CHAPTER 8 : QUARKS AND SU(3)

In view of the clear resemblance of SU(3) multiplets discussed in chapter 7 and the emperically discovered multiplets graphed in the $(T_3 Y)$-plane in chapter 6 it is fairly clear that SU(3) must be behind the patterns noted in chapter 6. Hence we identify that $\hat{T}_3$ and $\hat{Y}$ of chap. 7 are truly the same as the Isospin and Hypercharge. Indeed,

$$\boxed{\hat{Q} = \frac{1}{2}\hat{Y} + \hat{T}_3}$$

operators

* not true.
($\hat{Y}$ and $\hat{T}_3$ are Casimirs)

SU(3) is rank 2 and T_3, Y are suitable quantum #'s to label the multiplets. Of course for physical reasons there are extra quantities besides the SU(3) physics, we group these all-together as "α"

$$\boxed{|T_3 Y, \alpha\rangle}$$ ← typical SU(3) state, an eigenstate of both $\hat{T}_3$ and $\hat{Y}$.

This state ~~is~~ is labeled by it's $\hat{T}_3$ and $\hat{Y}$ eigenvalues,

$$\boxed{\begin{aligned} \hat{T}_3 |T_3 Y, \alpha\rangle &= T_3 |T_3 Y, \alpha\rangle \\ \hat{Y} |T_3 Y, \alpha\rangle &= Y |T_3 Y, \alpha\rangle \end{aligned}}$$

* **Remark:** Racah's Th^{m} said we could classify multiplets by a set of eigenvalues, the eigenvalues of the Casimirs. Here T_3 and Y are not Casimirs so what's the deal? I think that given (T_3, Y) we can find (C_1, C_2). That is the eigenvalues of $\hat{T}_3$ and $\hat{Y}$ uniquely correspond to the pair of e.v.'s of $\hat{C}_1$ and $\hat{C}_2$ which we know label the multiplets by Racah's Th^{m}.

The eigenvalues of $\hat{Q}$ gives the charge of the state, (Electric)

$$\begin{aligned} \hat{Q}\,|T_3 Y, \alpha\rangle &= \left(\tfrac{1}{2}\hat{Y} + \hat{T}_3\right)|T_3 Y, \alpha\rangle \\ &= \left(\tfrac{1}{2}Y + T_3\right)|T_3 Y, \alpha\rangle \\ &= Q\,|T_3 Y, \alpha\rangle \quad \Longrightarrow \quad \boxed{Q = \tfrac{1}{2}Y + T_3} \end{aligned}$$

eigenvalues

This means that "Q is a good quantum number". It makes sense to ask what is the charge of a SU(3) multiplet.

Quarks and SU(3) getting towards why charge is fractional

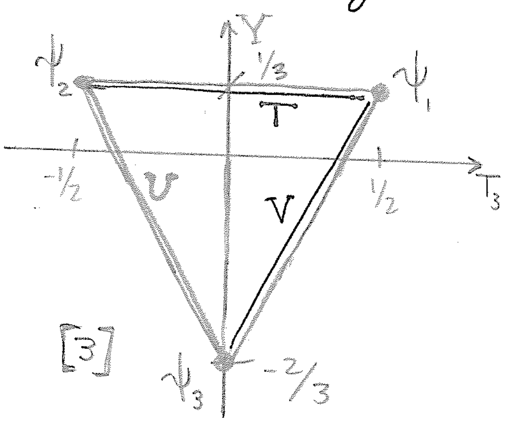
Notice the singlet $|00,\alpha\rangle$ has zero charge,

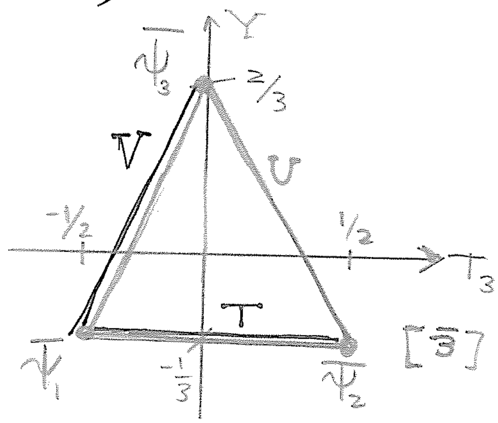
$$\hat{Q}\,|00,\alpha\rangle = \left(\tfrac{1}{2}(0)+0\right)|00,\alpha\rangle = 0\,|00,\alpha\rangle$$

The Λ^* hyperon with 1405 MeV (see 86 from Ex. 6.5) and spin $\frac{1}{2}$ and negative parity should be understood as such a singlet state. This is a <u>trivial</u> representation of SU(3). Recall that for su(2) the doublet $T=\frac{1}{2}$ was the smallest non-trivial representation, $T=0$ was trivial. All su(2) multiplets can be formed by coupling doublets thru the machine of Clebsch-Gordon.

The question then is what is the smallest non-trivial SU(3) representation? The T-U-V exchange symmetry $\Rightarrow$ it must contain 3-states at minimum, otherwise it wouldn't satisfy the $\pm 120°$ rotational invariance that T-U-V exchange requires. More than this <u>F-spin SU(3) has SU(2) subalgebras</u> $\Rightarrow$ the SU(3) rep. must contain SU(2) doublets, otherwise it couldn't replicate the rep. of the su(2) subalgebras. T-U-V $\Rightarrow$ must have a doublet for T, U and V

Hence the following is inescapable,





We have illustrated the fundamental SU(3) building-blocks $[3]$ & $[\bar{3}]$ and have noted the various T, U and V doublets within $[3]$ & $[\bar{3}]$.

$[3]$ are particles $\Rightarrow$ $[\bar{3}]$ are antiparticles

They have opposite charge because they have negative T_3 and Y of $[3]$.

~~In particular we denote~~

$$\hat{Q}\,\psi_i = Q_i\,\psi_i \quad \text{whereas} \quad \hat{Q}\,\bar{\psi}_i = -Q_i\,\psi_i$$

Fractional Charge and [3], [$\bar{3}$] the fundamental representations

The values of T_3 are known for the doublets in [3] and [$\bar{3}$]. Lets calculate the values of hypercharge Y. The T-isodoublet and singlet,

$$\boxed{\begin{array}{ll} \psi_1 \equiv |\tfrac{1}{2}\ Y\rangle & \text{SU(2) isodoublet} \\ \psi_2 \equiv |\tfrac{-1}{2}\ Y\rangle & \\ \psi_3 \equiv |0\ Y'\rangle \leftarrow & \text{SU(2) singlet} \end{array}}$$

We seek to determine Y and Y'. Notice ψ_1 is killed by U_3 because ψ_1 is a U-spin singlet see [3] on 111.

$$\hat{U}_3 \psi_1 = 0$$

$$\Rightarrow \tfrac{1}{4}(3\hat{Y} - 2\hat{T}_3)\psi_1 = 0$$

$$\Rightarrow \hat{Y}\psi_1 = \tfrac{2}{3}\hat{T}_3\psi_1 = \tfrac{2}{3}\tfrac{1}{2}\psi_1 = \tfrac{1}{3}\psi_1$$

$$\Rightarrow \boxed{Y = \tfrac{1}{3}}$$

(ψ_2 has the same hypercharge see [3] again from 111)

Recall that U_- lowers the hypercharge by 1 unit. Thus we can infer that ψ_3 has hypercharge $Y' = \frac{1}{3} - 1 = \boxed{\frac{-2}{3} = Y'}$

Thru similar arguments one can find the hyper charges of $\bar{\psi}_i$

$$\boxed{\hat{Y}\bar{\psi}_1 = -\tfrac{1}{3}\bar{\psi}_1 \qquad \hat{Y}\bar{\psi}_2 = -\tfrac{1}{3}\bar{\psi}_2 \qquad \hat{Y}\bar{\psi}_3 = \tfrac{2}{3}\bar{\psi}_3}$$

Given T_3 and the hypercharges of the states ψ_1, ψ_2 and ψ_3 we can deduce their charges thru the Gell-Mann-Nishijima relation,

$$\left.\begin{aligned}
\hat{Q}\psi_1 &= (\tfrac{1}{2}\hat{Y} + \hat{T}_3)\psi_1 = (\tfrac{+1}{6} + \tfrac{1}{2})\psi_1 = \tfrac{2}{3}\psi_1 \\
\hat{Q}\psi_2 &= (\tfrac{1}{2}\hat{Y} + \hat{T}_3)\psi_2 = (\tfrac{+1}{6} - \tfrac{1}{2})\psi_2 = -\tfrac{1}{3}\psi_2 \\
\hat{Q}\psi_3 &= (\tfrac{1}{2}\hat{Y} + \hat{T}_3)\psi_3 = (\tfrac{-1}{3} + 0)\psi_3 = -\tfrac{1}{3}\psi_3 \\
\hat{Q}\bar{\psi}_1 &= (\tfrac{1}{2}\hat{Y} + \hat{T}_3)\bar{\psi}_1 = (\tfrac{-1}{6} - \tfrac{1}{2})\bar{\psi}_1 = -\tfrac{2}{3}\bar{\psi}_1 \\
\hat{Q}\bar{\psi}_2 &= (\tfrac{1}{2}\hat{Y} + \hat{T}_3)\bar{\psi}_2 = (\tfrac{-1}{6} + \tfrac{1}{2})\bar{\psi}_2 = \tfrac{1}{3}\bar{\psi}_2 \\
\hat{Q}\bar{\psi}_3 &= (\tfrac{1}{2}\hat{Y} + \hat{T}_3)\bar{\psi}_3 = (\tfrac{1}{3} + 0)\bar{\psi}_3 = \tfrac{1}{3}\bar{\psi}_3
\end{aligned}\right\}\ \text{quarks have fractional charges.}$$

$$\boxed{\begin{array}{llll}
\psi_1 = \text{up quark} = q_1 & & \bar{\psi}_1 = \bar{q}_1 = \text{anti-up quark} \\
\psi_2 = \text{down quark} = q_2 & & \bar{\psi}_2 = \bar{q}_2 = \text{anti-down quark} \\
\psi_3 = \text{strange quark} = q_3 & & \bar{\psi}_3 = \bar{q}_3 = \text{anti-strange quark.}
\end{array}}$$

