Chapter 6 : Hypercharge

The validity of isospin is manifest in the existence of multiplets based on isospin. Moreover, we noticed that particles in a charge multiplet differed only by E/M properties like charge, magnetic momentum and so on... Further one can note that in a multiplet the electric charge wasn't always centered about zero. In fact we noted that all charge values between Q_{max} and Q_{min} were realized within a multiplet of isospin, but $Q_{max} + Q_{min} \neq 0$.

The total isospin describes a multiplet which has members which have $T_3 \in \{-T, -T+1, \dots, T-1, T\}$ but $-T$ goes to Q_{min} while T goes to Q_{max}. In order that $T_3 = 0$ be in the center of charge we will let T_3 be centered at $\frac{Q_{max} + Q_{min}}{2}$ thus as per our observations about charge inc. by e for T_3 increasing by 1 the charge Q is given by

$$\boxed{Q = \frac{1}{2}\left(Q_{min} + Q_{max}\right) + T_3}$$

Where T_3 can take values

$$T_3 = 0, \pm 1, \pm 2, \dots, \pm \frac{1}{2}\left(Q_{max} - Q_{min}\right)$$

But we know $T_3 = 0, \pm 1, \pm 2, \dots, \pm T$ hence charge and total isospin are related by $\frac{1}{2}(Q_{max} - Q_{min}) = T$ aka

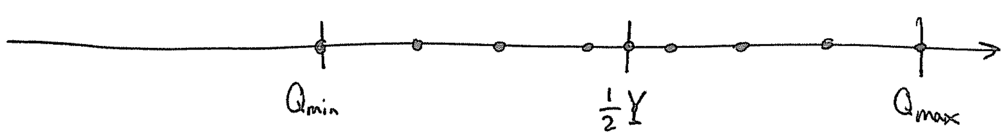
$$\boxed{2T + 1 = Q_{max} - Q_{min} + 1}$$

We define the center of charge to be $\frac{1}{2}Y$ where Y is the <u>hypercharge</u>

$$\frac{1}{2}Y \equiv Q_{max} - Q_{min}$$

Hence the above becomes the Gell-Mann-Nishijima relation

$$\boxed{Q = \frac{1}{2}Y + T_3 \qquad \text{where } T_3 = -T, -T+1, \dots, T}$$



EXERCISE 6.1 : HYPERCHARGE OF NUCLEI

Determine hypercharge of nuclei in an isospin multiplet, given that the charge operator $\hat{Q}$ for a nucleus with Z protons and N nuetrons ($A = N + Z$ is the total # of nucleons) is known from 5.21 recall

$$\hat{Q} = \sum_n \hat{Q}(n) = e \sum_{n=1}^{A} \tfrac{1}{2}\left(\hat{t}_3(n) + 1\right) = e\left(\hat{T}_3 + \tfrac{1}{2}A\right)$$

Then simply compare with $\hat{Q} = T_3 + \frac{1}{2}Y$ (where $e = 1$) to see

$$\boxed{A = Y}$$

The total # of nucleons is the hyper charge. Notice we have no idea which multiplet it belongs to, correct?

EXAMPLE 6.2 Hypercharge of Δ resonances

Back on (81) we discussed briefly the isospin quartet $\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$ as it is suggested by the notation $Q_{max} = 2e$ while $Q_{min} = -e$ or in unit charge mode, $Q_{max} = 2$ while $Q_{min} = -1$ thus

$$Y = Q_{max} + Q_{min} = 1$$

$$T = \tfrac{1}{2}\left(Q_{max} - Q_{min}\right) = \tfrac{3}{2}$$

Then the Gell-Mann-Nishijima relation reproduces the known charges above

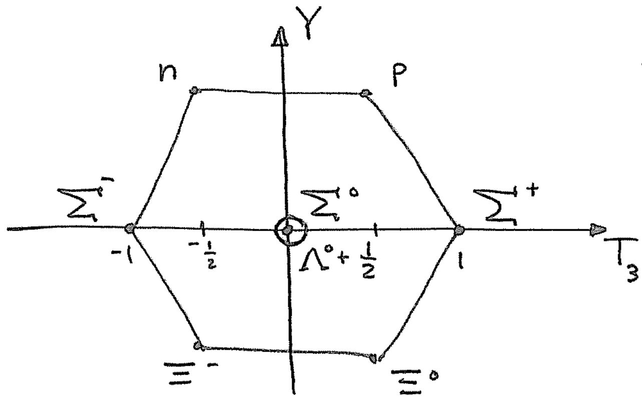
$$\boxed{Q = \tfrac{1}{2}Y + T_3 = \tfrac{1}{2} + T_3 = 2, 1, 0, -1 .}$$

$$\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}$$

- Hypercharge is in addition to the isospin symmetry. We began by studying isospin multiplets (which a priori could have had random charge correlations with isospin) and found the charge of isospin multiplets obeyed some rather regular patterns described aptly by introducing ~~isospin~~. hypercharge Y.

EXAMPLE 6.3 : BARYONS

Baryon # is conserved. When Baryons decay they decay into other Baryons. Thus the # of baryons is unchanged thru nuclear decays. We can graph Baryon Multiplets in the $(T_3\, Y)$-plane

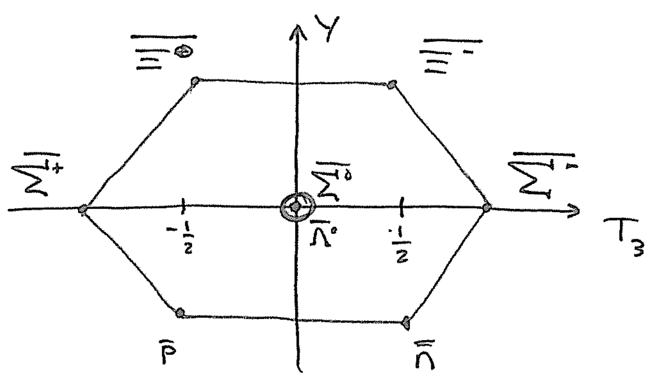


"Baryon nonet"

- there are all positive parity spin $\frac{1}{2}$ baryons.
- Ω has $\frac{3}{2}$ spin, it's not here.

ANTIPARTICLES: Every spin $\frac{1}{2}$ particle has an antiparticle of opposite charge but identical mass. Even nuetral particles $(n \neq \bar{n})$ have antiparticles but they're hard to observe. The same is not quite true for mesons ($\overline{\pi^+} = \pi^-$ and $\overline{\pi^0} = \pi^0$). The bar denotes the antiparticle.

$$Q = \frac{1}{2} Y + T_3$$



"Antibaryon octet"

$Y = 0 \Rightarrow Q \longrightarrow -Q \Leftrightarrow T_3 \longrightarrow -T_3$ $\left(\Sigma^+ \longrightarrow \overline{\Sigma^+} \text{ and } \Sigma^- \longrightarrow \overline{\Sigma^-} \quad Y=0 \text{ flip over } Y\text{-axis} \right)$

EXERCISE 6.5: ISOSPIN AND HYPERCHARGE OF BARYON RESONANCES

Consider the properties of the N^*, N', Λ^*, Σ^*, Ξ^* resonances

Name		J^P (P = PARITY)	Q	T	T_3	Mass [MeV]	LIFETIME [s]	Γ [MeV]	Decays (Main)	RESONANT PARTIAL WAVES
N^*	Δ^{++}	$3/2^+$	2	3/2	3/2	1232 ± 2	5.49×10^{-24}	120	$N\pi$	P_{33} πp
	Δ^+	$3/2^+$	1	3/2	1/2					
	Δ^0	$3/2^+$	0	3/2	-1/2					
	Δ^-	$3/2^+$	-1	3/2	-3/2					
N'	N'^+	$1/2^+$	1	1/2	1/2	1440 ± 40	3.13×10^{-24}	210	$N\pi, N\pi\pi$	P_{11} πp
	N'^-	$1/2^+$	0	1/2	-1/2					
Λ^*		$1/2^-$	0	0	0	1405 ± 05	1.65×10^{-23}	40	$\Sigma'\pi$	S_{01} K^-p
Σ^*	Σ^{*1}	$3/2^+$	1	1	1	1382.3 ± 0.4	1.78×10^{-25}	37	$\Lambda\pi, \Sigma\pi$	P_{13} K^-p
	Σ^{*0}	$3/2^+$	0	1	0	1382.0 ± 2.5				
	Σ^{*-1}	$3/2^+$	-1	1	-1	1387.4 ± 0.6				
Ξ^*	Ξ^{*0}	$3/2^+$	0	1/2	1/2	1531.8 ± 0.3	9.4×10^{-23}	7	$\Xi'\pi$	P
	Ξ^{*-}	$3/2^+$	-1	1/2	-1/2	1535.0 ± 0.6				

Now we can use the Gell-Mann – Nishijima formula and deduce from the hypercharge the isospin.

