Sequences and Series

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Abstract

In this article we study sequences and series. We study how to formulate sequences as a list following a pattern, or by a formula, or even by some recursive rule. For example, the factorial sequence is defined by the recursive rule $n! = n(n-1)!$ where $0! = 1$. We study limits of sequences and present limit laws which are in strong analogy to our previous work on limit laws in first semester calculus and we also learn how to borrow from the calculus of a continuous variable by a simple correspondence theorem. In short, we can extend sequences to a function of a real variable and apply calculus techniques to the extension. We also studied the bounded monotonic sequence theorem which gives us a method to calculate limits of recursively defined sequences.

Series are formed from adding the terms in a sequence. We use sequences to define series. In particular, a series **converges** or is **summable** if its **sequence of partial sums** converges. Usually the direct calculation of a series is an insurmountable task. There are just a few nice example where we can concretely calculate the sum of a summable series. Typical examples where the sum can be explicitly calculated include geometric series, telescoping series and series which correspond to a Riemann sum of an explicitly integrable function. Since summable series forbid the actual calculation of the sum it is important to understand a number of indirect method which affirm or deny the summability of a given series. The situation is much like you have already faced with integration. Consider this, **any** continuous function f on $[a, b]$ has a well-defined area function thus by the FTC I the area function is an antiderivative. But, can you find F for which $\frac{dF}{dx} = f$? You might say, the area function, yes, that is true, but can you find a formula for the antiderivative which is not based on calculating an infinite Riemann sum ? So the story goes for series, except, the criteria for a series to be summable is much more subtle than mere continuity in my analogy. It will take us about a week of lectures to just to detail the theory which allows us to decide the summability of a series. Understanding and implementing theorems is the main task of this material. We have to understand the theorems and know how and when to use them. However, first things first, we must understand the definition of a series and what we mean when we say it is summable.

1 Sequences

We begin by defining sequences of real numbers. Many texts define a real sequence as a function from $\mathbb{N} = \{1, 2, 3, \dots\}$ to $\mathbb{R}$. I'll give a slightly less elegant definition which reflects our actual practice; the sequence can start at any $n_o \in \mathbb{N}$.

Definition 1.1. *Sequences*

Let $S = \{n_o, n_o + 1, \dots\} \subseteq \mathbb{Z}$. A function $a : S \to \mathbb{R}$ is called a **sequence**. We denote $a(n) = a_n$ and we refer to a_n as the **n-th term in the sequence**. Alternatively, we also denote the sequence by $\{a_n\}$ or by an explicit list of values:

$$\{a_n\}_{n=n_o}^{\infty} = \{a_{n_o}, a_{n_o+1}, \dots\}.$$

There are various ways to define a sequence. I'll illustrate with a few examples.

Example 1.2. *If* $a_n = n^2$ *for* $n \in \mathbb{N}$ *then* $\{a_n\} = \{1, 4, 9, 16, \dots\}$

Example 1.3. *If* $\{a_n\}_{n=1}^{\infty} = \{3, 4, 5, 6, \dots\}$ *then* $a_n = n + 2$ *for* $n = 1, 2, \dots$.
Alternatively, we can write $\{b_k\}_{k=3}^{\infty} = \{3, 4, 5, 6, \dots\}$ *then* $b_k = k$ *for* $k = 3, 4, \dots$.

Example 1.4. *Let* a_n *for* $n = 0, 1, \dots$ *be defined* **recursively** *as follows* $a_0 = 1$*,* $a_1 = 1$ *and* $a_{n+1} = na_n$ *for* $n = 1, 2, \dots$. *The standard notation for this sequence is* $a_n = n!$*, which is read as* n**-factorial**. *This sequence is grows very large very quickly:*

$$0! = 1, \;\; 1! = 1, \;\; 2! = 2, \;\; 3! = 6, \;\; 4! = 24, \;\; 5! = 120, \;\; 6! = 720, \;\; 7! = 5,040, \;\; 8! = 40,320$$

$$50! = 30414093201713378043612608166064768844377641568960512000000000000$$

Example 1.5. *Let* $a_n = cr^n$ *for* $n = 0, 1, \dots$ *where* r, c *are nonzero constants;* $\{a_n\} = \{c, cr, cr^2, \dots\}$. *Such a sequence is called a* **geometric sequence**. *Notice* $a_{n+1}/a_n = (cr^{n+1})/(cr^n) = r$. *Infact, it is possible to define the geometric sequence recursively;* $a_0 = c$ *and* $a_n = ra_{n-1}$ *for all* $n \geq 1$.