§8.2 : Transformation Properties of Quark States

There are 3-states in the triplet representation [3] thus the F-spin operators $\hat{F}_\alpha$ are given by 3×3 matrices in the basis $|q_i\rangle$ $i=1,2,3$. That is,

$$\boxed{(\hat{F}_\alpha)_{ij} = \langle q_i | \hat{F}_\alpha | q_j \rangle}$$

Recall the graph of [3] in the $(T_3 Y)$-plane and the T, U and V lines. It follows,

$$\boxed{\begin{array}{llll}
\hat{T}_-|q_1\rangle = |q_2\rangle & \hat{V}_-|q_1\rangle = |q_3\rangle & \hat{T}_3|q_1\rangle = \frac{1}{2}|q_1\rangle & \hat{Y}|q_1\rangle = \frac{1}{3}|q_1\rangle \\
\hat{T}_+|q_2\rangle = |q_1\rangle & \hat{V}_+|q_3\rangle = |q_1\rangle & \hat{T}_3|q_2\rangle = -\frac{1}{2}|q_2\rangle & \hat{Y}|q_2\rangle = \frac{1}{3}|q_2\rangle \\
\hat{U}_-|q_2\rangle = |q_3\rangle & & \hat{T}_3|q_3\rangle = 0|q_3\rangle & \hat{Y}|q_3\rangle = -\frac{2}{3}|q_3\rangle \\
\hat{U}_+|q_3\rangle = |q_2\rangle & & &
\end{array}}$$

Now recall how the shifts and the F-spin operators are related

$$\boxed{\begin{array}{lll}
\hat{F}_1 = \frac{1}{2}(\hat{T}_+ + \hat{T}_-) & \hat{F}_4 = \frac{1}{2}(\hat{V}_+ + \hat{V}_-) & \hat{F}_6 = \frac{1}{2}(\hat{U}_+ + \hat{U}_-) \\
\hat{F}_2 = -\frac{i}{2}(\hat{T}_+ - \hat{T}_-) & \hat{F}_5 = -\frac{i}{2}(\hat{V}_+ - \hat{V}_-) & \hat{F}_7 = -\frac{i}{2}(\hat{U}_+ - \hat{U}_-) \\
\hat{F}_3 = \hat{T}_3 & & \hat{F}_8 = \frac{\sqrt{3}}{2}\hat{Y}
\end{array}}$$

Now we had previously a 3×3 matrix rep. of $\hat{F}_\alpha$ where $\hat{F}_\alpha = \frac{1}{2}\hat{\lambda}_\alpha \leftarrow$ Gell-Mann Matrices.

$$(\hat{F}_1)_{[3]} = \begin{pmatrix} \langle q_1|\hat{F}_1|q_1\rangle & \langle q_1|\hat{F}_1|q_2\rangle & \langle q_1|\hat{F}_1|q_3\rangle \\ \langle q_2|\hat{F}_1|q_1\rangle & \langle q_2|\hat{F}_1|q_2\rangle & \langle q_2|\hat{F}_1|q_3\rangle \\ \langle q_3|\hat{F}_1|q_1\rangle & \langle q_3|\hat{F}_1|q_2\rangle & \langle q_3|\hat{F}_1|q_3\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \langle q_1|\frac{1}{2}q_1\rangle & 0 \\ \langle q_2|\frac{1}{2}q_2\rangle & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\lambda_1 \qquad \text{for example}$$

An arbitrary unitary operator can be expressed as exponential of the F-spin op's,

$$\hat{U}(\vec{\Theta}) = \exp(-i\,\Theta_\alpha \hat{F}_\alpha)$$

this is a group operator of SU(3), it transforms the states within each SU(3)-multiplet. In the case of the triplet representation [3] they are 3×3 unitary matrices with $\det(U) = 1$. We could write explicitly the transformation

$$|q_i\rangle' = |q_j\rangle\langle q_j|\hat{U}(\theta)|q_i\rangle = |q_j\rangle U_{ji}(\theta)$$

We continue this study in the next excercise on 114 except that would be silly since we have explicitly started 8.1 w/o knowing it. Ex. 8.1 is simply doing the same for $i = 2,3,4,\dots,7,8$. The result is $\hat{F}$-spin is represented by Gell-Mann Matrices in [3].

EXERCISE 8.1: GENERATORS OF SU(3) in the representation [3]

See the text pgs. 222–223 GREINER — follows calculation on 113.

EXERCISE 8.2: Transformation Properties of the Antitriplet [3]

PROBLEM: Show that the $[\bar{3}]$ transforms by $\bar{U}_{ji}$ (instead of U_{ji} for [3]) that is

$$|\bar{q}_i\rangle' = \hat{\bar{U}}(\theta)|\bar{q}\rangle = |\bar{q}_j\rangle\langle\bar{q}_j|\hat{\bar{U}}(\theta)|\bar{q}_i\rangle = |\bar{q}_j\rangle \bar{U}_{ji}(\theta)$$

Where the unitary operator with the $\bar{U}_{ji}(\theta)$ means,

$$\hat{\bar{U}}(\theta) = \exp\left(-i\,\theta_\alpha \hat{\bar{F}}_\alpha\right) \quad \text{where } \hat{\bar{F}}_\alpha = -\tfrac{1}{2}\hat{\lambda}^*_\alpha$$

$$\bar{U}_{ji}(\theta) = \langle\bar{q}_j|\hat{\bar{U}}(\theta)|\bar{q}_i\rangle = U^*_{ji}(\theta)$$

curious, wasn't $F = F^\dagger$ hmm...

Begin with what we know from [3] namely for the transformation

$$\hat{U}(\theta) = \exp(-i\,\theta\cdot\hat{F})$$

we obtain the following transformation for $|q_i\rangle \xrightarrow{\hat{U}(\theta)} |q_i\rangle'$, (nothing profound yet)

$$|q_i\rangle' = \hat{U}(\theta)|q_i\rangle$$

Now take the complex congugate:

$$(|q_i\rangle')^* = \hat{U}^*(\theta)(|q_i\rangle)^* = \hat{U}^*(\theta)\,|q_i^*\rangle$$

$$\hat{U}^*(\theta) = \exp\left([-i\,\theta\cdot\hat{F}]^*\right) = \exp\left(i\,\theta\cdot\hat{F}^*\right) \equiv \exp\left(-i\,\theta\cdot\hat{\bar{F}}\right)$$

I believe that "—" is to denote some objects relation to the antitriplet $[\bar{3}]$. Anyway lets restate the last thought,

$$\hat{U}^*(\theta) = \exp\left(i\,\theta\cdot(\hat{F})^*\right) \equiv \exp\left(-i\,\theta\cdot\hat{\bar{F}}\right)$$

$$\Rightarrow \hat{\bar{F}} \equiv -\hat{F}^*$$

(typo in texts on 225 ? $\hat{F}$ should be $\hat{\bar{F}}$)

The generators of $[\bar{3}]$ are

$$\boxed{-\hat{F}^*_\alpha = -\tfrac{1}{2}\hat{\lambda}^*_\alpha = \hat{\bar{F}}_\alpha \leftarrow \text{generator of } [\bar{3}]}$$

$\Rightarrow T_3$ & Y have negative of [3] eigenvalues. $\Rightarrow$ Antitriplet $[\bar{3}]$

Also we say $|q_i\rangle^* \equiv |\bar{q}_i\rangle$.

Exercise 8.3 : Why [3] and [$\bar{3}$] are inequivalent rep. of SU(3)

PROBLEM: Show that [3] and [$\bar{3}$] cannot be transformed into each other.

If the generator of [3] can be transformed into the generators of [$\bar{3}$] then [3] and [$\bar{3}$] would be equivalent. Suppose $\exists$ such a unitary transformation $\hat{S}$ of $\hat{F}_\alpha \longmapsto \hat{\bar{F}}_\alpha$ well say $\hat{\lambda}_\alpha \longmapsto \hat{\bar{\lambda}}_\alpha = -\lambda_\alpha^*$ that is,

$$\hat{S}\hat{F}_\alpha\hat{S}^{-1} = \hat{\bar{F}}_\alpha \longleftrightarrow \hat{S}\hat{\lambda}_\alpha\hat{S}^{-1} = -\hat{\lambda}_\alpha^*$$

Lets see what poisoiness fruit this wrong assumption will bear,

$$\hat{S}\hat{\lambda}_\alpha|q_i\rangle = \hat{S}\lambda|q_i\rangle = \lambda\hat{S}|q_i\rangle = \hat{S}\hat{\lambda}_\alpha\hat{S}^{-1}\hat{S}|q_i\rangle$$

Now if $\hat{S}|q_i\rangle = |q_i\rangle'$ then we obtain using $\hat{S}\hat{\lambda}_\alpha\hat{S}^{-1} = -\hat{\lambda}_\alpha^*$ on the above,

$$-\hat{\lambda}_\alpha^*|q_i\rangle' = \lambda|q_i\rangle'$$

Thus $\hat{\lambda}_\alpha$ and $-\hat{\lambda}_\alpha^*$ have the eigenvalue λ. Don't forget the Gell-Mann matrices are Hermitian so

$$\hat{\lambda}_\alpha = (\hat{\lambda}_\alpha)^\dagger = (\hat{\lambda}_\alpha^*)^T$$

Notice that $\hat{\lambda}_\alpha$ and $\hat{\lambda}_\alpha^*$ have same determinant since $\det(A^T) = \det(A)$ more over the eigenvalues are given by the characteristic eq$^{\underline{n}}$,

$$\det(\hat{\lambda}_\alpha - \lambda I) = \det(\hat{\lambda}_\alpha^* - \lambda I) = 0$$

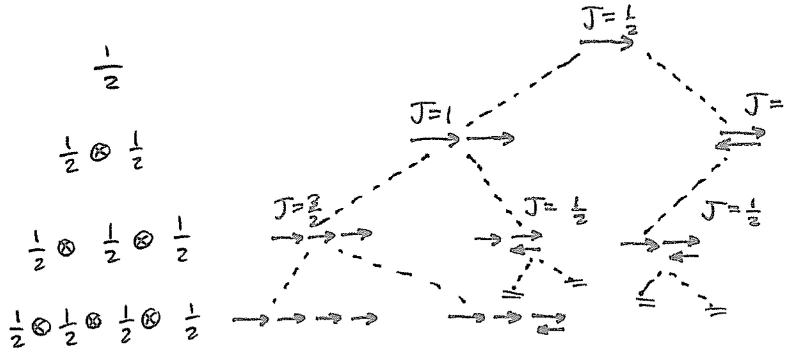
this implies that the eigenvalues of $-\hat{\lambda}_\alpha^*$ differ from $\hat{\lambda}_\alpha$ only by a sign. Consider that λ_8 has eigenvalues $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-2}{\sqrt{3}}$ thus $\hat{\lambda}_8$ and $-\hat{\lambda}_8^*$ have no common eigenvalues. No λ exists. $\Rightarrow$ no $\hat{S}$ exists....

λ_i $i = 1, 2, \ldots, 7$ are not the problem as they have eigenvalues $-1, 0, 1$ so $-\lambda_i^*$ likewise have $-1, 0, 1$ this is why SU(2) had $\approx$ doublet and antidoublet representations.

