATRIX ELEMENT) whatever the τ has to do with I suppose.

  As remarked previously can see if system follows some symmetry w/o caring about the full physical picture by %ing to cancel reduced matrix element.
- Wigner-Eckart in case of rank $k = 0$ scalar Hamiltonian,

$$\langle \tau' t' t_3' | \hat{H} | \tau t t_3 \rangle = (t\, 0\, t' | t_3\, 0\, t_3') \langle \tau' t' \| \hat{H} \| \tau t \rangle$$

$$= \delta_{tt'} \delta_{t_3 t_3'} \langle \tau' t' \| \hat{H} \| \tau t \rangle$$

Example 5.9: Pion Production in Proton – Deuteron Scattering

In proton – ducteron scattering one can produce the following (among many others) products,

$$\boxed{p + d \begin{array}{l} \longrightarrow \pi^0 + {}^3He \\ \longrightarrow \pi^+ + {}^3H \end{array}}$$

Now we know the gnd. state of d has isospin $T=0$ while (See 47–49) the proton has $T=\frac{1}{2}$ thus $T_{intial} = 0 + \frac{1}{2} = \frac{1}{2}$. Wherear the final state must also have total isospin $\frac{1}{2}$, the pions have $T=1$ while ${}^3He$ has $T=\frac{1}{2}$ and the ${}^3H$ has $T=\frac{1}{2}$. The nuclei ${}^3He$ and ${}^3H$ are "mirror" nuclei and form an iso-doublet,

$$\boxed{\begin{aligned} |\text{intial state}\rangle &= |p+d\rangle = |\tfrac{1}{2}\ \tfrac{1}{2}\rangle|0\,0\rangle \\ |\text{final state}\rangle &= |\pi^+ + {}^3H\rangle = |1\,1\rangle|\tfrac{1}{2}\ {-\tfrac{1}{2}}\rangle \\ |\text{final state}\rangle &= |\pi^0 + {}^3He\rangle = |1\,0\rangle|\tfrac{1}{2}\ \tfrac{1}{2}\rangle \end{aligned}}$$

Now we can take the ratio of the cross-sections σ for the two different "exit channels"

$$R = \frac{\sigma(p+d \longrightarrow \pi^+ + {}^3H)}{\sigma(p+d \longrightarrow \pi^0 + {}^3He)} = \frac{|(1\,\tfrac{1}{2}\,\tfrac{1}{2}|1\,{-\tfrac{1}{2}}\,\tfrac{1}{2})|^2}{|(1\,\tfrac{1}{2}\,\tfrac{1}{2}|0\,\tfrac{1}{2}\,\tfrac{1}{2})|^2} = \frac{2/3}{1/3} = 2$$

Experiment is with about 10 % of this.

Questions:

1. How do we know ${}^3He$ and ${}^3H$ have $T=\frac{1}{2}$?
2. Why is isospin not an exact symmetry ?

EXAMPLE 5.10 : PRODUCTION OF NUETRAL PIONS IN DEUTERON - DEUTERON SCATTERING

Another indication of the existence of isospin symmetry is the reaction

$$\boxed{d + d \longrightarrow {}^4He + \pi^0}$$

Where one may note that 4He and d are iso-singlets,

$$|{}^4He\rangle = |T=0, T_3=0\rangle \quad \text{and} \quad |d\rangle = |T=0, T_3=0\rangle$$

Hence the reaction above should <u>not be allowed</u> because of the <u>orthogonality of $|00\rangle|00\rangle$ and $|00\rangle|10\rangle$</u>. Indeed

$$\sigma(d+d \longrightarrow {}^4He + \pi^0) < 1.6 \times 10^{-32}\ cm^2 \qquad \left(10^{-26}\,cm^2 \text{ typical in nuclear experiments}\right)$$

Compared to the reaction which is observed

$$\boxed{d + d \longrightarrow {}^4He + \gamma}$$

as the photon <u>γ contains an iso-singlet aka $\gamma \sim |00\rangle$</u> to isospin, it also has triplet components because γ doesn't conserve isospin. (not the point here.)