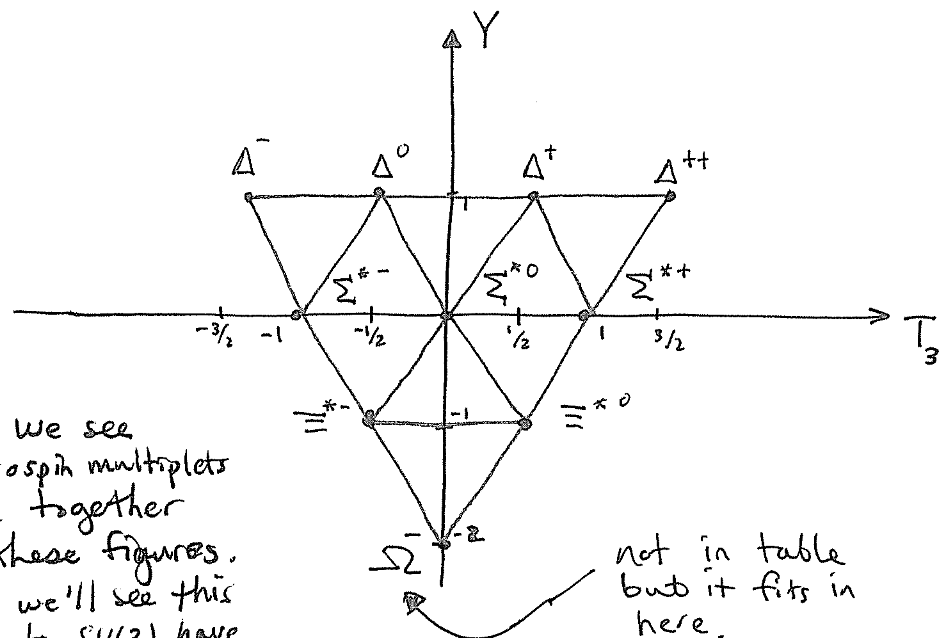
$$Y = Q_{max} + Q_{min}$$

$$T = \frac{1}{2}(Q_{max} - Q_{min})$$

$N^*(Y=1)$
$N'(Y=1)$
$\Lambda^*(Y=0)$
$\Sigma^*(Y=0)$
$\Xi^*(Y=-1)$

$$Q = \frac{1}{2}Y + T_3 \longrightarrow T_3 = Q - \frac{1}{2}Y$$

We use these formulas to fill in the table above's T T_3 columns.



not in table but it fits in here.

$(Y\,T_3)$-plot of the spin $J = 3/2$ Baryon Resonances. They don't last long notice the lifetimes are quite short. This sym. pattern seems to indicate some new underlying approximate symmetry (masses not equal!)

- **Remark**: We see many isospin multiplets strung together in these figures. Later we'll see this corres. to SU(3) have SU(2) rep. as building-blocks.

Chapter 7 : The SU(3) Symmetry

We saw that many particles fit into highly symmetric plots in the $Y-T_3$ plane. In a single plot several isospin multiplets were linked together by the hypercharge. Lets see how SU(3) and it's representations reproduce this phenomenon.

7.1 : Math of U(n) and SU(n)

A unitary quadratic matrix $\hat{U}$ with n rows an n collumns can be written,

$$\boxed{\hat{U} = \exp(i\hat{H})}$$

Here $\hat{H}$ is a Hermitian quadratic matrix ($n \times n$ as well). Meaning that

$$\boxed{H_{ii}^* = H_{ii}} \quad \text{and} \quad \boxed{H_{ij}^* = H_{ji}} \qquad \left(\begin{array}{c}\text{Components of} \\ \hat{H} \text{ don't get hats}\end{array}\right)$$

There are n- diagonal parameters and $2 \times \frac{1}{2} n(n-1)$ off diag. parameters (counting real d.o.f.) $n + n(n-1) = n + n^2 - n = n^2$. That is $U(n)$ is a n^2-dim'l <u>real</u> manifold.

- Claim its obviously continuously connected
- Its compact because $\lim_{\sigma \to \sigma^0} U_{ik}(\sigma) = U_{ik}(\sigma^0)$ which means that a sequence of matrices, well the limit of a matrix valued function with values in the group $U(n)$ again lands in $U(n)$.
- The trace of a <u>Hermitian</u> matrix is real since

$$\text{trace}(\hat{H}) = \sum_{i=1} H_{ii} = \sum_{i=1} H_{ii}^* = (\text{trace}(\hat{H}))^*$$

Mathematical overview of SU(3) continued

The group $U(3)$ is composed of matrices with $\hat{U} = e^{i\hat{H}}$ where $U^\dagger U = I \Rightarrow \hat{H} = \hat{H}^\dagger$ and trace $(\hat{H}) \in \mathbb{R}$. Consider then that

$$U^\dagger U = I \Rightarrow \det(U^\dagger)\det(U) = \det(I)$$
$$\Rightarrow \det(U^*)\det(U) = 1$$
$$\Rightarrow |\det(U)|^2 = 1$$

Which is consistent with the calculation below, letting $\alpha \equiv tr(\hat{H}) \in \mathbb{R}$

$$\det(U) = \det(e^{iH}) = e^{i\,tr(\hat{H})} = e^{i\alpha}$$

(using the well-known $\det(e^A) = e^{tr(A)}$)

- The proof on pg. 184 assumes that $\hat{U}$ is diagonalizable, is that obvious? $\exists$ nonsingular nondiag. matrices.

- $\boxed{\text{Def}^{n}/ \text{ The subset of } U(3) \text{ for which } \det(U) = 1 \text{ is called the special unitary group of order 3 aka } SU(3)}$

- Since $U(n)$ has dimension n^2 and $\det(U) = 1$ is 1 eq^{n}; it follows that the dimension of $SU(n)$ is $n^2 - 1$. Question: why did this complex eq^{n} only remove one real dof? Answer: the fact that $\det(U) = e^{i\alpha} \Rightarrow \det(U) = 1$ only restricts angular complex coordinate since modulus is not further restricted from $U(n)$ criterion.

- We can write anything in $U(n)$ via a particular $U_0 \in SU(n)$

pick some $U_0 \in SU(n)$ with $U_0 = e^{iH_0}$ so $\det(U_0) = 1$ then

$$\hat{U} = e^{i\hat{H}} = \exp\left(i\left(\hat{H}_0 + \frac{\alpha}{n}\mathbb{1}\right)\right) =$$
$$= \exp(i\hat{H}_0)\exp\left(\frac{i\alpha}{n}\mathbb{1}\right)$$
$$= \hat{U}_0 \exp\left(i\frac{\alpha}{n}\right)$$

: since $\mathbb{1}$ commutes with everything the matrix exp. behaves simply here.

Where $\alpha \equiv$ trace $(\hat{H})$ and as is easily seen this is consistent

$$\det(U) = \det\left(\hat{U}_0 \exp\left(i\frac{\alpha}{n}\right)\right) = \det(U_0)\det\left(\exp\left(i\frac{\alpha}{n}\right)\mathbb{1}\right)$$
$$= n\exp\left(i\frac{\alpha}{n}\right)\det(\mathbb{1}) \quad \text{(pulled out } n\text{-factors)}$$
$$= \exp(i\alpha)$$

$SU(n)$ factors $U(n)$ continued

As we were discussing if we pick some representative $U_0 \in SU(n)$ then we can write any $U \in U(n)$ via U_0 with

$$\boxed{\hat{U} = e^{i\hat{H}} = \exp\left(i\left(\hat{H}_0 + \frac{\alpha}{n}\mathbb{1}\right)\right) = U_0 \exp\left(i\frac{\alpha}{n}\right)\mathbb{1}}$$

Where $\alpha = \text{trace}(\hat{H})$ which is reasonable as $SU(n)$'s generators are traceless,

$$\begin{aligned} tr(\hat{H}) &= tr\left(\hat{H}_0 + \frac{\alpha}{n}\mathbb{1}\right) \\ &= tr(\hat{H}_0) + \frac{\alpha}{n} tr(\mathbb{1}) \\ &= 0 + \frac{\alpha}{n}\cdot n \\ &= \alpha \end{aligned}$$

$\left(\det(U_0) = 1 \Rightarrow \text{trace}(H_0) = 0 \text{ where } U_0 = \exp(iH_0)\right)$

Remark: the matrice $\exp\left(i\frac{\alpha}{n}\right)\mathbb{1}$ form a realization of $U(1)$ thus an arbitrairy element U of $U(n)$ can be written as the product of a $U(1)$ and $SU(n)$ factor as described above.

Subgroups of $U(m)$

Notice that $SU(n) \subset U(n) \subset U(m)$ when $n < m$. This can be understood via the following construction,

$$U \in U(n) \longmapsto \begin{pmatrix} U & 0 \\ 0 & \mathbb{1} \end{pmatrix} = U' \in U(m)$$

Where $\mathbb{1}$ is $(m-n)\times(m-n)$ so that U' is $m\times m$. Because of block multiplication we can easily see that $U'^{\dagger} = U'^{-1}$ Where it's not hard to deduce $e^{i\hat{H}'} = U' \Rightarrow \hat{H}' = \begin{pmatrix} \hat{H} & 0 \\ 0 & 0 \end{pmatrix}$.

The construction above reveals the existence of at least one $U(n)$ subgroup of $U(m)$, there could be other ways to independently find more subgroups in $U(m)$.

Generators of $U(n)$ and $SU(n)$

Generators of the group are determined by the group elements infinitesimally close to the identity. Call the n^2 group parameters φ_j and the corresponding n^2 generators $\hat{\lambda}_j$ then for a tiny group element near $\mathbb{1}$,

$$\hat{U}(\delta\varphi_j) = e^{i\hat{H}(\delta\varphi_j)}$$
$$= 1 + i\hat{H}(\delta\varphi_j)$$
$$= \mathbb{1} + i\sum_{j=1}^{n^2}\delta\varphi_j\hat{\lambda}_j$$

Where $\hat{H}^\dagger = \hat{H}$ and we assume these are the n^2 linearly indep. generators $\hat{\lambda}_j$

Notice that $\left(i[\hat{\lambda}_i,\hat{\lambda}_j]\right)^\dagger = -i[\hat{\lambda}_j^\dagger,\hat{\lambda}_i^\dagger] = i[\hat{\lambda}_i,\hat{\lambda}_j] \leftarrow$ Hermitian $n\times n$ matrix hence it must be in $U(n)$ which is spanned by the n^2 $\hat{\lambda}_j$ thus

$$\boxed{[\hat{\lambda}_i,\hat{\lambda}_j] = i\,c_{ijk}\hat{\lambda}_k}$$

Generators of $SU(n)$ are n^2-1 linearly independent ~~space~~ matrices with trace zero to insure $\det(SU(n)) = 1$.