For SU(2) we found it was possible to construct general multiplets from the angular momentum algebra, that is the Lie Algebra of SU(2). The result was for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ there corresponds a multiplet $|j\, m\rangle$ where m takes $2j+1$ values namely $-j, -j+1, \ldots, 0, 1, \ldots, j = m$. We then discussed how clebsch-gordon can be used to break these up. The point here is that we can build the $|j\, m\rangle$ by coupling doublets together (doublet $j = \frac{1}{2}$ and $m = \pm\frac{1}{2}$). For example,

$$\boxed{\begin{aligned} [\tfrac{1}{2}] \otimes [\tfrac{1}{2}] &= [\tfrac{1}{2}]^2 = [1] \oplus [0] \\ [\tfrac{1}{2}] \otimes [\tfrac{1}{2}] \otimes [\tfrac{1}{2}] &= [\tfrac{1}{2}]^3 = [\tfrac{3}{2}] \oplus [\tfrac{1}{2}] \oplus [\tfrac{1}{2}] \end{aligned}}$$

"$\otimes$" is the Kronecker Product. Notice that there are several copies of $[\frac{1}{2}]$ in the direct sum on the RHS.



$[1] \oplus [0]$

$[\frac{3}{2}] \oplus [\frac{1}{2}] \oplus [\frac{1}{2}]$

$[2] \oplus [1] \oplus [1] \oplus [1] \oplus [0] \oplus [0] \oplus \cdots$
(if I had drawn more of the $(\frac{1}{2})^4$ row.

Physically this corresponds to the way that composite particles can recieve spin from their constiuents.

Now we'll argue the same holds for SU(3), that is we can build multiplets from kroneckering [3] and [$\bar{3}$]'s together. Actually only [3] is needed, we can get [$\bar{3}$] later ...
For example some SU(3) irreps,

$$[3] \otimes [3] = [6] \oplus [\bar{3}]$$
$$[\bar{3}] \otimes [\bar{3}] = [\bar{6}] \oplus [3]$$

Generally we could consider,

$$\boxed{\Big(\underbrace{[3] \otimes [3] \otimes \cdots \otimes [3]}_{p\text{ - times}}\Big) \otimes \Big(\underbrace{[\bar{3}] \otimes [\bar{3}] \otimes \cdots \otimes [\bar{3}]}_{q\text{ - times}}\Big)}$$

Physically this is the process of combining p - quarks and q - antiquarks into a composite particle.

question: do p and q correspond to their meaning from chpt. 7 ?

§ 8.4 : Constructing $D(p,q)$ from Quarks and Antiquarks

Begin by recalling the innards of $[3]$ and $[\bar{3}]$

$$\boxed{\begin{array}{l} [3] \text{ has } |T_3, Y\rangle \text{ with } (T_3, Y) = (\tfrac{1}{2}, \tfrac{1}{3}), (\tfrac{-1}{2}, \tfrac{1}{3}), (0, \tfrac{-2}{3}) \\ \hline [\bar{3}] \text{ has } |\bar{T}_3, \bar{Y}\rangle \text{ with } (\bar{T}_3, \bar{Y}) = (\tfrac{-1}{2}, \tfrac{-1}{3}), (\tfrac{1}{2}, \tfrac{-1}{3}), (0, \tfrac{2}{3}) \end{array}}$$

Then the direct product (aka Kronecker) is the set of all product states of the form:

$$\psi \equiv \boxed{|T_3(1), Y(1)\rangle |T_3(2), Y(2)\rangle \cdots |T_3(p), Y(p)\rangle \otimes |\bar{T}_3(1), \bar{Y}(1)\rangle \cdots |\bar{T}_3(q)\, \bar{Y}(q)\rangle}$$

This is a <u>p-quark, q-antiquark state</u>. Now $\hat{T}_3$ and hypercharge $\hat{Y}$ are additive thus,

$$\hat{T}_3 = \sum_i{}' \hat{T}_3(i) \quad \text{and} \quad \hat{Y} = \sum_i{}' \hat{Y}(i)$$

Where i runs over all p-quarks and q-antiquarks. In other words the net isospin and hypercharge are found by simply adding the constituent parts

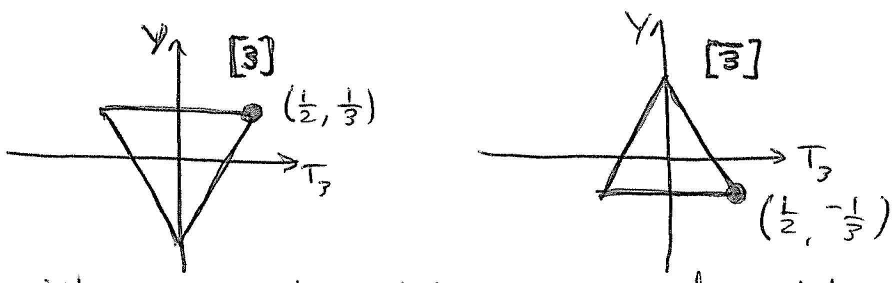
$$(T_3, Y) = \left(\sum_{i=1}^{p} T_3(i) + \sum_{j=1}^{q} \bar{T}_3(j),\ \sum_{i=1}^{p} Y(i) + \sum_{j=1}^{q} \bar{Y}(j) \right)$$

Following the math literature the pair (T_3, Y) is called the weight of the state ψ. We give weights the <u>dictionary</u> ordering meaning, $(T_3, Y) > (T_3', Y')$ if

$$\boxed{T_3 > T_3' \quad \text{OR} \quad T_3 = T_3' \ \text{and} \ Y > Y' \iff (T_3, Y) > (T_3', Y')}$$

<u>Example 8.4</u> $\quad (\tfrac{1}{2}, \tfrac{1}{3}) > (\tfrac{-1}{2}, \tfrac{1}{3})$ and $(\tfrac{1}{2}, \tfrac{1}{3}) > (0, \tfrac{-2}{3})$ and $(\tfrac{1}{2}, 1) > (\tfrac{1}{2}, -1)$. In words the highest weight state is to the right on a multiplet in the $(T_3 Y)$-plane.

<u>Example 8.5</u>



illustrating the states of maximal weight in $[3]$ & $[\bar{3}]$

Constructing $D(p, q)$ from quark/antiquark states $[3]/[\bar{3}]$

The state of highest weight is the same state we looked for in chapter 7. Its the state where $\hat{T}_3$ has it's largest eigenvalues and both shifts lead to states of smaller T_3 or vanish. We found thru geometric arguments involving the T-U-V exchange symmetry ~~that~~

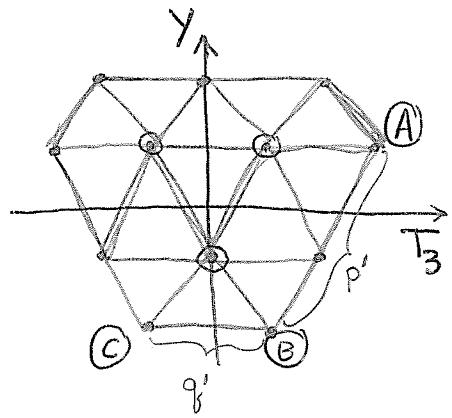
$$(T_3)_{max} = \left(\frac{p+q}{2}\right)$$

$$(Y)_{max} = \left(\frac{p-q}{3}\right)$$

The heighest-weight of the p-quark and q-antiquark state is simply $\left(\frac{p+q}{2}, \frac{p-q}{3}\right)$. Clearly this highest weight state is unique. States of lower weight need not be unique.

For example states with $(T_{3(max)}-1, Y_{max})$ are obtained by one of the factors $|\frac{1}{2},\frac{1}{3}\rangle$ or $|\frac{1}{2},-\frac{1}{3}\rangle$ in the highest weight state, replace it with $|-\frac{1}{2},\frac{1}{3}\rangle$ or $|-\frac{1}{2},-\frac{1}{3}\rangle$ so that the resulting state has T_3 reduced by one and Y the same. Clearly there are many lower states to be generated from a particular highest-weight state. The shift operators $T_\pm$, $U_\pm$ and $V_\pm$ can be used to systematically generate the whole ~~representation~~ multiplet from the highest weight state.

Example: take $p=2$ and $q=1$ lets find $D(2,1)$. Actually ~~of~~ our goal is to show that p and q gives the largest rep of SU(3). So suppose that we have (p', q')



$$(T_3)_{max} = \frac{q'}{2}\times 1 + p'\times\frac{1}{2}$$

The point (A) lies on U-doublet hence

$$U_3(A) = -\frac{1}{2}q'$$

$\frac{1}{2}q'$ 0 $-\frac{1}{2}q'$

Yet we also know $(T_3)_{max} = \frac{p+q}{2}$ and

$$(U_3)_{min} = \frac{1}{2}\left(\frac{3}{2}Y_{max} - (T_3)_{max}\right) = \frac{1}{2}\left(\frac{3}{2}\left(\frac{p-q}{3}\right) - \frac{p+q}{2}\right) = -\frac{1}{2}q$$

Thus p' and q' are found as fncts of p & q

$$\frac{p+q}{2} = \frac{p'+q'}{2} \quad \& \quad -\frac{q}{2} = \frac{-q'}{2} \implies \boxed{p = p' \;\&\; q = q'}$$

- We have arrived at these formulas from two directions. One the general SU(3) rep. theory of chpt. 7 and Two the construction of p-quark and q-antiquark states. They're the same.