EXAMPLE 5.11 : PION - NUCLEON SCATTERING

Pion - nucleon scattering involves reactions of type

$$\pi + N \longrightarrow \pi^? + N^?$$

Where π could be π^0, $\pi^\pm$ and N could be a nuetron, proton or some excited state of the nucleon. Diagramatically,

$$\underbrace{\underset{\text{PION}}{|T=1\rangle} \otimes \underset{\text{NUCLEON}}{|T=\tfrac{1}{2}\rangle}}_{\text{intial state}} \longrightarrow \underbrace{\underset{\text{PION}}{|T=1\rangle} \otimes \underset{\text{NUCLEON}}{|T=\tfrac{1}{2}\rangle}}_{\text{final state}}$$

I suppose that we can have total isospin $T = \frac{1}{2}$ or $T = \frac{3}{2}$ that is to say $[1] \otimes [\frac{1}{2}] = [\frac{1}{2}] \otimes [\frac{3}{2}]$ hmm... is this a typo?

Lets expand on what the above means, the intial state looks like

$$\boxed{|1\mu\rangle|\tfrac{1}{2}\nu\rangle = \left(1\,\tfrac{1}{2}\,\tfrac{1}{2}\,\middle|\,\mu\,\nu\,\mu+\nu\right)|\tfrac{1}{2}, \mu+\nu\rangle + \left(1\,\tfrac{1}{2}\,\tfrac{3}{2}\,\middle|\,\mu\,\nu\,\mu+\nu\right)|\tfrac{3}{2}, \mu+\nu\rangle}$$

Likewise the final state,

$$\boxed{|1\mu'\rangle|\tfrac{1}{2}\nu'\rangle = \left(1\,\tfrac{1}{2}\,\tfrac{1}{2}\,\middle|\,\mu'\,\nu'\,\mu'+\nu'\right)|\tfrac{1}{2}, \mu'+\nu'\rangle + \left(1\,\tfrac{1}{2}\,\tfrac{3}{2}\,\middle|\,\mu'\,\nu'\,\mu'+\nu'\right)|\tfrac{3}{2}, \mu'+\nu'\rangle}$$

Where $\mu = \pm 1, 0$ characterize the intial state having $\pi^\pm$ or π^0 whereas $\nu = \pm\frac{1}{2}$ characterize the intial state having a proton ($T_3 = \frac{1}{2}$) or nuetron ($T_3 = -\frac{1}{2}$).

EXAMPLE 5.19 $\pi + N \longrightarrow \pi' + N'$, PION-NUCLEON SCATTERING

$$\boxed{\begin{aligned}\langle 1\,\mu\,\tfrac{1}{2}\,\nu|\hat{S}|1\mu'\tfrac{1}{2}\nu'\rangle = &\left(1\,\tfrac{1}{2}\,\tfrac{1}{2}\middle|\mu\nu\,\mu+\nu\right)\left(1\,\tfrac{1}{2}\,\tfrac{1}{2}\middle|\mu'\nu'\mu'+\nu'\right)\langle \tfrac{1}{2}\,\mu+\nu|\hat{S}|\tfrac{1}{2}\,\mu'+\nu'\rangle \\ &+\left(1\,\tfrac{1}{2}\,\tfrac{3}{2}\middle|\mu\nu\mu+\nu\right)\left(1\,\tfrac{1}{2}\,\tfrac{3}{2}\middle|\mu'\nu'\mu'+\nu'\right)\langle \tfrac{3}{2}\,\mu+\nu|\hat{S}|\tfrac{3}{2}\,\mu'+\nu'\rangle\end{aligned}}$$

Using Wigner-Echart in the case explained back on 73, they're are two Clebsch-Gordon coefficients because the Th^{m} has been applied to the "bra" and the "ket" as opposed to the "ket" only like on 73. The cross-terms between states of different total isospin vanish, only $T=T'$ contribute.

Now we also know that $\langle T'T_3'|\hat{S}|TT_3\rangle = \delta_{TT'}\,\delta_{T_3T_3'}\,\langle T\|\hat{S}\|T\rangle$ which means $\mu+\nu=\mu'+\nu'$ only survives and $\langle TT_3|\hat{S}|TT_3\rangle = \langle T\|\hat{S}\|T\rangle$ thus the above becomes,