Example 1.6. *Consider,* $\{3, 6, 12, 24, 48, \dots\}$ *is geometric with* $c = 3$ *and* $r = 2$ *since*

$$2 = 6/3 = 12/6 = 24/12 = 48/24$$

and the first term is $c = 3$.

Example 1.7. *Let* a_n *be given by the decimal representation of* π *given to the* n*-th decimal place for* $n = 1, 2 \dots$. *Then* $\{a_n\} = \{3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, \dots\}$

In the example above, the limit of the sequence is simply π and we can write $a_n \to \pi$ as $n \to \infty$. We should define the limit of a sequence carefully:

Definition 1.8. *Limits of Sequences*

If for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ for which $n > N$ implies $|a_n - L| < \varepsilon$ then we say the limit of a_n is L and we denote this by $a_n \to L$ as $n \to \infty$. Equivalently, we write $\lim_{n\to\infty} a_n = L$. A sequence which has a limit is known as a **convergent sequence**. If the sequence does not converge then the sequence is said to **diverge**.

What this definition is saying is that a sequence converges to L then all the terms in the sequence get close to L if we go far enough out in the sequence.

Example 1.9. *Let's prove* $\lim_{n\to\infty} \frac{1}{n^2} = 0$*. Let* $\varepsilon > 0$ *and choose* $N \in \mathbb{N}$ *for which* $N > \frac{1}{\sqrt{\varepsilon}}$*. If* $n \in \mathbb{N}$ *and* $n > N > \frac{1}{\sqrt{\varepsilon}}$ *then* $n^2 > \frac{1}{\varepsilon}$ *implies* $\frac{1}{n^2} < \varepsilon$*. Thus*

$$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} < \varepsilon.$$

Therefore, $\frac{1}{n^2} \to 0$ *as* $n \to \infty$*.*

If $p > 0$ then we could make a similar argument to that given above to prove $\lim_{n\to\infty} \frac{1}{n^p} = 0$. Convergent sequences are necessarily **bounded**. To say $\{a_n\}_{n=n_o}^{\infty}$ is bounded means there exists $m, M \in \mathbb{R}$ for which $m \leq a_n \leq M$ for all $n \geq n_o$. Equivalently, $\{a_n\}$ is bounded if and only if there exists M for which $|a_n| \leq M$ for all n.

Theorem 1.10. *convergent sequences are bounded*

If $\{a_n\}$ is a convergent sequence then $\{a_n\}$ is bounded.

Proof: Consider the sequence $\{a_n\}_{n=n_o}^{\infty}$ for which $a_n \to L$ as $n \to \infty$. Let $\varepsilon = 1$ then note there exists $N \in \mathbb{N}$ for which $|a_n - L| < 1$ whenever $n > N$. Thus,

$$-1 < a_n - L < 1 \quad \Rightarrow \quad L - 1 < a_n < L + 1.$$

for each $n \in \mathbb{N}$ with $n > N$. Define

$$m = min(L - 1, a_{n_o}, a_{n_o+1}, \dots, a_N)$$

$$M = max(L - 1, a_{n_o}, a_{n_o+1}, \dots, a_N)$$

then we find $m \leq a_n \leq M$ for all $n \in \mathbb{N}$ with $n \geq n_o$. $\square$

Logically, if a sequence is not bounded then it cannot be convergent. However, there are sequences which are bounded and yet do not converge.

Example 1.11. *Let* $a_n = (-1)^{n+1}$ *for* $n \in \mathbb{N}$*. Notice* $-1 \leq a_n \leq 1$ *for all* n*, hence this is a bounded sequence. Note* $a_{2k} = (-1)^{2k+1} = -1$ *whereas* $a_{2k-1} = (-1)^{2k-1+1} = (-1)^{2k} = 1$*. For this sequence the* **even subsequence** *is the constant sequence* $-1, -1, \dots$ *whereas the* **odd subsequence** *is the constant sequence* $1, 1, \dots$*. Naturally* $a_{2k} \to -1$ *whereas* $a_{2k-1} \to 1$ *as* $k \to \infty$*. It follows the limit of* a_n *does not exist.*

A useful strategy for showing a sequence diverges is illustrated by the example above; if we can find two subsequences which converge to different values then it follows that the given sequence diverges. On the other hand, if the bounded sequence is also *monotonic* then convergence of the sequence is inevitable.