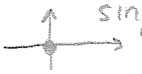
$$(T_3)_{max} = \frac{p+q}{2}, \quad Y_{max} = \frac{p-q}{3}$$

THE SMALLEST SU(3) REPRESENTATIONS

(PHYSICAL EXAMPE IN PARENTHESES)

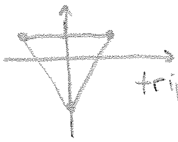
Lets list the simplest multiplets of SU(3),

$D(0,0) = [1]$



singlet

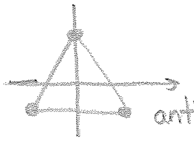
$D(1,0) = [3]$



triplet

(quarks)

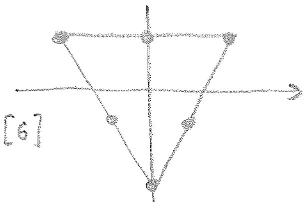
$D(0,1) = [\bar{3}]$



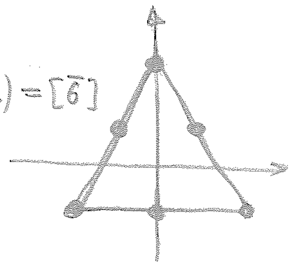
antitriplet

(antiquarks)

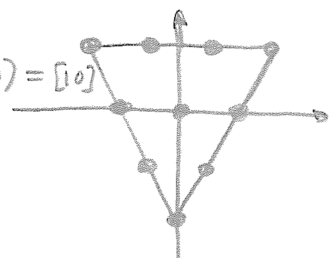
$D(2,0) = [6]$



$D(0,2) = [\bar{6}]$

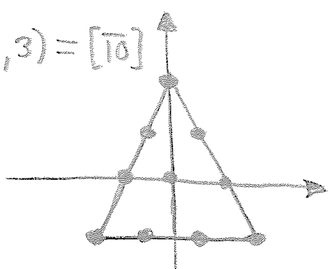


$D(3,0) = [10]$

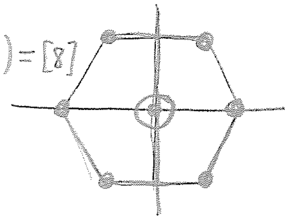


(Baryon Resonances Ex. 6.5)

$D(0,3) = [\overline{10}]$

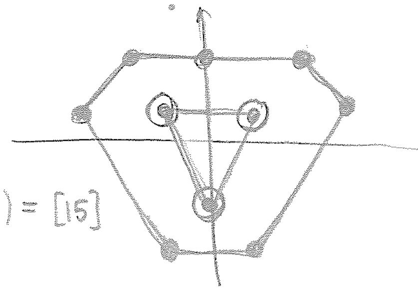


$D(1,1) = [8]$

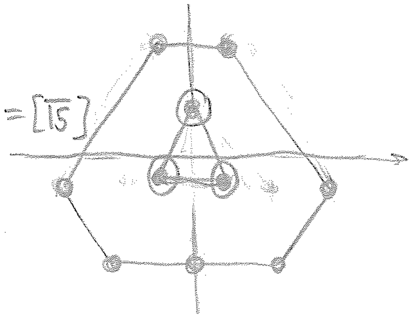


(Pseudoscalar Mesons)

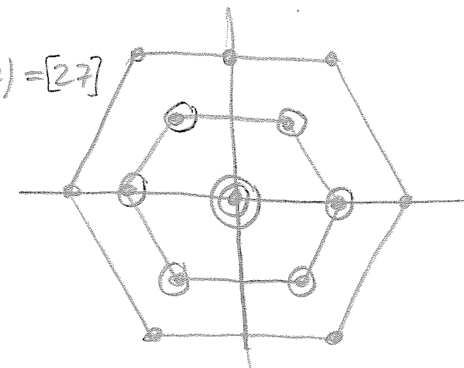
$D(2,1) = [15]$



$D(1,2) = [\overline{15}]$



$D(2,2) = [27]$



Notice Each Multiplet is labeled according to it's dimension

$$D(p,q) = \frac{1}{2}(p+1)(q+1)(p+q+2)$$

(which was calculated in Ex. 7.11., pg. 213-214)

for exp.

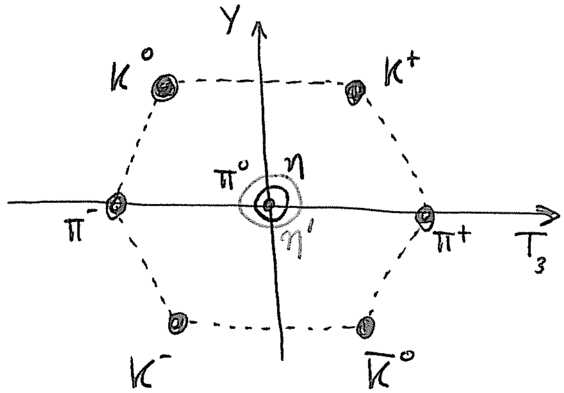
$$D(3,0) = \frac{1}{2}(3+1)(0+1)(3+0+2) = \frac{4}{2}\cdot 5 = 10.$$

$$D(2,2) = \frac{1}{2}(2+1)(2+1)(2+2+2) = \frac{1}{2}3\cdot 3\cdot 6 = 27.$$

- Also note the states within a Δ have same multiplicity, however inside a hexagon each level inc. degeneracy by one until Δ is reached.

Example 8.6 : The Pseudoscalar Mesons

By definition elementary particles with odd-parity and spin zero are called pseudoscalar mesons. In fact the pseudo-scalar mesons form an [8] of SU(3). The η' is a singlet which is often joined to this [8] to form the "meson nonet" (η' and the particles in [8] are in different multiplets, if SU(3) is exact they ought not interact, do they?)



The pseudoscalar "nonet"

- (SKIP the Kaon digression of 8.4 for now) lets focus on the group-theoretic aspects for now.

§8.5 Meson Multiplets

In contrast to Baryon Multiplets, meson multiplets contain both particles and antiparticles within a particular multiplet. Moreover the meson and its antiparticle possess the same spin and parity. In baryons we say a baryon has baryon # 1 while the antibaryon has baryon # −1, because particles & antiparticles appear with the same meson multiplet, and the multiplet must share a common baryon # it follows baryon # $B = 0$ for mesons. This is a direct result of quarks having baryon # 1/3 while antiquarks have baryon # $-1/3 = B$. This is clearly inferred from the fact that a baryon is constructed from 3-quarks has $B_{net} = 1$ and B is additive like T_3 & Y $\Rightarrow$ q has $B = \frac{1}{3}$. Similary for the antibaryon made of 3 $\bar{q}$'s $\Rightarrow$ $\bar{q}$ has $B = \frac{-1}{3}$.

$$\text{Mesons are } q\bar{q} \text{ pairs} \Rightarrow [3] \otimes [\bar{3}] = [8] \oplus [1]$$

That is group-theory suggest if mesons are made of q-$\bar{q}$ pairs then we should observe a meson-nonet. The mixing of [8] and [1] $\Rightarrow$ SU(3) is violated. The singlet state $|T_3 = 0, Y = 0\rangle$ mixes with the $|00\rangle$ state in [8] thus the [8] & [1] are connected, to make the nonet. In Baryon Multiplets there is <u>not as much</u> SU(3) mixing.

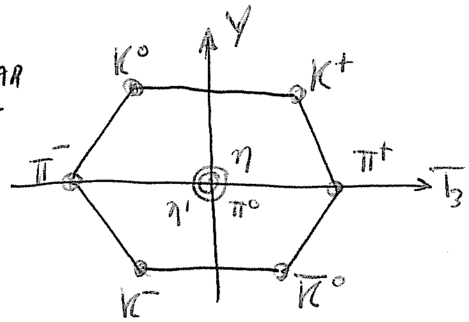
§8.5 Meson Multiplets Continued

So basically there are a # of meson multiplets. Each shares a common parity and spin, they are,

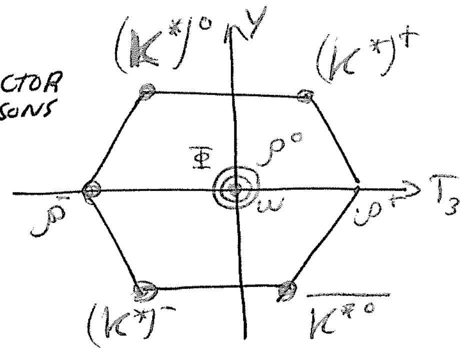
Scalar mesons	$J^P = 0^+$	vector mesons	$J^P = 1^-$
pseudoscalar mesons	$J^P = 0^-$	axial vector mesons	$J^P = 1^+$
tensor mesons	$J^P = 2^+$	yada, yada, yada...	
pseudotensor mesons	$J^P = 2^-$		

I gather each of these mixes with a singlet of same type to form a $[8] \oplus [1]$ nonet. (Question why mixing SU(3)? How serious a problem?)

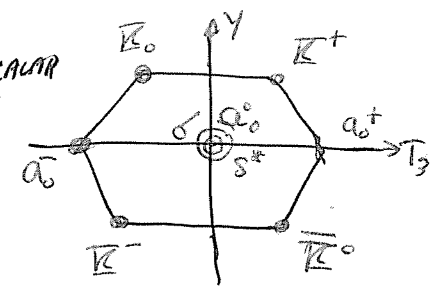
PSEUDO SCALAR MESONS



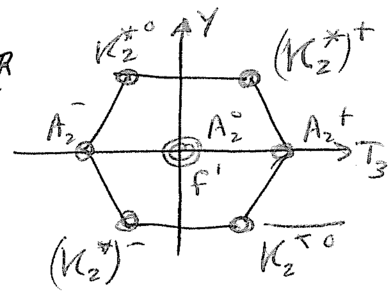
VECTOR MESONS



~~PSEUDO~~SCALAR MESONS



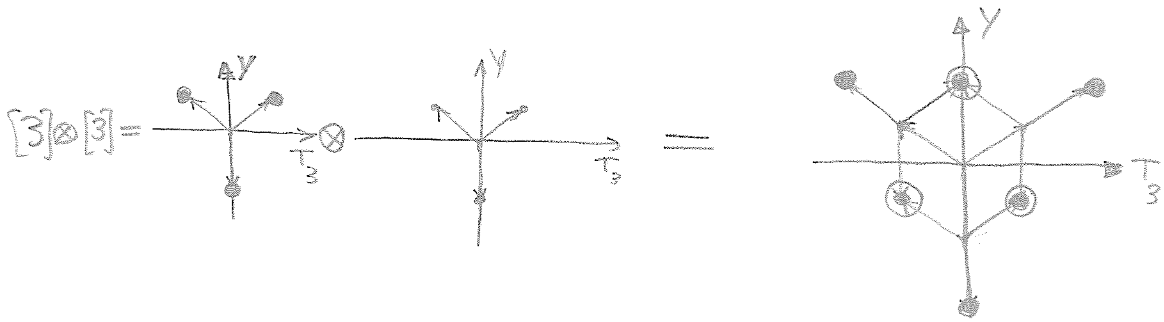
TENSOR MESONS



Each of these has a mixing of the $|00\rangle$ of [8] with the singlet [1]. There are some ambiguities because the mixing is considerable for the scalar mesons for example.