Lie Algebra of $SU(2)$

Need $n^2-1 = 4-1 = 3$ linearly indep., traceless matrices. Choose Pauli matrices

$$\boxed{\hat{\sigma}_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \hat{\sigma}_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad \hat{\sigma}_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

These are hermitian and traceless they generate $SU(2)$. Can calculate

$$[\hat{\sigma}_i,\hat{\sigma}_j] = 2i\,\epsilon_{ijk}\hat{\sigma}_k$$

Which becomes with $\hat{S}_j = \frac{1}{2}\hat{\sigma}_j$ simply.

$$[\hat{S}_i,\hat{S}_j] = i\,\epsilon_{ijk}\hat{S}_k$$

Clearly isomorphic to the Lie algebra $so(3)$ of the group $SO(3)$. For $U \in SU(2)$

$$\boxed{\hat{U} = \exp\left(-i\,\varphi_j\hat{S}_j\right)}$$

And the Casimir of $SU(2)$ (Only one because $SU(2)$ is rank one)

$$\boxed{\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + \hat{S}_3^2}$$

§7.2 The Generators of $SU(3)$

$SU(3)$ has $n^2 - 1 = 3^2 - 1 = 8$ generators. Call them $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_8$ and choose them wisely, we know $SU(2) < SU(3)$ so extend the Pauli Matrices to get

$$\boxed{\hat{\lambda}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{\lambda}_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \hat{\lambda}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

All are hermitian and traceless. Using block multiplication the fact that $\hat{\lambda}_i = \begin{pmatrix} \hat{\sigma}_i & 0 \\ 0 & 0 \end{pmatrix}$ and $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k \Rightarrow [\hat{\lambda}_i, \hat{\lambda}_j] = 2i\epsilon_{ijk}\hat{\lambda}_k$ where $i, j, k \in \{1, 2, 3\}$ and ϵ_{ijk} is the Levi-civita symbol in 3-dimensions. We still need 5 more generators. Physics conventionally uses, (Gell-Mann for example)

$$\boxed{\begin{array}{lll} \hat{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \hat{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \\ \hat{\lambda}_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \hat{\lambda}_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \hat{\lambda}_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\frac{1}{\sqrt{3}} \end{array}}$$

Where $\hat{\lambda}_4$ and $\hat{\lambda}_6$ are similar to $\hat{\lambda}_1$ while $\hat{\lambda}_5$ and $\hat{\lambda}_7$ are similar to $\hat{\lambda}_2$. Finally $\hat{\lambda}_8$ is the only other way (easy way) to make a traceless matrix the $\frac{1}{\sqrt{3}}$ was necessary in order to enforce the normilization

$$\boxed{\mathrm{Trace}(\hat{\lambda}_i \hat{\lambda}_j) = 2\delta_{ij}}$$

question, where is this trace taken? Is this an adjoint statement? see (93)

- Now we proceed to prove that these do in fact generate $su(3)$ the space of hermitian traceless 3×3 matrices. $\hat{H} \in su(3)$ has

$$\hat{H} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12}^* & h_{22} & h_{23} \\ h_{13}^* & h_{23}^* & -(h_{11}+h_{22}) \end{pmatrix} \underset{\text{claim}}{=\!=} \sum_{j=1}^{8} \lambda_j \hat{\lambda}_j$$

This leads to 9 eq$^{\underline{n}}$ going component by component. The 1-1 component is non-zero only for $\hat{\lambda}_3$ and $\hat{\lambda}_8$ hence:

$$\lambda_3 + \frac{1}{\sqrt{3}}\lambda_8 = h_{11}$$

Only $\hat{\lambda}_1$, $\hat{\lambda}_2$ have non zero (12) components thus

$$\lambda_1 - i\lambda_2 = h_{12}$$

Showing that $\{\hat{\lambda}_i \mid i=1,2,\dots,8\}$ generate $su(3) = \{ A \in \mathbb{C}^{3\times 3} \mid A^\dagger = A \text{ and } \operatorname{trace}(A) = 0 \}$

Continuing we find 9 eq$^{\underline{n}}$'s from $\hat{H} = \sum_{j=1}^{8} \lambda_j \hat{\lambda}_j$,

$$\begin{aligned}
\lambda_3 + \tfrac{1}{\sqrt{3}}\lambda_8 &= h_{11} \\
\lambda_1 - i\lambda_2 &= h_{12} \\
\lambda_4 - i\lambda_5 &= h_{13} \\
\lambda_1 + i\lambda_2 &= h_{12}^* \\
-\lambda_3 + \tfrac{1}{\sqrt{3}}\lambda_8 &= h_{22} \\
\lambda_6 - i\lambda_7 &= h_{23} \\
\lambda_4 + i\lambda_5 &= h_{13}^* \\
\lambda_6 + i\lambda_7 &= h_{23}^*
\end{aligned}
\quad \longleftrightarrow \quad A\vec{\lambda} = \vec{h} \text{ unique sol}^{\underline{n}} \text{ if } \det(A) \neq 0$$

Thus consider as on pg. 189,

$$\det \begin{vmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/\sqrt{3} \\
1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -i & 0 & 0 & 0 \\
1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1/\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 1 & -i & 0 \\
0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & i & 0
\end{vmatrix} = \frac{16}{\sqrt{3}} i \neq 0 \Rightarrow \exists \text{ a unique sol}^{\underline{n}} \text{ thus } \{\hat{\lambda}_i\} \text{ spans } su(3).$$

• I'd like to explicitly show how it spans, $a, b, c, d, e, f, g, h, k \in \mathbb{R}$

$$\hat{H} = \begin{pmatrix} a & d+ie & f+ig \\ d-ie & b & h+ik \\ f-ig & h-ik & c \end{pmatrix} \qquad (\text{where } a+b+c = 0)$$

$$= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + d\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & +i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + g\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$$+ h\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} + d\hat{\lambda}_1 + e\hat{\lambda}_2 + f\hat{\lambda}_4 + g\hat{\lambda}_5 + h\hat{\lambda}_6 + k\hat{\lambda}_7$$

We see that $A\hat{\lambda}_3 + B\hat{\lambda}_8 = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = A\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + B\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} A + B/\sqrt{3} & 0 & 0 \\ 0 & -A + B/\sqrt{3} & 0 \\ 0 & 0 & -2B/\sqrt{3} \end{pmatrix}$

Recall $\operatorname{trace}(H) = a+b+c = 0 \Rightarrow a = -b-c$ thus solve

$$\begin{aligned}
+\ \ -b-c &= A + B/\sqrt{3} \\
b &= -A + B/\sqrt{3} \\
\hline
-c &= 2B/\sqrt{3}
\end{aligned}$$

$$\therefore \boxed{B = -\frac{\sqrt{3}}{2}c} \Rightarrow \boxed{A = -b + \frac{\sqrt{3}}{2}c}$$

$$\therefore \boxed{\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = \left(-b + \frac{\sqrt{3}}{2}c\right)\hat{\lambda}_3 + \left(-\frac{\sqrt{3}}{2}c\right)\hat{\lambda}_8}$$

$$\Rightarrow \hat{H} \in \operatorname{span}_{\mathbb{R}}\{\hat{\lambda}_i \mid i = 1,2,\dots,8\}.$$

§7.3 The Lie algebra su(3) of the Lie Group SU(3).

We define the structure constants f_{ijk} for $i,j,k \in \{1,2,3,\dots,8\}$ as (sum ⇒'d)

$$\boxed{[\hat{\lambda}_i , \hat{\lambda}_j] = 2i f_{ijk} \hat{\lambda}_k}$$

Similarly one can calculate the anticommutator and find

$$\boxed{\{\hat{\lambda}_i , \hat{\lambda}_j\} = \frac{4}{3}\delta_{ij} \mathbb{1} + 2 d_{ijk} \hat{\lambda}_k}$$

The symbols f_{ijk} are completely antisymmetric while the symbols d_{ijk} are completely symmetric. One could calculate (non-zero terms listed.)

ijk	f_{ijk}
123	1
147	1/2
156	−1/2
246	1/2
257	1/2
345	1/2
367	−1/2
458	$\sqrt{3}/2$
678	$\sqrt{3}/2$

(antisymmetric in ijk)

ijk	d_{ijk}
118	$1/\sqrt{3}$
146	1/2
157	1/2
228	$1/\sqrt{3}$
247	−1/2
256	1/2
338	$1/\sqrt{3}$
344	1/2
355	1/2
366	−1/2
377	−1/2
448	$-1/2\sqrt{3}$
558	$-1/2\sqrt{3}$
668	$-1/2\sqrt{3}$
778	$-1/2\sqrt{3}$
888	$-1/\sqrt{3}$

(Symmetric in ijk)

Given that $\{\hat{\lambda}_i , \hat{\lambda}_j\} = \hat{\lambda}_i\hat{\lambda}_j + \hat{\lambda}_j\hat{\lambda}_i$ where that is equal to $\frac{4}{3}\delta_{ij}\mathbb{1} + 2 d_{ijk}\lambda_k$ it's not hard to calculate

$$\begin{aligned}
\mathrm{trace}(\hat{\lambda}_i\hat{\lambda}_j) &= \mathrm{trace}\left(\tfrac{1}{2}\{\hat{\lambda}_i,\hat{\lambda}_j\} + \tfrac{1}{2}[\hat{\lambda}_i,\hat{\lambda}_j]\right)\\
&= \mathrm{trace}\left(\tfrac{1}{2}\left(\tfrac{4}{3}\delta_{ij}\mathbb{1} + 2d_{ijk}\hat{\lambda}_k\right) + \tfrac{1}{2}\cdot 2i f_{ijk}\hat{\lambda}_k\right)\\
&= \mathrm{trace}\left(\tfrac{2}{3}\delta_{ij}\mathbb{1}\right) + d_{ijk}\,\underbrace{\mathrm{trace}(\hat{\lambda}_k)}_{0} + i f_{ijk}\,\underbrace{\mathrm{trace}(\hat{\lambda}_k)}_{0}\\
&= \tfrac{2}{3}\delta_{ij}\,\mathrm{trace}(\mathbb{1})\\
&= \boxed{2\delta_{ij} = \mathrm{trace}(\hat{\lambda}_i\hat{\lambda}_j)}
\end{aligned}$$

it is not an adjoint statement. Its the algebra itself, the adjoint would have 8×8 matrices here.