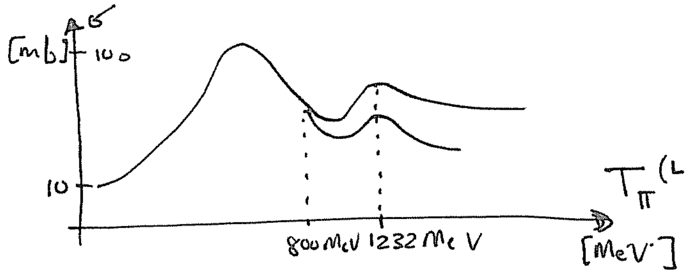
$$\begin{aligned}\langle 1\,\mu\,\tfrac{1}{2}\,\nu|\hat{S}|1\,\mu'\tfrac{1}{2}\nu'\rangle = \Big[&\left(1\,\tfrac{1}{2}\tfrac{1}{2}\middle|\mu\nu\mu+\nu\right)\left(1\,\tfrac{1}{2}\,\tfrac{1}{2}\middle|\mu'\nu'\mu+\nu\right)\langle \tfrac{1}{2}\|\hat{S}\|\tfrac{1}{2}\rangle + \\ &+\left(1\,\tfrac{1}{2}\,\tfrac{3}{2}\middle|\mu\nu\mu+\nu\right)\left(1\,\tfrac{1}{2}\,\tfrac{3}{2}\middle|\mu'\nu'\mu+\nu\right)\langle \tfrac{3}{2}\|\hat{S}\|\tfrac{3}{2}\rangle\Big]\delta_{\mu+\nu,\,\mu'+\nu'}\end{aligned}$$

If you consider the above carefully you'll find 10 possible reactions, its no accident that in each of the below total isospin is conserved.

$\mu=1$	$\nu=\frac{1}{2} \longrightarrow \mu'=1 \quad \nu'=\frac{1}{2}$	$\pi^+ + p \longrightarrow \pi^+ + p$
	$\nu=-\frac{1}{2}$: $\rightarrow \mu'=1 \quad \nu'=-\frac{1}{2}$; $\rightarrow \mu'=0 \quad \nu'=\frac{1}{2}$	$\pi^+ + n \rightarrow \pi^+ + n$; $\rightarrow \pi^0 + p$
$\mu=0$	$\nu=\frac{1}{2}$: $\rightarrow \mu'=0 \quad \nu'=\frac{1}{2}$; $\rightarrow \mu'=1 \quad \nu'=-\frac{1}{2}$	$\pi^0 + p \rightarrow \pi^0 + p$; $\rightarrow \pi^+ + n$
	$\nu=-\frac{1}{2}$: $\rightarrow \mu'=0 \quad \nu'=-\frac{1}{2}$; $\rightarrow \mu'=-1 \quad \nu'=\frac{1}{2}$	$\pi^0 + n \rightarrow \pi^0 + n$; $\rightarrow \pi^- + p$
$\mu=-1$	$\nu=\frac{1}{2}$: $\rightarrow \mu'=-1 \quad \nu'=\frac{1}{2}$; $\rightarrow \mu'=0 \quad \nu'=-\frac{1}{2}$	$\pi^- + p \rightarrow \pi^- + p$; $\rightarrow \pi^0 + n$
	$\nu=-\frac{1}{2} \longrightarrow \mu'=-1 \quad \nu'=-\frac{1}{2}$	$\pi^- + n \longrightarrow \pi^- + n$

Ex. 5.11 Pion - Nucleon Scattering

The 1st resonance, called the Δ resonance is at 1232 MeV. It's also called the $\frac{3}{2}-\frac{3}{2}$ resonance because the isospin is $\frac{3}{2}$ in and out. The figure one illustrates cross-section σ as a function of pion energy for



$$\pi^+ + P \longrightarrow \pi^+ + P$$

$T_\pi^{(L)} \equiv$ Kinetic Energy of π^+ in the lab-frame

Center of Mass System:

$P = (\vec{P}, E_N/c) = (\vec{P}, P_0)$ and $q = (\vec{q}, E_\pi/c) = (\vec{q}, q_0)$

are the 4-vectors of nucleon and pion in the intial state while W is total energy

$$W^2/c^2 = -(P+q)^2 \equiv (p^0+q^0)^2 - (\vec{P}+\vec{q})^2$$

Where in the c.m-frame $\vec{P}+\vec{q} = 0$

$$W^2 = (p^0+q^0)^2c^2 \Rightarrow W \text{ is really the total energy of } \pi-N \text{ in the c.o.m. frame.}$$

Laboratory System:

we have nucleus at rest $\Rightarrow \vec{P}_{lab} = 0$ hence

$$W^2/c^2 = -(P+q)^2 = -P^2 - q^2 - 2P\cdot q$$

$$= M^2c^2 + m_\pi^2c^2 + 2P_0q_0 \qquad P_0 = MC^2/c = MC$$

$$= M^2c^2 + m_\pi^2c^2 + 2ME_\pi^{(L)}(c^2)$$

(?) missing perhaps.