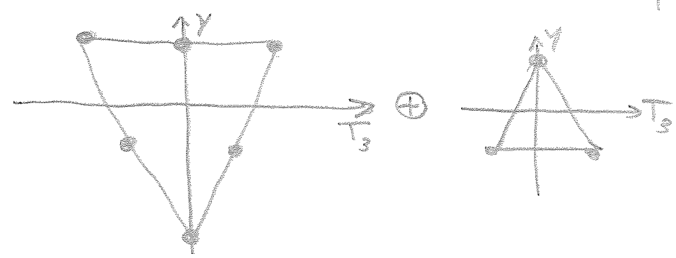
§8.6 : RULES FOR THE REDUCTION OF DIRECT PRODUCTS OF SU(3) MULTIPLETS

We develope a graphical method for explaining $[3]\otimes[3] = [6]\oplus[\bar{3}]$ and more. We do this by example, the product representation of $[3]\otimes[3]$ has weight vectors for the corresponding direct product states $|T_3(1)\,Y(1)\rangle\,|T_3(2)\,Y(2)\rangle$ according to vector addition in the (T_3, Y)-plane



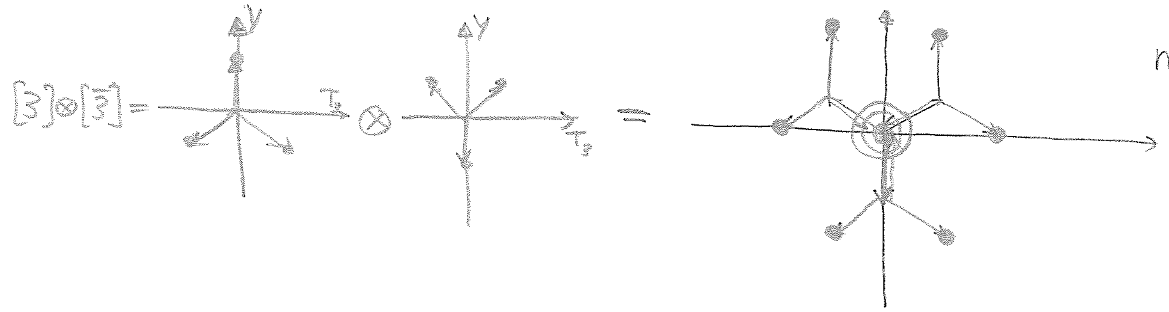
put [3] at each node of [3]. The ⊙ denotes double occupancy.

=



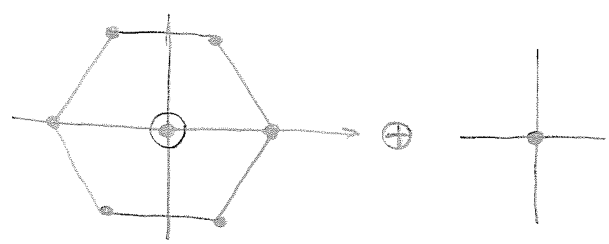
$= [6]\oplus[\bar{3}]$

the endpoints of the vector addition represent all the possible states of the direct product in the $(T_3 - Y)$ lattice. Next consider how to show $[\bar{3}]\otimes[3] = [8]\oplus[1]$,



notice the triple occupancy at $|00\rangle$.

=



$= [8]\oplus[1]$

We can show $[3]\otimes[6] = [10]\oplus[8]$ by this graphical method. see 245 if you wish.

TRIALITY and SU(3) representations

Analyzing $[3]\otimes[3] = [6]\oplus[\bar{3}]$, why is $[6]$ in $[3]\otimes[3]$? Recall that

$$\boxed{(T_3)_{max} = \frac{p+q}{2} \quad \& \quad (Y)_{max} = \frac{p-q}{3}}$$

Lets introduce τ the triality to focus on $(Y)_{max}$, we define

$$\boxed{Y_{max} = \frac{3n+\tau}{3} \quad \text{where} \quad \begin{array}{l} n = 0, \pm 1, \pm 2, \dots \\ \tau = 0, 1, 2 \text{ modulo } 3 \end{array}}$$

Usually Y_{max} is a "multiple" of $\frac{1}{3}$ although it can be an integer (which is still a multiple of $\frac{1}{3}$ where I come from...) Anyway clearly $\tau = 0, 1, 2$ split up SU(3) rep as follows

$\tau = 0$	Y integer (like $D(1,1) = [8]$ or $D(3,0) = [10]$)
$\tau = 1$	Y fraction (for example $[3]$)
$\tau = 2$	Y fraction (for example $[\bar{3}]$)

Remark: All representations which occur by reduction of a direct product of irreducible SU(3) muliplets have equal triality. More than this it can be shown that the triality is additive lets illustrate by example,

$$\underset{(\tau=1)}{[3]} \otimes \underset{(\tau=1)}{[3]} = \underset{(\tau=2)}{[6]} \oplus \underset{(\tau=2)}{[\bar{3}]}$$

notice that $1+1=2$ and $[6]$ & $[\bar{3}]$ have the same triality.

$$\underbrace{[3]\otimes[3]\otimes[3]}_{\substack{1+1+1=0 \\ (\text{mod } 3)}} = \underbrace{[1]\oplus[8]\oplus[8]\oplus[10]}_{\text{have } \tau=0}$$

$$\underbrace{[8]\otimes[8]}_{0+0=0} = \underbrace{[1]\oplus[8]\oplus[8]\oplus[10]\oplus[\overline{10}]\oplus 27}_{\text{have } \tau=0}$$

The $[8]\otimes[8]$ is symmetric around $(0,0)$ thus the direct sum must likewise be symmetric $\Rightarrow$ need $[10]$ and $[\overline{10}]$ (pg. 231 helps make this clear.)

> It seems all physically observed SU(3) multiplets have $\tau = 0$. This is a restatement of quark confinement.

Exercise 8.12 : Applying Graphical Method to SU(2) Multiplet Reduction

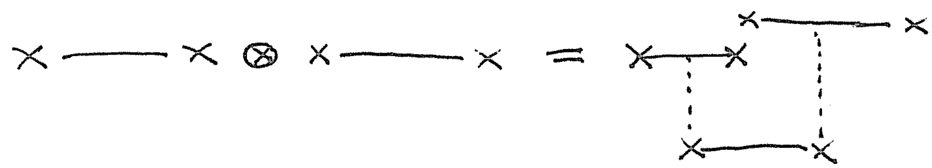
PROBLEM: Decompose $[2]\otimes[2]\otimes[2]$ into a direct sum of irred SU(2) reps.

SU(2) has rank one ⇒ one casimir ⇒ one dim'l weight space instead of $(T_3 Y)$-plane just have T_3 line. Some basic irreps

[2]	=	× $-\frac{1}{2}$	× $\frac{1}{2}$		
[3]	=	× -1	× 0	× 1	
[4]	=	× $-\frac{3}{2}$	× $-\frac{1}{2}$	× $\frac{1}{2}$	× $\frac{3}{2}$

The eigenvalues of diag. operators are additive, the addition can be graphically represented by putting the center point on each point of the other multiplet

Consider the graphical decomposition of $[2]\otimes[2]$

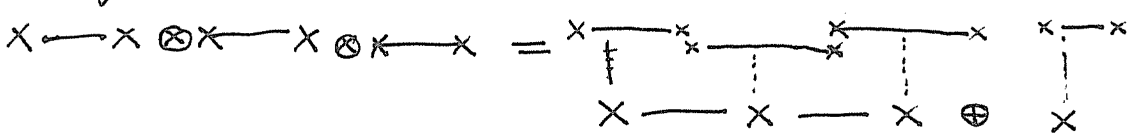


$= \times - \underset{(2)}{\times} - \times$

$= \times - \times - \times \oplus \times$

$= [3]\oplus[1]$

Now tackle the $[2]\otimes[2]\otimes[2]$ graphically using the above namely that $[2]\otimes[2]\otimes[2] = [2]\otimes([3]\oplus[1])$



$= \times - \underset{(2)}{\times} - \underset{(2)}{\times} - \times \oplus \times - \times$

$= [\times - \times - \times - \times] \oplus [\times - \times] \oplus [\times - \times]$

$= \boxed{[4]\oplus[2]\oplus[2] = [2]\otimes[2]\otimes[2]}$

neat.

§ 8.7 U-spin Invariance ...

Upto this point we have made extensive use of T-spin (isospin) which is a copy of SU(2) inside SU(3). Recall baryon octets can be classified with hypercharge, the isospin multiplets have constant Y over their states for example, See figure 8.11 for graphical rep.

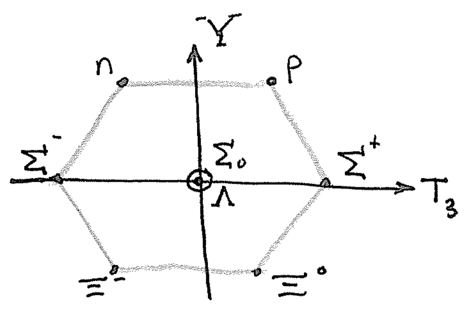
$T = \frac{1}{2}$: (n, p)	has $Y = 1$	and	$\Delta M = 2\,MeV$
$T = 1$: $(\Sigma^-, \Sigma^0, \Sigma^+)$	has $Y = 0$	and	$\Delta M = 8\,MeV$
$T = \frac{1}{2}$: (Ξ^-, Ξ^0)	has $Y = -1$	and	$\Delta M = 7\,MeV$

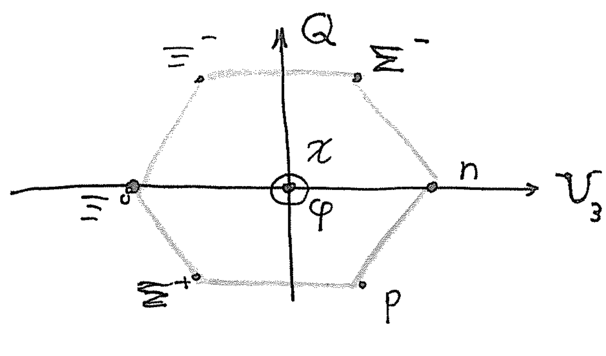
States of a given multiplet above are transformed amongst each other via the $\hat{T}_\pm$ operators. The mass difference ΔM is small compared to the masses of the particles involved this means T-isospin is weakly broken, its a good symmetry

As previously discussed SU(3) has T, U and V su(2) subalgebras. These are all called " "-spin, eg T-spin. The U-spin multiplets have states connected by $\hat{U}_\pm$ for examples

(Σ^-, Ξ^-)	with $\Delta M = 124\,MeV$	and	$Q = -1$	$U = \frac{1}{2}$
(n, Σ^0, Ξ^0)	with $\Delta M = 374\,MeV$	and	$Q = 0$	$U = 1$
(p, Σ^+)	with $\Delta M = 251\,MeV$	and	$Q = 1$	$U = -\frac{1}{2}$

See figure 8.12 for this (non standard rep. perhaps). Note that states on edge of rep. have unit multiplicity and are eigenstates of T, U, V whereas center states can have degeneracy and as such are not eigenstates of all 3 operators at once (see fig. 8.11 & 8.12).