The Lie Algebra of SU(3) continued — "F-spin"

As in the SU(2) case we rescale the generators; $\hat{F}_i \equiv \frac{1}{2}\hat{\lambda}_i$ hence

$$\boxed{[\hat{F}_i, \hat{F}_j] = i f_{ijk} \hat{F}_k}$$

So as $\hat{\sigma}_i$ is to $\hat{S}_i$ (the spin matrices) so to $\hat{\lambda}_i$ is to $\hat{F}_i$ (the F-spin matrices) the title "Spin" isn't very good here since this has nothing to do with intrinsic angular momentum! It wasn't difficult to prove $\{\hat{\lambda}_i\}$ was a basis of $su(3)$ but for further discussion much like the $su(2)$ case it's desirable to change to the "spherical rep. of the $\hat{F}$ operators"

$$\boxed{\begin{array}{lll} \hat{T}_\pm = \hat{F}_1 \pm i\hat{F}_2 & \hat{T}_3 = \hat{F}_3 & \\ \hat{V}_\pm = \hat{F}_4 \pm i\hat{F}_5 & \hat{Y} = \frac{2}{\sqrt{3}}\hat{F}_8 , & \hat{U}_\pm = \hat{F}_6 \pm i\hat{F}_7 \end{array}}$$

This new set of generators $\{\hat{T}_\pm, T_3, \hat{V}_\pm, \hat{Y}, \hat{U}_\pm\}$ has the following commutation relations (not surprising if you see how they relate to the Pauli-matrices implicit inside the $\hat{\lambda}_i$'s)

$$\boxed{\begin{array}{l}
[\hat{T}_3, \hat{T}_\pm] = \pm\hat{T}_\pm , \quad [\hat{T}_+, \hat{T}_-] = 2\hat{T}_3 \\
[\hat{T}_3, \hat{U}_\pm] = \mp\frac{1}{2}\hat{U}_\pm , \quad [\hat{U}_+, \hat{U}_-] = \frac{3}{2}\hat{Y} - \hat{T}_3 \overset{\text{def}}{\equiv} 2\hat{U}_3 \\
[\hat{T}_3, \hat{V}_\pm] = \pm\frac{1}{2}\hat{V}_\pm , \quad [\hat{V}_+, \hat{V}_-] = \frac{3}{2}\hat{Y} + \hat{T}_3 \overset{\text{def}}{\equiv} 2\hat{V}_3 \\
[\hat{Y}, \hat{T}_\pm] = 0 , \quad [Y, U_\pm] = \pm U_\pm , \quad [Y, V_\pm] = \pm V_\pm \\
[T_+, V_+] = [T_+, U_-] = [U_+, V_+] = 0 \\
[T_+, V_-] = -U_- \quad [T_+, U_+] = V_+ \\
[U_+, V_-] = T_- \quad [T_3, Y] = 0
\end{array}}$$

We also know from hermiticity of $\hat{F}_i$ that $\hat{T}_+ = (\hat{T}_-)^\dagger$, $U_+ = (U_-)^\dagger$ and $V_+ = (V_-)^\dagger$

The rank of SU(3) can be seen to be 2 because the max # of commuting generators is 2. (for example $[T_3, Y]$ or $[Y, T_\pm]$, ... ~~this is not an obvious statement~~) ~~the~~ ok. Casimirs are

rank 2 $\Longrightarrow$ 2 Casimirs.

$$\boxed{\begin{array}{l}
\hat{C}_1(\hat{F}_i) = \sum_{i=1}^{8} \hat{F}_i^2 = \frac{-2i}{3}\sum_{ijk} f_{ijk}\hat{F}_i\hat{F}_j\hat{F}_k \\
\hat{C}_2(\hat{F}_i) = \sum_{ijk} d_{ijk}\hat{F}_i\hat{F}_j\hat{F}_k
\end{array}}$$

how?

§7.3 : Symmetry of the coefficients d_{ijk}

Prove that $d_{ijl} \overset{?}{=} \frac{1}{4} Tr(\{\lambda_i, \lambda_j\} \lambda_l)$

$$= \frac{1}{4} Tr\left(\left(\frac{4}{3}\delta_{ij} \mathbb{1} + 2 d_{ijk} \hat{\lambda}_k\right)\hat{\lambda}_l\right)$$

$$= \frac{1}{4} Tr\left(\frac{4}{3}\delta_{ij}\hat{\lambda}_l + 2 d_{ijk}\hat{\lambda}_k \hat{\lambda}_l\right)$$

$$= \frac{2}{4} d_{ijk} \, Tr(\hat{\lambda}_k \hat{\lambda}_l)$$

$$= \frac{1}{2} d_{ijk} \cdot 2\delta_{kl}$$

$$= d_{ijl}$$

Now that $d_{ijk} = \frac{1}{4} Trace(\{\lambda_i, \lambda_j\}\lambda_k)$ is established why is it clear that d_{ijk} is symmetric in each pair of indices.

$$d_{ijk} = \frac{1}{4} Trace\left((\lambda_i\lambda_j + \lambda_j\lambda_i)\lambda_k\right)$$

$$= \frac{1}{4} Tr(\{\lambda_j, \lambda_i\}\lambda_k) = d_{jik}$$

$$= \frac{1}{4} Tr(\lambda_k\{\lambda_j, \lambda_i\})$$

$$= \frac{1}{4} Tr(\lambda_k\lambda_j\lambda_i + \lambda_k\lambda_i\lambda_j)$$

$$= \frac{1}{4} Tr(\lambda_k\lambda_j\lambda_i + \lambda_j\lambda_k\lambda_i)$$

$$= \frac{1}{4} Tr(\{\lambda_k, \lambda_j\}\lambda_i) = d_{kji}$$

Thus $\boxed{d_{ijk} = d_{jik}}$ and $\boxed{d_{ijk} = d_{kji}}$ hence ~~$d_{ijk} = d_{...}$~~

Likewise $d_{ijk} = d_{kji} = d_{jki} = d_{ikj}$ (follows from the other 2)

Antisymmetry of the Structure Constants f_{ijk}

Show that $f_{ijk} = \frac{1}{4i}\text{Trace}\left([\hat{\lambda}_i, \hat{\lambda}_j]\hat{\lambda}_k\right)$ where $[\hat{\lambda}_i, \hat{\lambda}_j] = 2i f_{ijk}\hat{\lambda}_k$

$$f_{ijk} \stackrel{?}{=} \frac{1}{4i}\text{Trace}\left([\hat{\lambda}_i, \hat{\lambda}_j]\hat{\lambda}_k\right)$$
$$= \frac{1}{4i}\text{Trace}\left(2i f_{ij\ell}\hat{\lambda}_\ell \hat{\lambda}_k\right)$$
$$= \frac{1}{2} f_{ij\ell}\,\underbrace{\text{Trace}\left(\hat{\lambda}_\ell\hat{\lambda}_k\right)}$$
$$= \frac{1}{2} f_{ij\ell}\cdot(2\delta_{\ell k})$$
$$= f_{ijk} \qquad \therefore \quad \boxed{f_{ijk} = \frac{1}{4i}\text{Trace}\left([\hat{\lambda}_i, \hat{\lambda}_j]\hat{\lambda}_k\right)}$$

We can then show that $f_{ijk} = -f_{jik} = -f_{ikj}$ by the antisymmetry of the bracket and the cyclicity of the trace

$$f_{ijk} = \frac{1}{4i}\text{Trace}\left([\hat{\lambda}_i, \hat{\lambda}_j]\hat{\lambda}_k\right)$$
$$= -\frac{1}{4i}\text{Trace}\left([\hat{\lambda}_j, \hat{\lambda}_i]\hat{\lambda}_k\right) = -f_{jik} \checkmark$$
$$= -\frac{1}{4i}\text{Trace}\left(\hat{\lambda}_j\hat{\lambda}_i\hat{\lambda}_k - \hat{\lambda}_i\hat{\lambda}_j\hat{\lambda}_k\right)$$
$$= -\frac{1}{4i}\text{Trace}\left(\hat{\lambda}_i\hat{\lambda}_k\hat{\lambda}_j - \hat{\lambda}_k\hat{\lambda}_i\hat{\lambda}_j\right)$$
$$= -\frac{1}{4i}\text{Trace}\left([\hat{\lambda}_i, \hat{\lambda}_k]\hat{\lambda}_j\right) = -f_{ikj} \checkmark$$

Exercise 7.6 : Relations between the Structure Constants and the symmetric coeff. d_{ijk}

Show that the following identities hold,

$$\boxed{\begin{aligned} f_{plm} f_{mkq} + f_{klm} f_{mqp} + f_{pkm} f_{mql} &= 0 \\ f_{pkm} d_{mlq} + f_{qkm} d_{mlp} + f_{lkm} d_{mpq} &= 0 \end{aligned}}$$

The first is the Jacobi identity in disguise,

$$[[\hat\lambda_p, \hat\lambda_l], \hat\lambda_k] + [[\lambda_l, \lambda_k], \lambda_p] + [[\lambda_k, \lambda_p], \lambda_l] = 0$$

$$2i f_{plm} [\hat\lambda_m, \hat\lambda_k] + 2i f_{lkm} [\lambda_m, \lambda_p] + 2i f_{kpm} [\lambda_m, \lambda_l] = 0$$

$$(2i)^2 \left[f_{plm} f_{mkn} + f_{lkm} f_{mpn} + f_{kpm} f_{mln} \right] \hat\lambda_n = 0$$

$\therefore \; f_{plm} f_{mkn} + f_{lkm} f_{mpn} + f_{kpm} f_{mln} = 0$ because $\hat\lambda_n$ is a generator.