Definition 1.12. *Monotonic Sequences*

We say the sequence $\{a_n\}_{n=n_o}^{\infty}$ is strictly increasing if $n_o \leq n < m$ implies $a_n < a_m$. We say the sequence $\{a_n\}_{n=n_o}^{\infty}$ is strictly decreasing if $n_o \leq n < m$ implies $a_n > a_m$. If a sequence is strictly increasing or strictly decreasing then the sequence is said to be **monotonic**.

The proof of this theorem belongs to real analysis[1], but we will apply it in this course.

[1]the proof of this relies on the completeness of the real numbers. Moreover, this is an abbreviation of the full theorem which also claims the limit is given by the supremum or infimum of the set of upper or lower bounds for the sequence.

Theorem 1.13. *Bounded Monotonic Sequence Theorem*

A bounded monotonic sequence converges. Furthermore,

(1.) if $\{a_n\}$ is increasing and $a_n \leq M$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \leq M$,

(2.) if $\{a_n\}$ is decreasing and $a_n \geq m$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \geq m$.

Let us see how this helps us find limits of recursively defined sequences.

Example 1.14. *Consider the geometric sequence with $0 < r < 1$ and $c = 1$. In particular, we define a_n recursively by $a_0 = 1$ and $a_n = ra_{n-1}$ for $n \geq 1$. Notice $a_n = ra_{n-1} < a_{n-1}$ implies $a_n < a_m$ whenever $m > n$. It is clear the sequence is strictly decreasing. We note $0 < cr^n < c$ for all $n \geq 0$ thus $\{a_n\}$ is bounded. Thus $a_n \to L \in \mathbb{R}$ by the Bounded Monotonic Sequence Theorem. To find the value of L we take the limit[2] of the recursion rule which defined the sequence:*

$$\lim_{n\to\infty}(a_n) = \lim_{n\to\infty}(ra_{n-1}) = r\lim_{n\to\infty} a_{n-1} \quad \Rightarrow \quad L = rL \quad \Rightarrow \quad L(r-1) = 0$$

thus $L = 0$ since $r \neq 1$.

Example 1.15. *Let $a_1 = \sqrt{2}$ and define $a_n = \sqrt{2a_{n-1}}$ for $n = 2, 3, \dots$.*

$$a_2 = \sqrt{2\sqrt{2}} = 1.6818, \;\; a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = 1.8340, \;\; a_4 = \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}} = 1.9152$$

continuing in this fashion we can approximate

$$a_5 = 1.9571, \;\; a_6 = 1.9785, \;\; a_7 = 1.9892, \;\; a_8 = 1.9946$$

We can guess $a_n \to 2$ from the data we've collected so far. We argue 2 serves as an upper bound for a_n. Observe $a_1 = \sqrt{2} < 2$. Suppose $a_n < 2$ and observe

$$a_{n+1} = \sqrt{2a_n} < \sqrt{2(2)} = 2$$

thus $a_n < 2$ for each $n \in \mathbb{N}$ by mathematical induction[3]. If $\lim_{n\to\infty} a_n = L$ then we also know $\lim_{n\to\infty} a_{n-1} = L$. Hence, as $n \to \infty$,

$$a_n = \sqrt{2a_{n-1}} \to L = \sqrt{2L}.$$

Algebra finishes the job here, $L^2 = 2L$ gives $L(L-2) = 0$ hence either $L = 0$ or $L = 2$. But, since the terms in the sequence are increasing and positive we find $L = 2$.

Sometimes we can use calculus to help verify a bound for a given sequence, the next example illustrates such a technique.

Example 1.16. *Consider $a_n = \sqrt{n+1} - \sqrt{n}$. Let $f(x) = \sqrt{x+1} - \sqrt{x}$. Observe for $x \geq 1$ we find:*

$$\frac{df}{dx} = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} < 0$$

[2] forgive me for using limit law **(2.)** before its official announcement in this article, look ahead to Theorem 1.22

[3] proof by mathematical induction requires we verify the base-step is true and that if the claim is true for n then the claim likewise follows for $n+1$. The claim in this example was $a_n < 2$. Anytime we want to prove something for all $n \in \mathbb{N}$ it is likely that a proof by induction is technically required.

