EXAMPLES OF SEQUENCES

1.) If $a_n = \frac{(-1)^{n+1}}{n^2+1}$ then the first 5 terms starting from $n=1$ is,

$$a_1 = \frac{1}{2}, \quad a_2 = \frac{-1}{5}, \quad a_3 = \frac{1}{10}, \quad a_4 = \frac{-1}{17}, \quad a_5 = \frac{1}{26}$$

2.) If $a_1 = 6$ and $a_{n+1} = 5a_n - 3$ then we can calculate the next terms as follows,

$$a_2 = 5a_1 - 3 = 30 - 3 = 27$$

$$a_3 = 5a_2 - 3 = 5(27) - 3 = 132$$

$$a_4 = 5a_4 - 3 = 5(132) - 3 = 657$$

Remark: not having an explicit formula for a_n does not stop us from making meaningful analysis. The recursively defined a_n above illustrates this point.

3.) $a_n = \frac{n}{n!}$ for

We define $n!$ by $0! = 1$, $1! = 1$, $\underbrace{n! = n(n-1)!}_{\text{for } n \geq 2}$

$$2! = 2 \cdot 1! = 2$$

$$3! = 3 \cdot 2! = 6$$

$$4! = 4 \cdot 3! = 24$$

$$a_0 = \frac{0}{0!} = \frac{0}{1} = 0.$$

$$a_1 = \frac{1}{1!} = \frac{1}{1} = 1.$$

$$a_2 = \frac{2}{2!} = \frac{2}{2} = 1.$$

$$a_3 = \frac{3}{3!} = \frac{3}{6} = \frac{1}{2}$$

$$a_4 = \frac{4}{4!} = \frac{4}{24} = \frac{1}{6}$$

In fact, $a_n = \begin{cases} 0 & : n = 0 \\ \frac{1}{(n-1)!} & : n \geq 1 \end{cases}$

Remark: $n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ and $0! = 1$, $1! = 1$ describes the factorial. In view of this 3. could be done by algebra instead of pattern matching,

$$a_n = \frac{n}{n!} = \frac{n}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} \overset{\text{if } n \neq 0}{=} \frac{1}{\underbrace{(n-1)(n-2)\cdots 3\cdot 2\cdot 1}_{(n-1)!}}$$

whereas when $n = 0$,

$$a_0 = \frac{0}{0!} = \frac{0}{1} = 0.$$

It's important to remember $0! = 1$ by definition and thus $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ is only for $n \geq 1$. The factorial is key to many central results in this course.

4.) $a_n = \cos(\pi n)$, $n \geq 0$

$$\{a_n\}_{n=0}^{\infty} = \{a_0, a_1, a_2, \ldots\}$$
$$= \{\cos 0, \cos \pi, \cos 2\pi, \cos 3\pi, \cos 4\pi, \ldots\}$$
$$= \{1, -1, 1, -1, 1, \ldots\}$$

In fact, $\cos(\pi n) = (-1)^n$ in contrast $\sin(\pi n) = 0$.

5.) Find formula a_n for $n \geq 2$ such that

$$\{a_n\}_{n=2}^{\infty} = \{1, -4, 9, -16, 25, -36, \ldots\}$$

 $\boxed{a_n = (-1)^n (n-1)^2}$

Remark: finding a_n from list requires thinking. There is no algorithm.

6.) $\{a_n\}_{n=1}^{\infty} = \{3, 5, 7, 9, 11, \ldots\} \Rightarrow \boxed{a_n = 2n+1}$

EXAMPLES OF SEQUENCES : LIMITS

- One important idea to keep in mind is for <u>continuous</u> $f(x)$ such that $\lim_{x\to\infty} f(x) = L$ then $\lim_{n\to\infty} f(n) = L$ and this extends to case $L = \pm\infty$ or more ugly d.n.e. This means we can use ℓ-Hop. rule for many sequences if there exists an <u>appropriate</u> continuous function f which extends a_n via the rule $f(n) = a_n$. By "appropriate" I mean the choice of $f(x)$ allows <u>calculus</u> with <u>ease</u>.