$\Rightarrow \; f_{plm} f_{mkq} + f_{lkm} f_{mpq} + f_{kpm} f_{mlq} = 0$ changing q for n.

$\Rightarrow \; \boxed{f_{plm} f_{mkq} + f_{klm} f_{mqp} + f_{pkm} f_{mql} = 0}$ using antisymmetry twice, twice.

Next consider that

$$\begin{aligned} \mathrm{Trace}\left([\hat\lambda_a, \hat\lambda_b] \{\hat\lambda_c, \hat\lambda_d\} \right) &= 2i f_{abe} \, \mathrm{Trace}\left(\hat\lambda_e \{ \hat\lambda_c, \hat\lambda_d \} \right) \\ &= 2i f_{abe} \, \mathrm{Trace}\left(\hat\lambda_e \left(\tfrac{4}{3} \delta_{cd} \mathbb{1} + 2 d_{cdm} \hat\lambda_m \right) \right) \\ &= 4i f_{abe} d_{cdm} \, \mathrm{Trace}\left(\hat\lambda_e \hat\lambda_m \right) \\ &= \boxed{8i f_{abm} d_{cdm} = \mathrm{Trace}\left([\hat\lambda_a, \hat\lambda_b] \{\hat\lambda_c, \hat\lambda_d\} \right)} \end{aligned}$$

Thus we can check the 2nd identity thru an explicit calculation which works out due to linearity and cyclicity of trace most likely.

$$\begin{aligned} 8i \left(f_{pkm} d_{mlq} + f_{qkm} d_{mlp} + f_{lkm} d_{mpq} \right) &= \\ = \mathrm{Trace}\Big([\lambda_p, \lambda_k]\{\lambda_l, \lambda_q\} &+ [\lambda_q, \lambda_k]\{\lambda_l, \lambda_p\} + [\lambda_l, \lambda_k]\{\lambda_p, \lambda_q\} \Big) \\ = \mathrm{Trace}\Big(& \overset{1}{\cancel{\lambda_{pklq}}} + \overset{2}{\cancel{\lambda_{pkql}}} - \overset{3}{\cancel{\lambda_{kplq}}} - \overset{4}{\cancel{\lambda_{kpql}}} \\ & + \overset{5}{\cancel{\lambda_{qklp}}} + \overset{3}{\cancel{\lambda_{qkpl}}} - \overset{2}{\cancel{\lambda_{kqlp}}} - \overset{6}{\cancel{\lambda_{kqpl}}} \\ & + \overset{4}{\cancel{\lambda_{lk}\lambda_{pq}}} + \overset{6}{\cancel{\lambda_{lk}\lambda_{qp}}} - \overset{5}{\cancel{\lambda_{klpq}}} - \overset{1}{\cancel{\lambda_{klqp}}} \Big) \end{aligned}$$

$\leftarrow$ shorthand with obvious meaning. cancellations in view of cyclicity of trace!

Exercise 7.7 Casimir Operators of SU(3)

$\boxed{\text{Verify that } \hat{C}_1 \equiv \hat{F}_i \hat{F}_i \text{ and } \hat{C}_2 = d_{ijk} \hat{F}_i \hat{F}_j \hat{F}_k \text{ are Casimirs.}}$

Recall that $[\hat{F}_i, \hat{F}_j] = i f_{ijk} \hat{F}_k$ these are the F-spin generators from before. Consider how to prove $\hat{C}_1$ commutes with whole algebra, that is each generator

$$
\begin{aligned}
[\hat{C}_1, \hat{F}_k] &= [\hat{F}_i \hat{F}_i, \hat{F}_k] \\
&= \hat{F}_i [\hat{F}_i, F_k] + [\hat{F}_i, \hat{F}_k] \hat{F}_i \\
&= i f_{ikm} \hat{F}_i \hat{F}_m + i f_{ikm} \hat{F}_m \hat{F}_i \\
&= i f_{ikm} \underbrace{(\hat{F}_i \hat{F}_m + \hat{F}_m \hat{F}_i)}_{\text{symmetric}} \quad \text{whereas } f_{ikm} = -f_{mki}. \\
&= 0
\end{aligned}
$$

A similar calculation ought to prove $\hat{C}_2$ is a Casimir,

$$
\begin{aligned}
[\hat{C}_2, \hat{F}_k] &= d_{lmn} [\hat{F}_l \hat{F}_m \hat{F}_n, \hat{F}_k] \\
&= d_{lmn} \left(F_l F_m [F_n, F_k] + [F_n, F_k] F_l F_m \right) \\
&= d_{lmn} \left(F_l F_m \, i f_{nkq} \hat{F}_q + i f_{nkq} F_q F_l F_m \right) \\
&= i \left(d_{lmn} f_{nkq} + d_{mqn} f_{nkl} \right) F_l F_m F_q \\
&= \cancel{i (d_{lmn} f_{nkq} - d_{lqn} f_{nkm} - d} \\
&= i \left(f_{nkq} d_{lmn} + f_{nkl} d_{mqn} \right) F_l F_m F_q \\
&= \cancel{i (f_{nkq} d_{lmn} - f} \\
&= i \left(f_{kqn} d_{nlm} + f_{kln} d_{nmq} \right) F_l F_m F_q \\
&= i \left(f_{kqn} d_{nlm} - f_{qln} d_{nmk} - f_{mln} d_{nkq} \right) F_l F_m F_q \quad \text{using (97)'s identity} \\
&= i \left(-f_{mqn} d_{nlk} - f_{lqn} d_{nkm} - f_{qln} d_{nmk} - f_{mln} d_{nkq} \right) F_l F_m F_q \\
&=
\end{aligned}
$$

See pg. 197 it is cleaner.

Exercise 7.7 : Show $\hat{C}_2 = d_{ijk} \hat{F}_i \hat{F}_j \hat{F}_k$ is a casimir of SU(3)

The identity on 97 of GREINER is not obvious to me, hmm...

$$[\hat{C}_2, \hat{F}_k] = [d_{ijn} F_i F_j F_n , F_k]$$
$$= d_{ijn} [F_i F_j F_n , F_k]$$
$$= d_{ijn}(F_i F_j F_n F_k - F_k F_i F_j F_n)$$
$$= d_{ijn}(F_i F_j F_n F_k - F_i F_j F_k F_n + F_i F_j F_k F_n - F_k F_i F_j F_n)$$
$$= d_{ijn}(F_i F_j [F_n, F_k] + F_i F_j F_k F_n - F_i F_k F_j F_n + F_i F_k F_j F_n - F_k F_i F_j F_n)$$
$$= d_{ijn}(F_i F_j [F_n, F_k] + F_i [F_j, F_k] F_n + [F_i, F_k] F_j F_n)$$
$$= d_{ijn}(F_i F_j i f_{nkm} F_m + F_i i f_{jkm} F_m F_n + i f_{ikm} F_m F_j F_n)$$
$$= i d_{ijn}(f_{nkm} F_i F_j F_m + f_{jkm} F_i F_m F_n + f_{ikm} F_m F_j F_n)$$
$$= i F_i F_j F_m (d_{ijn} f_{nkm}) + i d_{ijn} f_{jkm} F_i F_m F_n + i d_{ijn} f_{ikm} F_m F_j F_n$$
$$= i F_i F_j F_m d_{ijn} f_{nkm} + i F_i F_j F_m d_{inm} f_{nkj} + i F_i F_j F_m d_{njm} f_{nki}$$
$$= i F_i F_j F_m (d_{ijn} f_{nkm} + d_{inm} f_{nkj} + d_{njm} f_{nki})$$
$$= i F_i F_j F_m (f_{nkm} d_{ijn} + f_{nkj} d_{inm} + f_{nki} d_{njm})$$
$$= i F_i F_j F_m (f_{kmn} d_{nij} + f_{kjn} d_{nmi} + f_{kin} d_{njm})$$
$$= -i F_i F_j F_m (f_{mkn} d_{nij} + f_{jkn} d_{nmi} + f_{ikn} d_{njm})$$
$$= -i F_i F_j F_m (f_{mkn} d_{nij} + f_{jkn} d_{nim} + f_{ikn} d_{nmj})$$
$$= 0$$

Same index pattern as the identity on 97.

$$\Rightarrow \boxed{\hat{C}_2 \text{ commutes with } SU(3)}$$

Exercise 7.8: Some useful Casimir Identities for SU(3)

Our goal is to show that $\hat{C}_1 = -\frac{2i}{3}\hat{F}_i\hat{F}_j\hat{F}_k f_{ijk}$ and $\hat{C}_2 = \hat{C}_1(2\hat{C}_1 - \frac{11}{6})$

We know that $[F_i, F_j] = if_{ijk}F_k$ while $\{F_i, F_j\} = \frac{1}{3}\delta_{ij}\mathbb{1} + d_{ijk}F_k$

Which follows from (2) of Ex. 7.3 replacing $\hat{F} = \frac{1}{2}\hat{\lambda}$,

$$[F_i, F_j] + \{F_i, F_j\} = 2F_iF_j = if_{ijk}F_k + \frac{1}{3}\delta_{ij}\mathbb{1} + d_{ijk}F_k$$

$$\therefore \boxed{F_iF_j = \frac{i}{2}f_{ijk}F_k + \frac{1}{6}\delta_{ij}\mathbb{1} + \frac{1}{2}d_{ijk}F_k}$$