Therefore, if $n < m$ then $f(n) > f(m)$ and hence $a_n > a_m$. Thus $\{a_n\}$ is strictly decreasing. Furthermore, since $g(x) = \sqrt{x}$ has $g'(x) = \frac{1}{2\sqrt{x}} > 0$ for $x > 0$ we likewise find the squareroot function is a strictly increasing function. Note $n < n+1$ thus implies $\sqrt{n} < \sqrt{n+1}$ which means $0 < a_n = \sqrt{n+1} - \sqrt{n}$. Thus $\{a_n\}$ is a bounded monotonic sequence which must converge. In fact, a bit more algebra would have already revealed the limit is exactly 0:

$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0$$

I should admit, unlike the last example, the Bounded Monotonic Sequence Theorem is not really needed to solve this limit. The purpose of this example is to explore the ideas, not to coach you in optimally efficient calculation.

Often the divergence fits into the categories defined below:

Definition 1.17. *Sequences diverging to $\pm\infty$*

If for each $M > 0$ there exists $N \in \mathbb{N}$ for which $a_n > M$ for all $n > N$ then we write $a_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$. If for each $M < 0$ there exists $N \in \mathbb{N}$ for which $a_n < M$ for all $n > N$ then we write $a_n \to -\infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = -\infty$.

Example 1.18. *Let's prove $\lim_{n\to\infty} n^2 = \infty$. Suppose $M > 0$ and let $N \in \mathbb{N}$ be the next integer after $\sqrt{M}$. By construction, $N \geq \sqrt{M}$. If $n > N$ then $n > \sqrt{M}$ thus $n^2 > M$ and we conclude $\lim_{n\to\infty} n^2 = \infty$.*

If $p > 0$ then we could make a similar argument to that given above to prove $\lim_{n\to\infty} n^p = \infty$. There is an obvious parallel between the limit at infinity of a sequence and the limit at infinity for a function. I'll state this theorem without proof.

Theorem 1.19. *exchange with continuous limit*

If f is a continuous function on $[n_o, \infty)$ and $\{a_n\}_{n=n_o}^{\infty}$ is a sequence for which $f(n) = a_n$ for all $n = n_o, n_o + 1, \dots$ then $\lim_{n\to\infty} a_n = L$ if and only if $\lim_{x\to\infty} f(x) = L$ where is either a real value or $\pm\infty$.

The variable n of a sequence is known as a **discrete variable** because it take values which jump from one integer to another. In contrast, if x is a real variable then we can call it a **continuous variable** since it can take on a continuous range of values we picture on the real number line. One way to call on the theorem above is simply to write that we are *extending n to be a continuous variable*. It is important theoretically to make this logical step before we use the tool of L-Hopital's Rule since it is formal nonsense to differentiate the discrete variable n. I'll illustrate both formalisms in the examples which follow next:

Example 1.20. *Consider $a_n = \tan^{-1}(n)$ for $n \in \mathbb{N}$. If $f(x) = \tan^{-1}(x)$ then f is continuous on $\mathbb{R}$ and $f(n) = \tan^{-1}(n) = a_n$ for each $n \in \mathbb{N}$. Since the graph $y = f(x)$ has horizontal asymptote $y = \frac{\pi}{2}$ we find $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$ thus $\lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2}$.*

Example 1.21. *Consider $a_n = ne^{-n}$ for $n \in \mathbb{N}$. Extend n continuously to be a real variable and observe*

$$a_n = ne^{-n} = \frac{n}{e^n} \to \frac{\frac{d}{dn}(n)}{\frac{d}{dn}(e^n)} = \frac{1}{e^n} \to 0$$

as $n \to \infty$ by the application of L'Hopital's Rule on type ∞/∞.

Limit laws for sequential limits should be familar from their analogs for continuous limits.