We now construct the eigenstates of U using the fundamental isospinors α and β we find ($T_3\alpha = +\frac{1}{2}\alpha$ and $T_3\beta = -\frac{1}{2}\beta$ I think)

$\Sigma^+ = \alpha\alpha$ $\Sigma^0 = \frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha)$ $\Sigma^- = \beta\beta$	the Triplet $(Y=0)$
$\Lambda = \frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha)$	the Singlet $(Y=0)$

Since T-, U- and V-spin are $SU(2)$ algebras, their generators must be the Pauli Matrices. Three distinct copies of them, for T-spin say generators are $\hat{\tau}_i$ and for U-spin the generators are $\hat{\mu}_i$ where

$$\hat{\mu}_3 = \hat{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{\mu}_- = \hat{\tau}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \hat{\mu}_+ = \hat{\tau}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- Although I wonder if these should be in different spaces, at least it seems they should act in different spaces. Maybe $\hat{\tau}_3 = \left(\begin{array}{c|c} \sigma_z & 0 \\ \hline 0 & 1 \end{array}\right)$ would be more appropriate, I could be off here.

For multiple particles we let $\hat{\tau}_3(i)$ be for the i^{th} particle and find the total $\hat{T}_3$ and $\hat{U}_3$ by summing,

$$\boxed{\begin{array}{ll} \hat{T}_3 = \frac{1}{2}\sum_i \hat{\tau}_3(i) & \hat{U}_3 = \frac{1}{2}\sum_i \hat{\mu}_3(i) \\ \hat{T}_+ = \cancel{\tfrac{1}{2}}\sum_i \hat{\tau}_+(i) & \hat{U}_+ = \cancel{\tfrac{1}{2}}\sum_i \hat{\mu}_+(i) \\ \hat{T}_- = \cancel{\tfrac{1}{2}}\sum_i \hat{\tau}_-(i) & \hat{U}_- = \cancel{\tfrac{1}{2}}\sum_i \hat{\mu}_-(i) \end{array}}$$

Next we'll see how to find the triplet eigenstates of U-spin which we denote η, x and Ξ^0

- Remark: $\sum_i$ is over how ever many isospinors are necessary to construct the states. I don't believe its over the # of particles.

Continuing § 8.7, find the U-triplet

We denote them (n, χ, Ξ^0) and (ϕ) where χ and ϕ are orthogonal linear combinations of the isospin eigenstates Λ and Σ^0 (the center states in $T_3 Y$ - diagram.) . Writing a, b to insure orthogonality $\langle \chi, \phi \rangle = 0$.

$$\boxed{\begin{aligned} \chi &= a\Sigma_0 + b\Lambda \\ \phi &= b\Sigma_0 - a\Lambda \end{aligned}}$$

Consider then

$$\begin{aligned} \hat{T}_- \Sigma^+ &= \tfrac{1}{2}\sum_i \hat{t}_-(i)[\alpha\alpha] \\ &= \tfrac{1}{2}\left(\hat{t}_-(1) + \hat{t}_-(2)\right)[\alpha\alpha] \\ &= \tfrac{1}{2}(\beta\alpha + \alpha\beta) \end{aligned} \Longrightarrow \hat{T}_- \Sigma^+ = \frac{\sqrt{2}}{\sqrt{2}}(\alpha\beta + \beta\alpha) = \sqrt{2}\,\Sigma^0$$

$$\therefore \quad \boxed{\hat{T}_- \Sigma^+ = \sqrt{2}\,\Sigma^0}$$

Interesting argument : If $\hat{T}_-$ connects Σ^+ to $\sqrt{2}\,\Sigma^0$ in the T-spin multiplet then so must $\hat{U}_-$ connect n to $\sqrt{2}\,\chi$ in the U-spin triplets.

$$\Rightarrow \quad \boxed{\hat{U}_- n = \sqrt{2}\,\chi}$$

Also since $\hat{T}_+ n = P$ this means that

$$\begin{aligned} [U_-, T_+]n &= U_- T_+ n - T_+ U_- n \\ &= U_- P - T_+ \sqrt{2}\,\chi \\ &= U_- P - \sqrt{2}\, T_+ (a\Sigma_0 + b\Lambda) \\ &= \Sigma^+ - 2a\Sigma^+ - \sqrt{2}\, b\, \cancel{T_+\Lambda} \to 0 \\ &= (1-2a)\Sigma^+ \end{aligned}$$

$$\begin{aligned} U_- P &= \Sigma^+ \\ T_+ \Sigma_0 &= \Sigma^+ \\ T_+ \Lambda &= \tfrac{1}{2\sqrt{2}} \sum_i \hat{t}_+(i)[\alpha\beta - \beta\alpha] \\ &= \tfrac{1}{2\sqrt{2}}(\beta\beta - \beta\beta) = 0. \end{aligned}$$

Because $[U_-, T_+] = 0 \Rightarrow 1 - 2a = 0 \Rightarrow \boxed{a = \tfrac{1}{2}}$

normalizing $\chi^* \chi = 1 = a^2 + b^2$ (assuming Σ_0 and Λ are normalized to one) gives $b^2 = 1 - a^2 = \frac{3}{4}$ choosing b real and positive $\Rightarrow b = \frac{\sqrt{3}}{2}$ (convention)

$$\boxed{\chi = \tfrac{1}{2}\Sigma_0 + \tfrac{\sqrt{3}}{2}\Lambda \quad \text{and} \quad \phi = \tfrac{\sqrt{3}}{2}\Sigma^0 - \tfrac{1}{2}\Lambda}$$

- Note $[U_-, T_+] = 0 \Rightarrow a^2 = \frac{1}{4} \Rightarrow b \neq 0$ or else $|\chi|^2 \neq 1$. This is a proof of our claim that the degenerate states are not mutually U, V and T spin states.

§8.8 - U-spin Invariance

The Electromagnetic interaction violates isospin symmetry since it splits T-multiplets. eg. for the isodoublet $\Delta M \cong 2\,MeV$. On the other hand since U-spin multiplets have the same charge thru-out a rep $\Leftrightarrow [Q, U_3] = 0$ charge is an eigenvalue of U_3-spin ~~states~~ eigenstates. $\Rightarrow$ U-spin multiplets shouldn't have the same symmetry breaking as in the T-spin reps. But this only follows if U-spin is a symmetry of physics, in particular the Strong Interaction. As we observed this is only approximately because $\Delta M \approx 10\%$ of mass particles involved. So it seems U-spin conservation $\Rightarrow$ similar Electromagnetic properties of U-spin multiplet particles. For example (Σ^-, Ξ^-) (n, Σ^0, Ξ^0) and (p, Σ^+) should have common E&M props. Experiment says,

$$\boxed{\begin{array}{ll} \mu_p = 2.79\,\mu_0 & \mu_{\Sigma^+} = (2.33 \pm 0.13)\,\mu_0 \\ \mu_{\Xi^-} = [(-0.69) \pm 0.04]\,\mu_0 & \mu_{\Sigma^-} = (-1.41 \pm 0.25]\,\mu_0 \\ \mu_n = -1.91\,\mu_0 & \mu_{\Xi^0} = (-1.253 \pm 0.014)\,\mu_0 \end{array}}$$

So we can observe that $\mu_p \approx \mu_{\Sigma^+}$ and $\mu_{\Sigma^-} \approx \mu_{\Xi^-}$ and $\mu_n \approx \mu_{\Xi^0}$ only very crudely. Allegedly quark-model will explain this later.

Now assume that mass of baryon results from from strong interaction which conserves isospin and on the other hand from EM interaction which conserves U-spin that means EM contributions to mass must be equal within a U-spin multiplet, Thus

$$\boxed{\begin{array}{l} \delta M_p = \delta M_{\Sigma^+} \\ \delta M_n = \delta M_{\Xi^0} \\ \delta M_{\Sigma^-} = \delta M_{\Xi^-} \end{array}} \longrightarrow \delta M_n - \delta M_p + \delta M_{\Xi^-} - \delta M_{\Xi^0} = \delta M_{\Sigma^-} - \delta M_{\Sigma^+}$$

if besides E&M $\nexists$ any other interaction to remove degeneracy of isomultiplet then,

$$\boxed{\begin{array}{l} \delta M_n - \delta M_p = M_n - M_p \\ \delta M_{\Sigma^-} - \delta M_{\Sigma^+} = M_{\Sigma^-} - M_{\Sigma^+} \\ \delta M_{\Xi^-} - \delta M_{\Xi^0} = M_{\Xi^-} - M_{\Xi^0} \end{array}} \longrightarrow \boxed{M_n - M_p + M_{\Xi^-} - M_{\Xi^0} = M_{\Sigma^-} - M_{\Sigma^+}}$$

"Coleman - Glashow" relation.

Experimentally the Coleman - Glashow relation is nearly confirmed. So U-spin is useful in this sense for studying SU(3) multiplets. On the other hand V-spin is not a good symmetry & has no easy commonality like charge. So ppl. don't use V-spin to classify states.

Experiment shows $\Delta M/M \cong 10\%$ for SU(3) and this is probably due to E/M splitting.
So SU(3) sym. is broken more severely than isospin symmetry which is good to 1%.
That is $M \approx 1000\,MeV$ and $\Delta M_{SU(3)} \cong 100\,MeV$ while $\Delta M_{isospin} \approx 10\,MeV$ where ΔM describes the mass difference found in an multiplet of the given symmetry.

We then start with the idea that the Hamiltonian of the strong interaction splits into a "super strong" part H_{ss} which is SU(3) invariant and another part which is "medium strong" called H_{ms} which breaks the SU(3) symmetry. Since the mass splitting amounts to about 10% of the mean mass so the contribution to the mass from H_{ms} is small that is to say

$$\boxed{M = \langle \hat{H}_{ss} \rangle + \langle \hat{H}_{ms} \rangle \quad \text{where } \langle H_{ss} \rangle \gg \langle H_{ms} \rangle}$$

In other words the $\hat{H}_{ms}$ removes the degeneracy from the SU(3) multiplets. Of course $\hat{H}_{ss}$ is degenerate accross a given multiplet. Mathematically $\hat{H}_{ss}$ must be constructed from SU(3) Casimirs while $\hat{H}_{ms}$ is constructed by generators, which don't commute with everything (hence the sym. breaking)

For now ignore the E/M mass splitting, assume isospin is a perfect symmetry which means even H_{ms} must commute with isospin T_3

$$\boxed{[H_{ss} + H_{ms}, T_3] = 0 \Rightarrow \underbrace{[H_{ss}, T_3] = 0}_{H_{ss}\ \text{SU(3) casimirs}} \ \& \ \underbrace{[H_{ms}, T_3 = 0]}_{\text{to insure isospin symmetry holds.}}}$$

Then in constructing H_{ms} we can use generators of SU(3) which commute with T_3 $\Longrightarrow$ use $\tilde{F}_8 = \frac{\sqrt{3}}{2}\hat{Y} \Rightarrow \boxed{\hat{H}_{ms} = b\hat{Y}}$

Now calculate $\langle \hat{H}_{ms} \rangle$ between the unperturbed wave fncts. $|T\,T_3\,Y\rangle$ in terms of 1st order perturbation theory ($\langle H_{ms} \rangle \ll \langle H_{ss} \rangle$)

$$\boxed{\langle T T_3 Y | \hat{H}_{ms} | T T_3 Y \rangle = bY \qquad \text{where } M = a_{ss} + bY}$$

So "a_{ss}" is the mean mass of the multiplet and b again is constant thru-out a multiplet (as we know $T_3\,Y$ characterize an SU(3) multiplet)
For the decuplet it should follow

$$\boxed{M_{\Omega^-} - M_{\Xi^*} = M_{\Xi^*} - M_{\Sigma^*} = M_{\Sigma^*} - M_{\Delta}}$$

But this is difficult to see experimentally because of the ~~E~~ E/M ~~M~~ mass splitting associated with isospin symmetry ... But if we compare U-spin multiplets which share a common charge perhaps we could see the effect.