Lets try to use this to calculate $\hat{C}_1$

$$\hat{C}_1 = F_\ell F_\ell$$

$$= \frac{i}{2}f_{\ell\ell k}F_k + \frac{1}{6}\delta_{\ell\ell}\mathbb{1} + \frac{1}{2}d_{\ell\ell k}F_k$$

$$= \frac{1}{2}\left(\mathbb{1} + d_{\ell\ell k}F_k\right)$$

Curious but not to the point of this exercise lets follow GREINER,

He begins by recalling $[F_i, F_j] = if_{ijk}F_k$ and claims $\boxed{f_{ijk}f_{ij\ell} = 3\delta_{k\ell}}$

$$f_{ijk}F_iF_jF_k = f_{ijk}\left([F_i, F_j] + F_jF_i\right)F_k$$

$$= f_{ijk}\left(if_{ij\ell}F_\ell + F_jF_i\right)F_k$$

$$= f_{ijk}F_jF_iF_k + if_{ijk}f_{ij\ell}F_\ell F_k$$

$$= -f_{ijk}F_iF_jF_k + 3i\delta_{k\ell}F_\ell F_k$$

$$\Rightarrow \frac{2}{3i}f_{ijk}F_iF_jF_k = F_\ell F_\ell = \boxed{\hat{C}_1 = \frac{-2i}{3}f_{ijk}F_iF_jF_k}$$

Exercise 7.8 Continued, Show $\hat{C}_2 = \hat{C}_1 (2\hat{C}_1 - \frac{11}{6})$

As we began $\{F_i, F_j\} = \frac{1}{3}\delta_{ij} \mathbb{1} + d_{ijk} F_k$

$$\Rightarrow d_{ijk} F_k = \{F_i, F_j\} - \frac{1}{3}\delta_{ij}\mathbb{1}$$

Now recall the definition of $\hat{C}_2$

$$
\begin{aligned}
\hat{C}_2 &= d_{ijk} F_i F_j F_k \\
&= F_i F_j \, d_{ijk} F_k \\
&= F_i F_j \left(\{F_i, F_j\} - \frac{1}{3}\delta_{ij}\mathbb{1} \right) \\
&= F_i F_j F_i F_j + F_i F_j F_j F_i - \frac{1}{3}\delta_{ij} F_i F_j \\
&= F_i (F_i F_j + [F_j, F_i]) F_j + F_i F_i F_j F_j - \frac{1}{3}\hat{C}_1 \\
&= 2 F_i F_i F_j F_j + F_i [F_j, F_i] F_j - \frac{1}{3}\hat{C}_1 \\
&= 2\hat{C}_1^2 - \frac{1}{3}\hat{C}_1 + i f_{jik} F_i F_k F_j \\
&= 2\hat{C}_1^2 - \frac{1}{3}\hat{C}_1 + i f_{kij} F_i F_j F_k \\
&= 2\hat{C}_1^2 - \frac{1}{3}\hat{C}_1 + i f_{ijk} F_i F_j F_k \\
&= 2\hat{C}_1^2 - \frac{1}{3}\hat{C}_1 - \frac{3}{2}\hat{C}_1 \qquad \left(\text{using } C_1 = \frac{-2i}{3} f_{ijk} F_i F_j F_k\right) \\
&= \boxed{\hat{C}_1 \left(2\hat{C}_1 - \frac{11}{6}\right) = \hat{C}_2} \qquad \frac{1}{3} + \frac{3}{2} = \frac{2}{6} + \frac{9}{6} = \frac{11}{6}
\end{aligned}
$$

(?)

§7.4 : The Subalgebras of the SU(3) – Lie Algebra and the Associated Shift Operators.

It is evident from the algebra of $SU(3)$ given on (94) that there are several $SU(2)$ subalgebras in $SU(3)$. Explicitly note that

$$\boxed{\begin{aligned} [\hat{T}_+ , \hat{T}_-] &= 2\hat{T}_3 \quad : \text{where } \hat{T}_\pm \equiv \hat{F}_1 \pm i\hat{F}_2 \\ [\hat{T}_3 , \hat{T}_\pm] &= \pm \hat{T}_\pm \qquad \hat{F}_i = \tfrac{1}{2}\hat{\lambda}_i \leftarrow \text{Gell-Mann Matrices.} \end{aligned}}$$

Clearly $\{T_1, T_2, T_3\}$ form a closed subalgebra. Likewise for

$$\boxed{\begin{aligned} [\hat{U}_+ , \hat{U}_-] &= 2\hat{U}_3 \quad : \text{where } \hat{U}_\pm \equiv \hat{F}_6 \pm i\hat{F}_7 \\ [\hat{U}_3 , \hat{U}_\pm] &= \pm \hat{U}_\pm \qquad \hat{F}_i = \tfrac{1}{2}\hat{\lambda}_i \text{ as discussed on (94).} \end{aligned}}$$

The last commutator follows from $\hat{U}_3 = \frac{1}{2}\left(\frac{3}{2}\hat{Y} - \hat{T}_3\right)$ and the known commutators from (94), $[\hat{T}_3, \hat{U}_\pm] = \mp\frac{1}{2}U_\pm$ and $[\hat{Y}, \hat{U}_\pm] = \pm U_\pm$ hence

$$\begin{aligned} [\hat{U}_3 , \hat{U}_\pm] &= \left[\tfrac{3}{4}\hat{Y} - \tfrac{1}{2}\hat{T}_3 , \hat{U}_\pm\right] \\ &= \tfrac{3}{4}[\hat{Y}, \hat{U}_\pm] - \tfrac{1}{2}[\hat{T}_3, \hat{U}_\pm] \\ &= \pm\tfrac{3}{4}\hat{U}_\pm - \tfrac{1}{2}\left(\mp\tfrac{1}{2}\hat{U}_\pm\right) \\ &= \pm\left(\tfrac{3}{4} + \tfrac{1}{4}\right)\hat{U}_\pm = \boxed{\pm\hat{U}_\pm = [\hat{U}_3, \hat{U}_\pm]} \end{aligned}$$

Thru similar arguments based on $\hat{V}_3 = \frac{1}{2}\left(\frac{3}{2}\hat{Y} + \hat{T}_3\right)$ and associated facts from (94),

$$\boxed{\begin{aligned} [\hat{V}_+ , \hat{V}_-] &= 2\hat{V}_3 \quad \text{where } \hat{V}_\pm \equiv \hat{F}_4 \pm i\hat{F}_5 \\ [\hat{V}_3 , \hat{V}_\pm] &= \pm \hat{V}_\pm \qquad \text{again } \hat{F}_i \text{ are the "F-spin" matrices, rescaled Gell-Mann's.} \end{aligned}}$$

We see that we have 3 copies of the angular momentum algebra inside $SU(3)$. Next we'll further explore what exactly is raised and lowered by these shift operators...

$$\left.\begin{aligned} &T - \text{spin algebra} \\ &U - \text{spin algebra} \\ &V - \text{spin algebra} \end{aligned}\right\} \approx SU(2)$$

§7.4: T, U and V - spin subalgebras and their shift operators.

In view of 94. and 102. what quantum #'s lable the SU(3) states? Notice

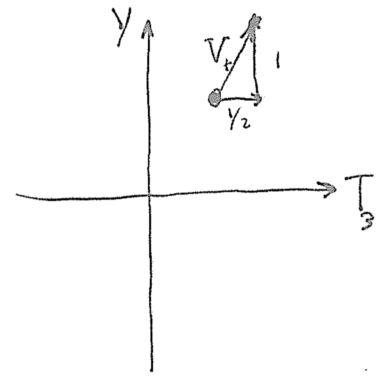
$$\boxed{[\hat{Y}, \hat{T}_3] = 0}$$

Hence $\hat{Y}$ and $\hat{T}_3$ can be simultaneously diagonalized. Denote these common eigenstates by $|T_3, Y\rangle$ with the obvious meaning,

$$\boxed{\begin{aligned} \hat{T}_3 |T_3, Y\rangle &= T_3 |T_3, Y\rangle \\ \hat{Y} |T_3, Y\rangle &= Y |T_3, Y\rangle \end{aligned}}$$

Recall that $[\hat{T}_3, \hat{V}_\pm] = \pm \frac{1}{2} V_\pm$ which clearly means that

$$(\hat{T}_3 \hat{V}_\pm - \hat{V}_\pm \hat{T}_3) |T_3, Y\rangle = \pm \frac{1}{2} V_\pm |T_3 Y\rangle$$



$$\hat{T}_3 \hat{V}_\pm |T_3 Y\rangle - \hat{V}_\pm \hat{T}_3 |T_3 Y\rangle = \pm \frac{1}{2} V_\pm |T_3 Y\rangle$$

$$\Rightarrow \hat{T}_3 \hat{V}_\pm |T_3 Y\rangle - \hat{V}_\pm T_3 |T_3 Y\rangle = \pm \frac{1}{2} V_\pm |T_3 Y\rangle \qquad \text{Using def}^{\text{n}} \text{ of states.}$$

$$\Rightarrow \hat{T}_3 (V_\pm |T_3 Y\rangle) = (T_3 \pm \tfrac{1}{2}) (V_\pm |T_3 Y\rangle$$

$\Rightarrow$ <u>the state $V_\pm |T_3 Y\rangle$ has eigenvalue $T_3 \pm \frac{1}{2}$ for $\hat{T}_3$.</u>