Theorem 1.22. *sequential limit laws*

Let a_n, b_n be convergent real sequences for which $a_n \to A$ and $b_n \to B$ as $n \to \infty$ then

(1.) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty}(a_n) \pm \lim_{n\to\infty}(b_n)$,

(2.) if $c \in \mathbb{R}$ then $\lim_{n\to\infty}(ca_n) = c\lim_{n\to\infty}(a_n)$,

(3.) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty}(a_n)\lim_{n\to\infty}(b_n)$,

(4.) if $b_n \neq 0$ for all n then $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty}(a_n)}{\lim_{n\to\infty}(b_n)}$,

I will likely prove **(1.)** and (**2.**) in lecture. Proofs of (3.) and (4.) are more challenging. The proof of the squeeze theorem for sequences is also left to the reader:

Theorem 1.23. *sequential limit squeeze theorem*

If $\{b_n\}, \{a_n\}, \{c_n\}$ are sequences for which there exists $M > 0$ for which $n > M$ implies $b_n \leq a_n \leq c_n$ and if $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L \in \mathbb{R}$ then $\lim_{n\to\infty} a_n = L$.

Example 1.24. *Suppose $|a_n| \to 0$ as $n \to \infty$. Observe*

$$-|a_n| \leq a_n \leq |a_n|$$

for all n. Since $|a_n| \to 0$ and $-|a_n| \to 0$ as $n \to \infty$ the sqeeze theorem provides $a_n \to 0$ as $n \to \infty$.

Example 1.25. *Let $a_n = (-1)^{n+1}/n$. Observe $|a_n| = 1/n \to 0$ as $n \to \infty$. Thus by the previous example, $a_n \to 0$ as $n \to \infty$.*

Example 1.26. *Following Example 1.14 we now $a_n = r^n$ for $n = 0, 1, \dots$ where $r \in \mathbb{R}$. If $-1 < r < 0$ then observe $|a_n| = |r|^n = (-r)^n$ where $0 \leq -r < 1$ thus by Example 1.14 we find $|a_n| \to 0$ hence $a_n \to 0$. If $r = 0$ then the sequence has the form $\{1, 0, 0, 0 \dots\}$ which clearly has limit 0. Thus, in summary, $r^n \to 0$ whenever $-1 < r < 1$. Next, if $r = 1$ then $r^n = 1$ for $n \geq 0$ thus the limit is clearly 1. If $|r| > 1$ then the sequence is not bounded thus a_n diverges. In particular, if $r > 1$ then $r^n \to \infty$ whereas if $r < -1$ then the limit of r^n does not exist due to oscillation. In summary:*

$$\lim_{n\to\infty}(r^n) = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \\ d.n.e. & \text{if } r \leq -1 \end{cases}$$

Example 1.27. *Let $a_n = \frac{5^n}{n!}$. This limit is not obvious because both the numerator and denominator grow without bound. Observe for $n \geq 6$,*

$$0 \leq \frac{5^n}{n!} = \frac{5}{1}\cdot\frac{5}{2}\cdot\frac{5}{3}\cdot\frac{5}{4}\cdot\frac{5}{5}\cdot\underbrace{\frac{5}{6}\frac{5}{7}\cdots\frac{5}{n-2}\frac{5}{n-1}}_{\text{each factor is most 1}}\frac{5}{n} \leq \frac{5^4}{4!}\frac{5}{n}$$

Therefore, as $\frac{5}{n} \to 0$ and $0 \to 0$ as $n \to \infty$ we find $\frac{5^n}{n!} \to 0$ as $n \to \infty$ by the squeeze theorem.

Example 1.28. *Consider $a_n = \frac{R^n}{n!}$. We can prove the limit is zero by a clever*[4] *application of the squeeze theorem. We begin by supposing $R > 0$ and choosing the positive integer M for which $M \leq R < M+1$. Such an integer clearly exists for each real number. If you wish, consider the decimal expansion of R, setting all the decimals to zero yields M. Notice, for $n > M$ we have that:*

$$0 \leq \frac{R^n}{n!} = \underbrace{\left(\frac{R}{1}\frac{R}{2}\cdots\frac{R}{M}\right)}_{\text{let this constant be } C} \underbrace{\left(\frac{R}{M+1}\frac{R}{M+2}\cdots\frac{R}{n}\right)}_{\text{each factor smaller than } 1} \leq C\frac{R}{n} \to 0$$

as $n \to \infty$. Thus, by squeeze theorem, $\frac{R^n}{n!} \to \infty$ as $n \to \infty$. If $R = 0$ the sequence is the constant sequence 0 for $n \geq 1$ hence it limits to zero. If $R < 0$ then $\left|\frac{R^n}{n!}\right| = \frac{|R|^n}{n!} \to 0$ by our previous argument since $|R| > 0$. Thus $\lim_{n\to\infty}\frac{R^n}{n!} = 0$ for each $R \in \mathbb{R}$.