$M_{\Omega^-} - M_{\Xi^*}$	$=$	$(137 \pm 1)\,MeV$
$M_{\Xi^*} - M_{\Sigma^*}$	$=$	$(148 \pm 1)\,MeV$
$M_{\Sigma^*} - M_{\Delta}$	$=$	$(148 \pm 5)\,MeV$

$\longrightarrow$ hmm. seems to work but it can't be quite right because $Y_{\Sigma^0} = Y_{\Lambda} \Rightarrow M_{\Sigma^0} = M_{\Lambda}$ but 77 MeV seperate these.

§8.9 Gell-Mann - Okubo Mass Formula

$M = a + bY$ works but can't be quite right so we modify it to try to account for $M_{\Sigma_0} \neq M_\Lambda$. Besides Y we could use T^2 and Y^2 which commute with T_3,

$$\hat{H}_{ms} = b\hat{Y} + c\hat{T}^2 + d\hat{Y}^2$$

$$\langle T T_3 Y | \hat{H}_{ms} | T T_3 Y \rangle = bY + cT(T+1) + dY^2$$

$$\langle T T_3 Y | \hat{H}_{ms} + \hat{H}_{ss} | T T_3 Y \rangle = \boxed{a + bY + cT(T+1) + dY^2 = M}$$

Again for a particular multiplet the coefficients c and d are constants. This modification now would seem to ruin our good result for $M = a + bY$, it doesn't yield a constant mass splitting for the decuplet. This can be corrected by demanding $cT(T+1) + dY^2 = x + yY$

$$\boxed{cT(T+1) + dY^2 = x + yY} \quad \boxed{8.57a}$$

Consider then the values of Δ^-, Ξ^{*-} and Ω^-

	T	Y
Δ^-	3/2	1
Ξ^{*-}	1/2	-1
Ω^-	0	-2

$\xrightarrow{\boxed{8.57a}}$

$$\left.\begin{aligned} \tfrac{15}{4}c + d &= x + y \\ \tfrac{3}{4}c + d &= x - y \end{aligned}\right\} \longrightarrow -\tfrac{6}{8}c + d = x - 2y$$

$$4d = x - 2y$$

$$\Rightarrow -\tfrac{6}{8}c + d = 4d$$

$$\Rightarrow -\frac{6c}{8} = 3d$$

$$\Rightarrow \boxed{d = -\frac{c}{4}}$$

Thus we find that we should write the mass M as

$$\boxed{M = a + bY + c\left(T(T+1) - \tfrac{1}{4}Y^2\right)}$$ mass formula of Gell-Mann (applied) & Okubo (invented)

Apply this to baryon octet, considering nuetral particles to minimize E/M splitting,

$$\underset{(1127.1 \pm 0.7)\,\mathrm{MeV}}{\underset{\parallel}{\tfrac{1}{2}\left(M_N + M_\Xi\right)}} = \underset{(1134.8 \pm 0.2)\,\mathrm{MeV}}{\underset{\parallel}{\tfrac{1}{4}\left(3M_\Lambda + M_\Sigma\right)}}$$

A difference of 7.7 MeV which isn't bad compared to the mean mass splitting which is ≈ 100 MeV in the baryon octet. This formula works well for other baryon multiplets and even meson multiplets provided we modify it to apply to M^2 instead of M because M is the natural measure of energy for particles satisfying DIRAC Eq^n whereas the Klein Gordon Eq^n is quadratic in mass (mesons)

§ 8.10 THE CLEBSCH – GORDAN COEFFICIENTS OF SU(3) (CGC)

Relative probabilities of two reactions taking place within an isospin multiplet are given by their C.G.C.. Likewise to examine reactions within SU(3) multiplets we should find the C.G.C. for SU(3). We know how to figure out the general split of the tensor product $\longrightarrow$ direct sum of irrep. eg.

$$\boxed{\begin{aligned} [3]\otimes[3] &= [6]\oplus[\bar{3}] \\ [3]\otimes[3]\otimes[3] &= [1]\oplus[8]\oplus[8]\oplus[10] \end{aligned}}$$

Let $\alpha = D(P_1, q_1)$ and $\beta = D(P_2, q_2)$ be two irreducible representations of SU(3), which are generated by basis fncts. $\psi_\mu^{(\alpha)}$ and $\psi_\nu^{(\beta)}$ where μ & ν are $(T\ T_3\ Y)$ multindices.

$$\mu = \{y, t, t_3\}$$
$$\nu = \{y', t', t_3'\}$$
$$m = \{Y, T, T_3\}$$

Take N to be dimension of $D(P,q)$. Now construct eigenfncts. of the isospin fom $\psi_\mu^{(\alpha)}$ ~~from~~ and $\psi_\nu^{(\beta)}$ by

$$\boxed{\chi^{(T,T_3)}_{yty't'} = \sum_{t_3 t_3'} (tt'T\,|\,t_3 t_3' T_3)\,\psi^{(\alpha)}_{ytt_3}\,\psi^{(\beta)}_{y't't_3'}}$$

where $(tt'T\,|\,t_3 t_3' T_3)$ are the usual SU(2) C.G.C. Now construct similarly eigenstates of irrep. γ of dimension N

$$\boxed{\psi^{(N\gamma)}_{YTT_3} = \sum_{yty't'} \underbrace{(\alpha\beta\, yty't'\,|\,NYT\gamma)}_{\text{"isoscalar factors"}}\,\chi^{(T,T_3)}_{yty't'}}$$

The isoscalar factors depend on T but not T_3. Also we need that $y+y'=Y$ for nonzero contributions to sum above. Inserting expression for χ into the above

$$\underbrace{(\alpha\beta\, ytt_3 y't't_3'\,|\,Nm\gamma)}_{\text{C.G.C of SU(3)}} = (\alpha\beta\mu\nu\,|\,Nm\gamma)$$
$$= \underbrace{(\alpha\beta\, yty't'\,|\,NYT\gamma)}_{\text{Isoscalar factor}}\underbrace{(tt'T\,|\,t_3t_3'T_3)}_{\text{old SU(2) C.G.C.}}$$

Claim: SU(3). CGC are obtainable from SU(2) C.G.C. and some isoscalar factors.

Goal: construct SU(3) - symmetric wave fncts. [wave fncts. with well-defined SU(3) symmetry] for the simple quark model.

$[3]$ & $[\bar{3}]$ are the smallest nontrivial SU(3) representations

$$[3] \longleftrightarrow \text{quarks} \longleftrightarrow q_1\ q_2\ q_3 \longleftrightarrow u\ d\ s$$

$$[\bar{3}] \longleftrightarrow \text{antiquarks} \longleftrightarrow \bar{q}_1\ \bar{q}_2\ \bar{q}_3 \longleftrightarrow \bar{u}\ \bar{d}\ \bar{s}$$

Physically then $[3]\otimes[\bar{3}] = [8]\oplus[1]$ $\Rightarrow$ quarks/antiquarks compose mesons in the "meson nonet"

$[3]\otimes[3]\otimes[3] = [1]\oplus[8]_1\oplus[8]_2\oplus[10]$ $\Rightarrow$ baryons are made of 3 quarks.

This composition also explains why mesons have isospin $I = 0$ or 1 and why ~~quarks~~ baryons have isospin $I = \frac{1}{2}$ or $\frac{3}{2}$ (since $I = \frac{1}{2}$ for quarks)

- Now consider the Meson Nonent in particular the pion triplet

π^+ must be a $u\bar{d}$ state (π^+ has $T_3 = 1$, $Y = 0$)

Because u has $T_3 = \frac{1}{2}$ and $Y = \frac{1}{3}$ while $\bar{d}$ has $T_3 = \frac{1}{2}$ and $Y = \frac{-1}{3}$ which add together to give what π^+ has. These follow from the table below I guess these are definitions.

	T	T_3	Q	Y
$q_1\ (u)$	$1/2$	$1/2$	$2/3$	$1/3$
$q_2\ (d)$	$1/2$	$-1/2$	$-1/3$	$1/3$
$q_3\ (s)$	0	0	$-1/3$	$-2/3$
$\bar{q}_1\ (\bar{u})$	$1/2$	$-1/2$	$-2/3$	$-1/3$
$\bar{q}_2\ (\bar{d})$	$1/2$	$1/2$	$1/3$	$-1/3$
$\bar{q}_3\ (\bar{s})$	0	0	$1/3$	$2/3$

The isospin function for $\pi^+ \approx u\bar{d}$

$$\chi^{(1,1)}_{\frac{1}{3}\frac{1}{2},\ \frac{-1}{3}\frac{1}{2}} = \underbrace{\left(\tfrac{1}{2}\ \tfrac{1}{2}\ 1 \,\middle|\, \tfrac{1}{2}\ \tfrac{1}{2}\ 1\right)}_{1.}\ \psi^{(3)}_{\frac{1}{3}\frac{1}{2}\frac{1}{2}}\ \psi^{(\bar{3})}_{\frac{-1}{3}\frac{1}{2}\frac{1}{2}}$$

Also the isoscalar factor $(3\bar{3}\ \tfrac{1}{3}\tfrac{1}{2}\ \tfrac{-1}{3}\tfrac{1}{2} \mid 8\,0\,1\,8) = 1$. This is good for all the pion states which have same T.

$$|\pi^0\rangle = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

$$|\pi^-\rangle = d\bar{u}$$

§8.10: Clebsch Gordon Coefficient and Building Particles from [3] and [$\bar{3}$],

Thru arguments outlined on 255 – 256 there are only a ~~few~~ certain way to construct the states found in the meson nonet from quarks,

SU(3) - multiplet	Quark Content	Y	T	T_3	NAME
[8]	$u\bar{d}$	0	1	1	π^+
[8]	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$	0	1	0	π^0
[8]	$d\bar{u}$	0	1	-1	π^-
[8]	$u\bar{s}$	1	½	½	K^+
[8]	$d\bar{s}$	1	½	-½	K^0
[8]	$s\bar{d}$	-1	½	½	$\bar{K}^0$
[8]	$s\bar{u}$	-1	½	-½	K^-
[8]	$1/\sqrt{6}\,(u\bar{u} + d\bar{d} - 2s\bar{s})$	0	0	0	η, η'
[1]	$1/\sqrt{3}\,(u\bar{u} + d\bar{d} - 2s\bar{s})$	0	0	0	η, η'

The coefficients here actually follow from SU(2) CGC because the isoscalar coeff. is just one here.