Consequently it must be formed by a linear combination of such states,

$$\boxed{\hat{V}_\pm |T_3 Y\rangle = \sum_{Y'} N(T_3, Y, Y') \, |T_3 \pm \tfrac{1}{2}, Y'\rangle}$$

§7.4 : The states $|T_3, Y\rangle$ indexed by hypercharge Y and isospin's 3rd component T_3 and how they are acted on by T, U, V shift operators

- Now recall that $[\hat{T}_3, \hat{U}_\pm] = \mp \frac{1}{2}\hat{U}_\pm$ and consider that

$$\left(\hat{T}_3 \hat{U}_\pm - \hat{U}_\pm \hat{T}_3\right)|T_3 Y\rangle = \mp \tfrac{1}{2}\hat{U}_\pm |T_3 Y\rangle$$

$$\Rightarrow \hat{T}_3\left(\hat{U}_\pm |T_3 Y\rangle\right) = \left(T_3 \mp \tfrac{1}{2}\right)\left(\hat{U}_\pm |T_3 Y\rangle\right)$$

Hence $\boxed{\hat{U}_\pm |T_3 Y\rangle \text{ has } \hat{T}_3 \text{ eigenvalue } T_3 \mp \tfrac{1}{2}}$

- Next consider that $\hat{V}_i$ and $\hat{Y}$ are related by $[\hat{Y}, V_\pm] = \hat{Y}$ which will mean that $V_\pm$ raises/lowers Y by one unit see how,

$$\left(\hat{Y}\hat{V}_\pm - \hat{V}_\pm \hat{Y}\right)|T_3 Y\rangle = \pm \hat{V}_\pm |T_3 Y\rangle$$

$$\hat{Y}\left(\hat{V}_\pm |T_3 Y\rangle\right) = V_\pm \hat{Y}|T_3 Y\rangle \pm \hat{V}_\pm |T_3 Y\rangle$$

$$= (Y \pm 1)\left(\hat{V}_\pm |T_3 Y\rangle\right) = \hat{Y}\left(V_\pm |T_3 Y\rangle\right)$$

$$\therefore \boxed{\hat{V}_\pm |T_3 Y\rangle \text{ has } \hat{Y} \text{ eigenvalue } Y \pm 1}$$

- Continuing lets see how $U_\pm$ changes the $\hat{Y}$ eigenvalue. We'll find that $[\hat{Y}, \hat{U}_\pm] = \pm \hat{U}_\pm \Rightarrow \hat{Y}$ eigenvalue changes by ± 1 like the $\hat{V}_\pm$ case, for the same reasons as above

$$\hat{Y}\left(\hat{U}_\pm |T_3 Y\rangle\right) = (Y \pm 1)\left(\hat{U}_\pm |T_3 Y\rangle\right)$$

$$\Rightarrow \boxed{\hat{U}_\pm |T_3 Y\rangle \text{ has } \hat{Y} \text{ eigenvalue } Y \pm 1}$$

<u>Remark</u>: $[T_\pm, \hat{Y}] = 0$ hence $\hat{T}_\pm$ do not change the quantum # Y (compared to $U_\pm$ and $V_\pm$ which do.)

§7.4 : Graphical Representation of the Shift operators in the $(T_3\,Y)$ - plane.

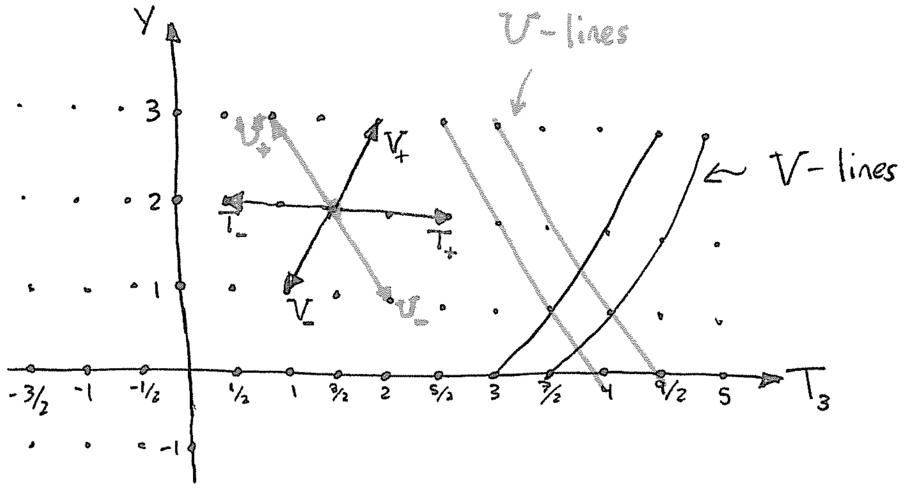
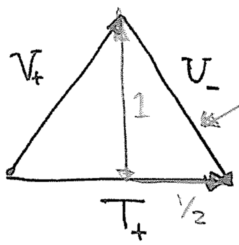


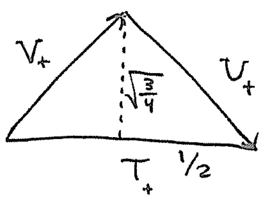
figure 7.1
pg. 200
GREINER



hypoteneuse has length:

$$\sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

If we do not scale the Y-axis.



hypoteneuse has length

$$\sqrt{\left(\sqrt{\frac{3}{4}}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

If we scale units of Y by $\sqrt{\frac{3}{4}}$ (relative to T_3 units)

- We rescale Y the hypercharge by $\sqrt{\frac{3}{4}}$ so that the U and V lines will form equilateral triangles in the $(T_3\,Y)$ - plane.

§7.5: Coupling the T, U, V Multiplets, a convenient method to describe SU(3) multiplets

1.) The SU(3) algebra has T, U and V subalgebras (isomorphic to SU(2) ang. mom.) and consequently SU(3) multiplets can be formed by coupling T, U and V multiplets together.

2.) SU(3) has rank 2 and so 2 quantum #'s label its states(?) The operators $\hat{T}_3$, $\hat{Y}$ are simultaneously diagonalizable as $[\hat{T}_3, \hat{Y}] = 0$. Likewise the linear combinations $\hat{U}_3 = \frac{1}{2}(\frac{3}{2}\hat{Y} - \hat{T}_3)$ and $\hat{V}_3 = \frac{1}{2}(\frac{3}{2}\hat{Y} + \hat{T}_3)$ are also diagonalized concurrently with $\hat{Y}$ and $\hat{T}_3$. Eigenvalues are

$$T_3 \quad , \quad Y \quad , \quad \tfrac{1}{2}(\tfrac{3}{2}Y - T_3) \quad , \quad \tfrac{1}{2}(\tfrac{3}{2}Y + T_3)$$

for the state $|T_3, Y\rangle$ of the obvious operators $\hat{T}_3$, $\hat{Y}$, exc... .

3.) The $U_\pm$, $V_\pm$ and $T_\pm$ shift operators act on the state $|T_3 Y\rangle$ illustrated by a point (T_3, Y) in figure 7.1 on 105. The endpoints form a regular hexagon.

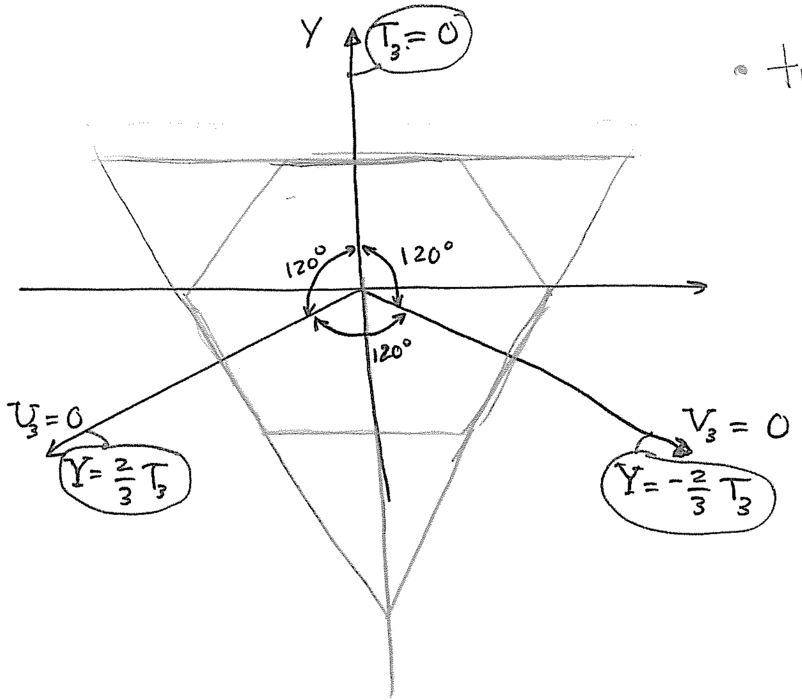
4.) T-multiplets are paralell to T_3-axis while U and V multiplets follow the U and V lines. The SU(3) multiplets are formed thru coupling these which must happen since $[\hat{T}_+, \hat{V}_-] = -\hat{U}_-$ and $[\hat{T}_+, \hat{U}_+] = -\hat{V}_+$ which means SU(3) mixes up the sub-multiplets of T, U and V.

5.) This mixing is done in a very regular way due to the equivalence of the T, U and V subalgebras. Specifically <u>the SU(3) multiplets must form regular triangles or hexagons in the $(T_3 Y)$-plane</u>.

One can infer symmetry accross Y-axis due to the fact a T-multiplet must take values $T_{3(max)} \geq T_3 \geq -T_{3(max)}$ (whatever the Y-value for that multiplet) this means that T-multiplets must form a right angle with Y-axis. T-lines are indeed horizontal.