Often the following theorem is very helpful in the calculation of sequential limits:

Theorem 1.29. *composition of limits with continuous function*

> If f is a continuous function at $L \in \mathbb{R}$ and a_n is a real sequence for which $a_n \to L$ as $n \to \infty$ then $f(a_n) \to f(L)$ as $n \to \infty$. In other words, $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right)$.

Continuous functions allow us to pass the limit of a convergent sequence inside the argument of the function.

Example 1.30. *Observe the sine function is continuous on $\mathbb{R}$ hence:*

$$\lim_{n\to\infty} \sin\left(\ln(1+2n) - \ln(1+n)\right) = \sin\left(\lim_{n\to\infty}\left[\ln(1+2n) - \ln(1+n)\right]\right) = \sin(\ln(2)).$$

Where the last step follows from the arguments below. Consider, using properties of the natural log and its continuity,

$$\begin{aligned}\lim_{n\to\infty}\left[\ln(1+2n) - \ln(1+n)\right] &= \lim_{n\to\infty}\left[\ln\left(\frac{1+2n}{1+n}\right)\right] \\ &= \ln\left[\lim_{n\to\infty}\left(\frac{1/n+2}{1/n+1}\right)\right] \\ &= \ln\left[\frac{0+2}{0+1}\right] \\ &= \ln(2).\end{aligned}$$

Example 1.31. *Since the exponential function is everywhere continuous,*

$$\lim_{n\to\infty} exp(\tanh(n)) = exp\left(\lim_{n\to\infty}\tanh(n)\right) = exp(1) = e.$$

To calculate the limit above I used the following algebra for the hyperbolic tangent:

$$\tanh(n) = \frac{\sinh n}{\cosh n} = \frac{e^n - e^{-n}}{e^n + e^{-n}} = \frac{1 - e^{-2n}}{1 + e^{-2n}} \to \frac{1-0}{1+0} = 1$$

as $n \to \infty$.

[4] thanks to Rogawski's text on calculus for this argument

Example 1.32. *If $a_n = \frac{n+\ln n}{n^2}$ then the limit is not immediately obvious since both the numerator and denominator limit to infinity. Extending n continuously we may apply L'Hopital's Rule:*

$$\frac{n+\ln n}{n^2} \to \frac{\frac{d}{dn}(n+\ln n)}{\frac{d}{dn}(n^2)} = \frac{1+\frac{1}{n}}{2n} = \frac{1}{2n} + \frac{1}{2n^2} \to 0$$

as $n \to \infty$.

Example 1.33. *Analyze the limit of the sequence $b_n = \ln(5^n) - \ln(n!)$. Apply properties of the logarithm:*

$$b_n = \ln 5^n - \ln n! = \ln\left(\frac{5^n}{n!}\right) \to -\infty$$

since we know from Example 1.27 that $\frac{5^n}{n!} \to 0$ and the natural log function has a vertical asymptote which tends to $-\infty$ as we approach $x = 0^+$.

Indeterminant powers are a little tricky. My usual approach is to use the identity $f^g = \exp(\ln(f^g)) = \exp(g\ln(f))$ which allows us to trade indeterminant forms of type $0^0, 1^\infty, \infty^0$ for indeterminant forms of type $0 \cdot \infty, \infty \cdot 0, 0 \cdot \infty$ respectively because[5] $\ln(0^+) = -\infty, \ln(1) = 0, \ln(\infty) = \infty$

Example 1.34. *Consider $b_n = \left(1 + \frac{r}{n}\right)^{nt}$ where $t, r \in \mathbb{R}$. Observe,*

$$b_n = \exp\left(\ln\left(1+\frac{r}{n}\right)^{nt}\right) = \exp\left(tn\ln\left(1+\frac{r}{n}\right)\right) \qquad \star$$

Extending n to be a continuous variable, focus on the expression within the above exponential

$$tn\ln\left(1+\frac{r}{n}\right) = \frac{t\ln\left(1+\frac{r}{n}\right)}{\frac{1}{n}} \to \frac{\left(\frac{t}{1+\frac{r}{n}}\right)\left(\frac{-r}{n^2}\right)}{\frac{-1}{n^2}} = \frac{tr}{\frac{1}{1+\frac{r}{n}}} \to \frac{rt}{1+0} = rt$$

as $n \to \infty$ where we have used L'Hopital's Rule on the type $0/0$ limit faced after the leftmost equality. Returning to $\star$ we find

$$\lim_{n\to\infty}(b_n) = \exp\left(\lim_{n\to\infty}\ln\left(1+\frac{r}{n}\right)^{nt}\right) = \exp(rt) = e^{rt}$$

Notice, if we set $r = t = 1$ we find a possible formula for use of the definition of the constant e:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

If we already knew the limit above then we could use it to calculate limits of other indeterminant powers.