7.) $\lim_{n\to\infty}(ne^{-n}) = \lim_{n\to\infty}\left(\frac{n}{e^n}\right)$: extending n-continuously apply ℓ-Hop rule next

$$\overset{\ell}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty}\left(\frac{1}{e^n}\right)$$

$$= \boxed{0}$$

Remark

The notation above is equivalent to writing $f(x) = xe^{-x}$ and noting $\lim_{x\to\infty}(f(x)) = \lim_{x\to\infty}\left(\frac{x}{e^x}\right) \overset{\ell}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{x\to\infty}\left(\frac{1}{e^x}\right) = 0$

thus $\lim_{n\to\infty} f(n) = \lim_{x\to\infty} f(x) = 0.$

8.) $\lim_{n\to\infty}\left(\frac{n^2+4n+6}{ne^n}\right) \overset{\ell}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty}\left(\frac{2n+4}{(n+1)e^n}\right)$: extending n-continuously

$$\overset{\ell}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{n\to\infty}\left(\frac{2}{(n+2)e^n}\right) = \boxed{0}$$

9.) $\lim_{n\to\infty} \left(1 + 10^n(-3)^{2n}\right) = \lim_{n\to\infty}\left(1 + \frac{10^n}{((-3)^2)^n}\right)$

$$= \lim_{n\to\infty}\left(1 + \frac{10^n}{9^n}\right)$$

$$= \lim_{n\to\infty}\left(1 + \left(\frac{10}{9}\right)^n\right)$$

$$= \boxed{\infty.}$$

10.) $\lim_{n\to\infty}\left(\frac{2^n + 3^n}{4^n}\right) = \lim_{n\to\infty}\left(\frac{2^n}{4^n} + \frac{3^n}{4^n}\right)$

$$= \lim_{n\to\infty}\left[\frac{1}{2^n} + \left(\frac{3}{4}\right)^n\right]$$

$$= \boxed{0.}$$

Th^m/ Every bounded monotonic sequence is convergent

Certain problems are best approached via the Th^m above.

11.) Let $0 < r < 1$ and consider $a_n = r^n$

observe $0 < r^n < 1$ thus $\{a_n\}_{n=1}^{\infty}$ is bounded below by 0 and above by 1 $\therefore$ $\{r^n\}_{n=1}^{\infty}$ is bounded.

Also,

$$a_{n+1} = r^{n+1} = r r^n = r a_n < a_n$$

thus $\{a_n\}_{n=1}^{\infty}$ is decreasing. It follows the sequence converges to its greatest lower bound of 0.

[usually we simply assume the result of 11.) is known, in fact,

$$\lim_{n\to\infty}(r^n) = \begin{cases} 0 & : \ |r| < 1 \\ 1 & : \ r = 1 \\ \text{d.n.e.} & : \ r \leq -1 \\ \infty & : \ r > 1 \end{cases}$$
]

12.) Consider $a_n = \frac{1-n}{2+n}$. Let $f(x) = \frac{1-x}{2+x}$

notice $\frac{df}{dx} = \frac{-1(2+x) - (1-x)(1)}{(2+x)^2} = \frac{-3}{(2+x)^2} < 0$

for $x \geq 1$ we find $f(x)$ is decreasing thus the corresponding sequence $a_n = f(n) = \frac{1-n}{2+n}$ is likewise decreasing. Furthermore,

$$a_n = \frac{1-n}{2+n} = \frac{3-(2+n)}{2+n} \quad \text{(algebra !)}$$
$$= \frac{3}{2+n} - 1$$

Clearly, $\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}\left(\underbrace{\frac{3}{2+n}}_{\to 0} - 1\right) = -1.$

This is in good agreement with the Bounded Monotonic Sequence Th$^{\underline{m}}$. Observe a <u>decreasing</u> sequence has $a_1 \geq a_2 \geq a_3 \geq \cdots$ thus a_1 serves as an <u>upper bound</u> for $\{a_n\}_{n=1}^{\infty}$. Here $a_1 = \frac{0}{3} = 0$.

The lower bound for a_n which is largest is -1

$$\underbrace{-1 < -1 + \frac{3}{2+n}}_{\text{this is clear as } \frac{3}{2+n} > 0 \text{ for } n \geq 1.} = a_n \leq 0$$

<u>Remark:</u> the point of 12.) is not to simply calculate $\lim_{n\to\infty}\left(\frac{1-n}{2+n}\right)$, rather the point is to illustrate the ideas/concepts present in the BOUNDED MONOTONIC SEQ. Th$^{\underline{m}}$.

Btw, -1 is the <u>greatest lower bound</u> for a_n.