Where $E_\pi^{(L)}$ = total energy of the pion in the laboratory frame given by

$$(E_\pi^{(L)})^2 = q^2c^2 + m_\pi^2c^4$$

Compared to the Kinetic energy $T_\pi^{(L)}$ in lab-frame

$$T_\pi^{(L)} = E_\pi^{(L)} - m_\pi c^2$$

$$\Rightarrow \boxed{W^2 = (Mc^2 + m_\pi c^2)^2 + 2Mc^2T_\pi^{(L)}}$$

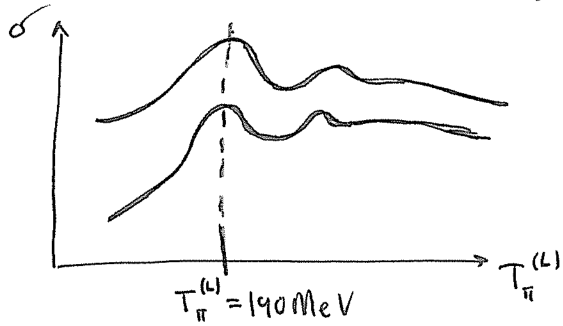
Question: How is this a resonance? Is there some direct connection between the $T_\pi^{(L)}-\sigma$ graph and W^2 formula?

Ex. 5.11 – Nucleon-Pion Scattering

A resonance can be interpreted as a short-lived intermediate state N^* where the nucleon and pion are jumbled up into N^*

$$\pi + N \longrightarrow N^* \longrightarrow \pi + N$$

the masses of the intermediate particles are given by the total energies W_{max} at the resonance maxima. The W can be calculated from formula on 80 given $T_\pi^{(L)}$.



$T_\pi^{(L)} = 190\,MeV \Rightarrow \boxed{W = 1232\,MeV}$

- Somehow you can see the intermediate in $\pi^+ p$ exhibit double positive charge while they are nuetral in $\pi^- p$ (intuitively sensible but how does it follow from graph?)

Another reaction with an intermediate with a single charge is

$$\gamma + p \longrightarrow N^* \longrightarrow \pi^0 + p$$

also has $W = 1232\,MeV$

Likewise the reaction below has intermediate N^* with same mass (See 170)

$$\pi^- p \longrightarrow \pi^+ \pi^+ \pi^- \pi^- n$$

These intermediates N^* form a charge quartet

$$\Delta^{++}(1232),\ \Delta^{+}(1232),\ \Delta^{0}(1232),\ \Delta^{-}(1232)$$

This is a $T = \frac{3}{2}$ isospin multiplet, it is not obvious that there are just these four from the scattering π-N alone, but we'll give more argument later to asure this.

An intermediate state is seen as a resonance in the cross-section,

$$e^- + e^+ \longrightarrow \rho^0 \longrightarrow \pi^- + \pi^+$$ (Resonance at $M = 770\,MeV$, must be nuetral

$$e^- + e^+ \longrightarrow \omega \longrightarrow \pi^- + \pi^+ + \pi^0$$ (Resonance at $M = 780\,MeV$ also nuetral but G-parity different)

Notice that because $\pi\pi$ has positive G-parity and $\pi\pi\pi$ has negative G-parity it follows that

ρ^0 has positive G-parity

ω has negative G-parity

There are also $\rho^\pm$ intermediates with $M = 770\,MeV$ in

$$\pi^\pm + p \longrightarrow \rho^\pm + p \longrightarrow \pi^\pm + \pi^0 + p$$

Summarizing, experiment has revealed,

	SPIN	G-PARITY	MASS [MeV]	LIFETIME [s]	WIDTH [MeV]	CHARGE
ρ^+	1	+	770	4.3×10^{-24}	153	e
ρ^-	1	+	770	4.3×10^{-24}	153	-e
ρ^0	1	+	770	4.3×10^{-24}	153	0
ω	1	−	783	6.5×10^{-23}	10	0

One last note consider the decay of ρ^0 into pions

$$\rho^0 \longrightarrow \pi^+ + \pi^-$$

$$\rho^0 \longrightarrow 2\pi^0$$

Given probabilities according to

$$\langle 1\mu 1{-\mu} | \hat{S} | T=1, T_3=0 \rangle = (1 1 1 | \mu {-\mu}\, 0) \langle 1 \| \hat{S} \| 1 \rangle$$

what possible μ values are there, well for the processes listed above we get two coefficients

$$(1 1 1 | 1 {-1}\, 0) \quad \text{and} \quad \cancel{(1 1 1 | 0 0 0)} \rightarrow 0$$ By properties of Clebsch Gordon.

$$\Rightarrow \rho^0 \not\rightarrow 2\pi_0$$

forbidden