[5] take off 10dpts from my quiz for not writing limits here

2 Finite Sums

The concept of a series is to extend finite sums without end. The finite sum is defined recursively,

$$\sum_{k=1}^{1} a_k = a_1 \qquad \& \qquad \sum_{k=1}^{n} a_k = a_n + \sum_{k=1}^{n-1} a_k$$

We call k the **index of summation** and we can trade explicit $\sum$-notation for $+\cdots$ as appropriate:

$$\sum_{k=1}^{n} a_k = \underbrace{a_1 + a_2 + \cdots + a_n}_{n\text{-summands}}.$$

Notice, the letter k does not appear in the explicit sum which the Σ-notation represents. Logically this means we can change the letter of the summation without changing the value of the sum:

$$\sum_{k=1}^{n} a_k = \sum_{j=1}^{n} a_j = a_1 + \cdots + a_n.$$

Example 2.1. *We can also make substitutions to* **re-index** *a given sum. For example, if we wish to write the sum $\sum_{k=1}^{n} a_k$ to start at 0 rather than 1 then we introduce $j = k-1$ which makes $j = 0$ when $k = 1$. Likewise, if $k = n$ then $j = n - 1$. Lastly, note $j = k - 1$ implies $k = j + 1$ thus $a_k = a_{j+1}$. Put it all together:*

$$\sum_{k=1}^{n} a_k = \sum_{j=0}^{n-1} a_{j+1}$$

I'll forego proof of the following proposition, the details are all proofs by mathematical induction anchored to the basic algebraic properties of real numbers such as associativity and commutativity of addition and the distributive properties for addition and multiplication.

Proposition 2.2. *properties of finite sum*

Let $a_k, b_k, c \in \mathbb{R}$ then

$$(\mathbf{1}.)\ \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} (a_k + b_k) \qquad \& \qquad (\mathbf{2}.)\ c\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} c a_k$$

$$(\mathbf{3}.)\ \sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_j b_k \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} a_j b_k \right) = \sum_{k=1}^{n} b_k \sum_{j=1}^{n} a_j$$

To summarize, finite sums are very nice and work just like you would expect. In the interest of saying at least something seemingly nontrivial about finite sums before we go on, let me share a result from Gauss which I mentioned in passing in a past article.

Proposition 2.3. *Gauss' formulas for finite sums*

$$\sum_{k=1}^{n} 1 = n, \qquad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The proof of the assertions above rest on mathematical induction. These are the formulas we need to explicitly calculate Riemann sums directly from the definition (without the incredible help of FTC II).

3 Infinite Series

Let us begin by carefully defining summability or convergence of a series:

Definition 3.1. *series*

The series $\sum_{k=n_o}^{\infty} a_k = a_{n_o} + a_{n_o+1} + \cdots$ has n-th partial sum $\sum_{k=n_o}^{n} a_k = a_{n_o} + a_{n_o+1} + \cdots + a_n$. We say the series $\sum_{k=n_o}^{\infty} a_k$ **converges** or is **summable** if its sequence of partial sums converges. The limit of the sequence of partial sums is known as the **sum** of the series and we write

$$\sum_{k=n_o}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=n_o}^{n} a_k.$$

If the series is not convergent then we say the series is **divergent**. If the sequence of partial sums diverges to $\pm\infty$ then we write $\sum_{k=n_o}^{\infty} a_k = \pm\infty$.