6.) Because T, U and V are equivalent symmetry w.r.t. Y-axis aka $T_3 = 0$ axis $\Rightarrow$ symmetry of SU(3) multiplet w.r.t. $U_3 = 0$ and the $V_3 = 0$ axis. These are $0 = \frac{3}{2}Y - T_3$ and $0 = \frac{3}{2}Y + T_3$ which we'll illustrate by fig. 7.4 on 107 or (201 in Greiner)

§ 7.5 Coupling T, U and V multiplets to form SU(3) multiplets

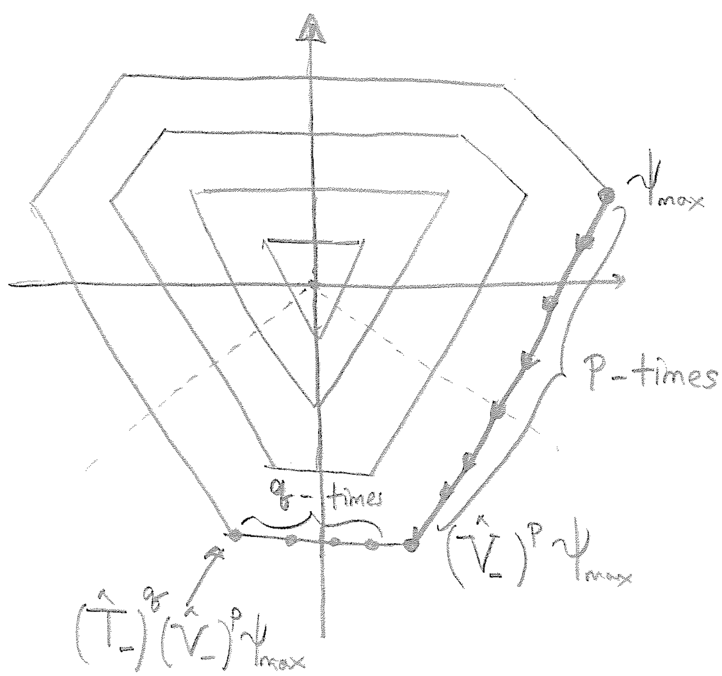


- triangles and hexagons respect the T-U-V exchange symmetry.

- Units of Y scaled by $\sqrt{\frac{3}{4}}$ as discussed before.

7.) The origin ($Y=0$, $T_3=0$) is at the center of each SU(3) multiplet. Each SU(3) multiplet is centered at the origin of (T_3 Y)-plane and is ~~symmetric~~ invariant w.r.t. rotations of ±120° around the origin.

Boundary of SU(3) Multiplets: meaning of "p" and "q"



The numbers p and q are given to describe a particular SU(3) multiplet. If the multiplet is finite there must be a unique state with $T_{3max} \Rightarrow \psi_{max}$. Then p and q are given by

$$(V_-)^{p+1}\psi_{max} = 0$$

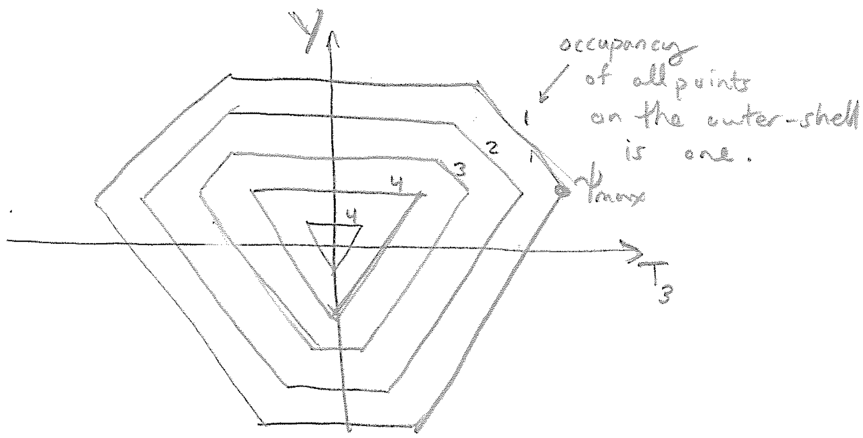
$$(\hat{T}_-)^{q+1}(V_-)^p\psi_{max} = 0$$

The lowest p and q that accomplish these vanishings.

- Read pg. 202 – 204 for why all the inner-states must be occupied and the shape of the multiplet is convex. It seems kinda inescapable given the T-U-V exchange symmetry and the existence of T_{3max} state.

§ 7.8 : Multiplicities of interior points on SU(3) multiplets

The outer boundary is occupied just once. However as we go into the hexagon each shell will increase occupancy by one. We can go q - steps then that shell will be a triangle in which all the points will be $(q+1)$ - occupied. We illustrate with the $(p, q) = (7, 3)$ multiplet



- You can read 205-206 for an argument in terms of some combinatorial counting of paths inward. Apparently Ex 7.9 will explain the multiplicities as well. We'll endeavor to understand 7.9 in detail.

Example 7.9

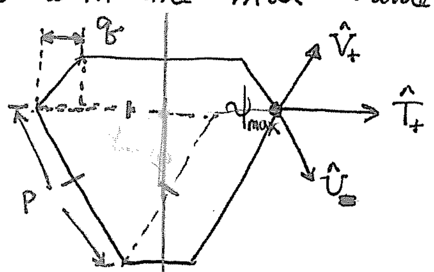
Maybe later. lets move ons to Ex. 7.10 for now, well lets skip ahead to 7.12.

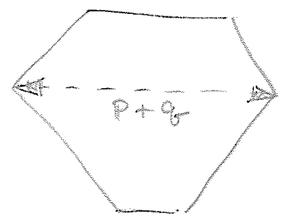
Exercise 7.12: Finding the quadratic Casimir for the $D(p,q)$ rep. of $SU(3)$

Recall that $\hat{C}_1 = \sum_{i=1}^{8} \hat{F}_i^2$. Recall that one can construct the squared generators from products of the shifts, $T_1^2 + T_2^2 = \frac{1}{2}(T_+T_- + T_-T_+)$ for example. Then,

$$\boxed{\hat{C}_1 = \sum_{i=1}^{8} (\hat{F}_i)^2 = \frac{1}{2}(\hat{T}_+\hat{T}_- + \hat{T}_-\hat{T}_+) + \hat{T}_3^2 + \frac{1}{2}(\hat{V}_+\hat{V}_- + \hat{V}_-\hat{V}_+) + \frac{1}{2}(\hat{U}_+\hat{U}_- + \hat{U}_-\hat{U}_+) + \frac{3}{4}\hat{Y}^2}$$

By the Th$^{\underline{m}}$ of Racah and related discussion we know that for the $D(p,q)$ multiplet the casimir $\hat{C}_1$ has the same eigenvalue througout the multiplet. A convenient state to begin our evaluation is ψ_{max} the unique state with the max value of T_3.





$p+q = 2T_{3max}$

$\Rightarrow \boxed{T_{3max} = \frac{p+q}{2}}$

$\left(\hat{T}_3\psi_{max} = \frac{p+q}{2}\psi_{max}\right)$

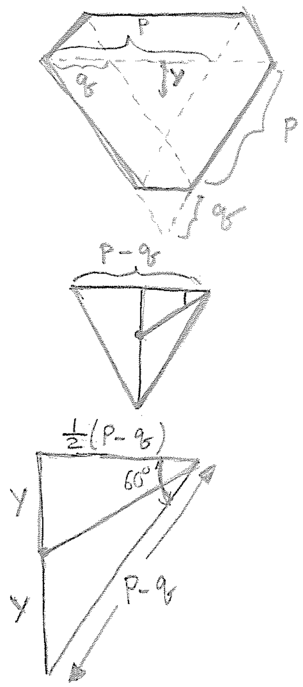
We know that these operators vanish on ψ_{max}. We also know how to trade related shifts via the relations,

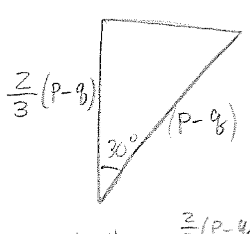
$$\boxed{\begin{aligned} T_+T_- &= T_-T_+ + 2T_3 \\ V_+V_- &= V_-V_+ + \tfrac{3}{2}Y + T_3 \\ U_-U_+ &= U_+U_- - \tfrac{3}{2}Y + T_3 \end{aligned}}$$

Now substitute these into our expression for $\hat{C}_1$ above,

$$\hat{C}_1 = \frac{1}{2}\left(2\hat{T}_-\hat{T}_+ + 2\hat{T}_3\right) + \hat{T}_3^2 + \frac{1}{2}\left(2\hat{V}_-\hat{V}_+ + \frac{3}{2}\hat{Y} + \hat{T}_3\right) + \frac{1}{2}\left(2\hat{U}_+\hat{U}_- - \frac{3}{2}\hat{Y} + \hat{T}_3\right) + \frac{3}{4}\hat{Y}^2$$

We just need to find the eigenvalue of $\hat{Y}$ for ψ_{max}. Well I don't see it at the moment but $Y = \frac{p-q}{3}$





$\cos(30°) = \frac{\frac{2}{3}(p-q)}{p-q} = \frac{2}{3} \neq \frac{\sqrt{3}}{2}$

$\cos(30°) = \frac{\sqrt{\frac{3}{4}}\,\frac{2}{3}(p-q)}{p-q} = \frac{\sqrt{3}}{2}$ ✓

guess its the scale-factor obscuring things.

Finally then we can calculate

$$\begin{aligned} \hat{C}_1\psi_{max} &= \\ &= 2T_{3max} + T_{3max}^2 + \frac{3}{4}Y^2 \\ &= \frac{p+q}{2} + \left(\frac{p+q}{2}\right)^2 + \frac{3}{4}\left(\frac{p-q}{3}\right)^2 \\ &= \boxed{\frac{p^2+pq+q^2}{3} + p + q = C_1} \end{aligned}$$