$T = \frac{1}{2}$ and $Y = 1 \longrightarrow d\bar{s},\ u\bar{s}$

$T = 1$ and $Y = 0 \longrightarrow d\bar{u},\ \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}),\ u\bar{d}$

$T = \frac{1}{2}$ and $Y = -1 \longrightarrow s\bar{u},\ s\bar{d}$

The above set of states reps. SU(3) because SU(3) transformation takes [8] into [8] and [1] into [1].

- **Remark**: I think the way to look at this is just that given [3] and [$\bar{3}$] we can form $[8] \oplus [1]$ thru the combinations above....

§ 8.11 Quark Models with Inner Degrees of Freedom

Upto now our quark model has been a bit naive. We have neglected the spin and angular momentum of the Hadrons and the quarks that compose them. Once quark spin enters the discussion we have 6 states (u_1 spin up u, u_2 is spin down u)

$$\boxed{q = \{u_1, u_2, d_1, d_2, s_1, s_2\}} \quad \text{quarks with spin } \uparrow\downarrow$$

This is the [6] of SU(6), the smallest nontrivial SU(6) rep. We can classify [6] or other SU(6) rep. by their relation to the SU(3) x SU(2) ⊂ SU(6). We denote,

$$[6]_{SU(6)} \longrightarrow [\{3\}, \tfrac{1}{2}]_{SU(3)\times SU(2)}$$

$$[\bar{6}]_{SU(6)} \longrightarrow [\underset{\substack{\uparrow \\ \text{quark} \\ \text{content}}}{\{\bar{3}\}}, \underset{\substack{\uparrow \\ \text{total} \\ \text{spin}}}{\tfrac{1}{2}}]_{SU(3)\times SU(2)}$$

Mesons are formed by $q\bar{q}$ pair so they should be inside the $[6]\otimes[\bar{6}]$ which can have spin 0 or 1 giving the psuedo-scalar or vector mesons

$$[6]\otimes[\bar{6}] = [1]\oplus[35]$$

$$[\{3\}, \tfrac{1}{2}]\otimes[\{\bar{3}\}, \tfrac{1}{2}] = [\{1\}, 0]\oplus[\{1\}, 1]\oplus\underbrace{[\{8\}, 1]}_{\substack{\text{vector} \\ \text{mesons}}}\oplus\underbrace{[\{8\}, 0]}_{\substack{\text{pseudo scalar} \\ \text{mesons}}}$$

§ 8.11 Quarks with Spin

Baryons as an example $[6]\otimes[6]\otimes[6] = [20]\oplus[56]\oplus[70]\oplus[70]$ (huh?)
Instead view these in terms of the $\{\text{flavor}\}\otimes\{\text{spin}\}\subset SU(6)$ where $[6]\otimes[6]\otimes[6]$ becomes,

$$[\{3\},\tfrac{1}{2}]\otimes[\{3\},\tfrac{1}{2}]\otimes[\{3\},\tfrac{1}{2}] = ([3]\otimes[3]\otimes[3])\otimes([2]\otimes[2]\otimes[2])$$
$$= ([10]\oplus[8]\oplus[8]\oplus[1])\otimes([4]\oplus[2]\oplus[2])$$
$$= [\{10\},\tfrac{3}{2}]\oplus[\{8\},\tfrac{3}{2}]\oplus[\{1\},\tfrac{3}{2}]$$
$$\oplus[\{10\},\tfrac{1}{2}]\oplus[\{8\},\tfrac{1}{2}]\oplus[\{1\},\tfrac{1}{2}]$$
$$\oplus[\{10\},\tfrac{1}{2}]\oplus[\{8\},\tfrac{1}{2}]\oplus[\{1\},\tfrac{1}{2}]$$

We've already encountered the baryon octet & decuplet which are

$$[\{8\},\tfrac{1}{2}]\oplus[\{10\},\tfrac{3}{2}]$$

These are $16\oplus 40$ states form 56 <u>symmetric</u> states of 3-quarks. Only these are physical because they together with the antisymmetric <u>color</u> wavefnct. lead to an <u>antisymmetric total wave fnct</u> which is req^{d} for a massive fermion which is after all what we're constructing.

$$\psi^{(3)}_{Ytt_3} \longmapsto \psi^{(3)}_{Ytt_3}\,\chi_{\frac{1}{2}\mu}$$
$$\psi^{(\bar{3})}_{Ytt_3} \longmapsto \psi^{(\bar{3})}_{Ytt_3}\,\chi_{\frac{1}{2}\nu}$$

} wavefncts discussed before times a spin piece χ.

Thru some simple C.G.C. arguments on pg. 259 → 260 we arive at

$$\boxed{\begin{aligned}
|\pi^+\rangle &= \tfrac{1}{\sqrt{2}}(u_\uparrow \bar{d}_\downarrow - u_\downarrow \bar{d}_\uparrow)\\
|\pi^\circ\rangle &= \tfrac{1}{2}(u_\uparrow \bar{u}_\downarrow - u_\downarrow \bar{u}_\uparrow - d_\uparrow \bar{d}_\downarrow + d_\downarrow \bar{d}_\uparrow)\\
|\pi^-\rangle &= \tfrac{1}{\sqrt{2}}(d_\uparrow \bar{u}_\downarrow - d_\downarrow \bar{u}_\uparrow)\\
|\rho^+_{\pm1}\rangle &= (u_{\uparrow\downarrow}\bar{d}_{\uparrow\downarrow})\\
|\rho^+_\circ\rangle &= \tfrac{1}{\sqrt{2}}(u_\uparrow \bar{d}_\uparrow + u_\downarrow \bar{d}_\uparrow)\\
|\rho^\circ_{\pm1}\rangle &= \tfrac{1}{\sqrt{2}}(u_{\uparrow\downarrow}\bar{u}_{\uparrow\downarrow} - d_{\uparrow\downarrow}\bar{d}_{\downarrow\uparrow})\\
|\rho^\circ_\circ\rangle &= \tfrac{1}{2}(u_\uparrow \bar{u}_\downarrow + u_\downarrow \bar{u}_\uparrow - d_\uparrow \bar{d}_\downarrow - d_\downarrow \bar{d}_\uparrow)\\
|\rho^-_{\pm1}\rangle &= (d_{\uparrow\downarrow}\bar{u}_{\uparrow\downarrow})\\
|\rho^-_\circ\rangle &= \tfrac{1}{\sqrt{2}}(d_\uparrow \bar{u}_\downarrow + d_\downarrow \bar{u}_\uparrow
\end{aligned}}$$

§ 8.11 — Quarks with Spin

$$\Delta^{++}_{jz} = \begin{cases} \Delta^{++}_{+\frac{3}{2}} = u_\uparrow u_\uparrow u_\uparrow \\ \Delta^{++}_{+\frac{1}{2}} = \frac{1}{\sqrt{3}}(u_\uparrow u_\uparrow u_\downarrow + u_\uparrow u_\downarrow u_\uparrow + u_\downarrow u_\uparrow u_\uparrow \end{cases}$$

This would seem to violate the Pauli-principal, hence we say quarks have color (red, green, blue). See pgs 262-263 for more detailed argument...

$$|P_\uparrow\rangle = \frac{1}{\sqrt{18}}\Big[2|u_\uparrow d_\downarrow u_\uparrow\rangle + 2|u_\uparrow u_\uparrow d_\downarrow\rangle + 2|d_\downarrow u_\uparrow u_\uparrow\rangle$$
$$- |u_\uparrow u_\downarrow d_\uparrow\rangle - |u_\uparrow d_\uparrow u_\downarrow\rangle - |u_\downarrow d_\uparrow u_\downarrow\rangle$$
$$- |d_\uparrow u_\downarrow u_\uparrow\rangle - |d_\uparrow u_\uparrow u_\downarrow\rangle - |u_\downarrow u_\uparrow d_\uparrow\rangle \Big]$$

Before we might have simply said $P = uud$ but physically we need the symmetrization above plus color to deal with spin & pauli-principle.

$$|n_\downarrow\rangle = \frac{1}{\sqrt{18}}\left[2 d_\uparrow d_\uparrow u_\downarrow - d_\uparrow d_\downarrow u_\uparrow - d_\downarrow d_\uparrow u_\uparrow + \text{permut.} \right]$$

I've done these calculations before in PY 507 I think.

pgs. 264–266 show how to get $|n_\uparrow\rangle$ from $\hat{T}_- |P_\uparrow\rangle$ since isospin relates P and n as we said from the beginning. Pgs. 267–275 explain why the baryon resonances have quark structure

$$\boxed{\begin{aligned} |\Delta^{++}\rangle &= u_\uparrow u_\uparrow u_\uparrow \\ |\Delta^{+}\rangle &= \tfrac{1}{\sqrt{3}}(d_\uparrow u_\uparrow u_\uparrow + u_\uparrow d_\uparrow u_\uparrow + u_\uparrow u_\uparrow d_\uparrow \\ |\Delta^{0}\rangle &= \tfrac{1}{\sqrt{3}}(d_\uparrow d_\uparrow u_\uparrow + d_\uparrow u_\uparrow d_\uparrow + u_\uparrow d_\uparrow d_\uparrow \\ |\Delta^{-}\rangle &= d_\uparrow d_\uparrow d_\uparrow \\ |\Sigma^{*0}\rangle &= \tfrac{1}{\sqrt{3}}(s_\uparrow u_\uparrow u_\uparrow + u_\uparrow s_\uparrow u_\uparrow + u_\uparrow u_\uparrow s_\uparrow) \\ &\;\vdots \\ |\Omega^{-}\rangle &= s_\uparrow s_\uparrow s_\uparrow \end{aligned}}$$

See pg. 275 from ex. 8.14

Then 8.15 argues how to construct the baryon octet with spin from the proton (the highest weight state) on down using the symmetrizer to insure the wave fnct. of flavor/spin is symmetric with exchange of quarks. Pgs. 283–286 discuss mass degeneracy/splitting in SU(6) and magnetic moments which are well explained by the quark model. Pg. 286–287 considers possibility of $qqq\bar{q}$ or $qqqq\bar{q}$ states to explain high spin resonances with $\ell = 0$ quarks but then we find qqq with $\ell \neq 0$ can better explain on 288–289, color finishes story later.