Let me express the sequence of partial sums in the case $n_o = 1$,

$$\left\{ \sum_{k=1}^{n} a_k \right\} = \{a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ \dots\}$$

Example 3.2. *Consider* $\sum_{k=1}^{\infty} 1 = 1 + 1 + \cdots$. *In this case the* n*-th partial sum is simply*

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n\text{-summands}} = n$$

Thus $\displaystyle\sum_{k=1}^{\infty} 1 = \lim_{n\to\infty} \sum_{k=1}^{n} 1 = \lim_{n\to\infty} n = \infty.$

Example 3.3. *Observe* $\displaystyle\sum_{k=1}^{n} 0 = 0 + 0 + \cdots + 0 = 0$ *thus* $\displaystyle\sum_{k=0}^{\infty} 0 = \lim_{n\to\infty} 0 = 0.$

Theorem 3.4. *n-th term test for divergence*

If $\sum_{k=n_o}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{k=n_o}^{\infty} a_k$ diverges.

Proof: Let $S_n = \sum_{k=n_o}^{n} a_k$ and suppose $S_n \to S$ as $n \to \infty$. Notice that the n-term in the series can be written as the difference of partial sums $S_n - S_{n-1} = \sum_{k=n_o}^{n} a_k - \sum_{k=n_o}^{n-1} a_k = a_n$. Thus,

$$a_n = S_n - S_{n-1} \to S - S = 0.$$

Therefore, if $\sum_{k=n_o}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$. Notice the second sentence in the Theorem follows by logic from the first. $\square$

Example 3.5. *Consider the series* $\tan^{-1}(1)+\tan^{-1}(2)+\dots$. *Observe the n-th term in the series is* $\tan^{-1}(n)$. *Therefore this series diverges by the n-th term test since* $\tan^{-1}(n)\to\frac{\pi}{4}$ *as* $n\to\infty$.

I should mention now that the converse to the n-th term test does not hold. In particular, it is possible to have a series with n-term $a_n\to 0$ as $n\to\infty$, yet the series still diverges. The most famous example of this is the harmonic series:

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots=\infty.$$

I will withold proof of the claim above until a later section. I just want you to understand why I include the term *divergence* in heading for the n-th term test. It is a test which only gives certitude of divergence. The n-th term test does not prove summability of the series.

3.1 telescoping series

Example 3.6. *Consider the series* $\sum_{k=2}^{\infty}\left[\frac{1}{\ln(k+2)}-\frac{1}{\ln(k)}\right]$. *Let* $S_n=\sum_{k=2}^{n}[1/\ln(k+2)-1/\ln(k)]$ *and calculate:*

$$\begin{aligned}S_n&=[1/\ln(4)-1/\ln(2)]+[1/\ln(5)-1/\ln(3)]+[1/\ln(6)-1/\ln(4)]+\cdots\\&\qquad\cdots+[1/\ln(n)-1/\ln(n-2)]+[1/\ln(n+1)-1/\ln(n-1)]+[1/\ln(n+2)-1/\ln(n)]\\&=-1/\ln(2)-1/\ln(3)+1/\ln(n+1)+1/\ln(n+2)\end{aligned}$$

the cancellation which occurs above is known as **telescoping** *and series with this sort of pattern are usually called* **telescoping series**. *Observe* $1/\ln(n+1), 1/\ln(n+2)\to 0$ *as* $n\to\infty$ *thus*

$$\sum_{k=2}^{\infty}\left[\frac{1}{\ln(k+2)}-\frac{1}{\ln(k)}\right]=\lim_{n\to\infty}(S_n)=-\frac{1}{\ln(2)}-\frac{1}{\ln(3)}.$$

Example 3.7. *Consider* $\sum_{k=1}^{\infty}\frac{4}{(4k-3)(4k+1)}$. *This is a telescoping series in disguise. We need to use the partial fractions algebra to properly understand the pattern. A short calculation reveals:*

$$\frac{4}{(4k-3)(4k+1)}=\frac{1}{4k-3}-\frac{1}{4k+1}$$

Thus,using $S_n=\sum_{k=1}^{n}\frac{4}{(4k-3)(4k+1)}$,

$$\begin{aligned}S_n&=\sum_{k=1}^{n}\left[\frac{1}{4k-3}-\frac{1}{4k+1}\right]\\&=\frac{1}{4-3}-\frac{1}{4+1}+\frac{1}{4(2)-3}-\frac{1}{4(2)+1}+\frac{1}{4(3)-3}-\frac{1}{4(3)+1}+\cdots\\&\qquad\cdots+\frac{1}{4(n-2)-3}-\frac{1}{4(n-2)+1}+\frac{1}{4(n-1)-3}-\frac{1}{4(n-1)+1}+\frac{1}{4n-3}-\frac{1}{4n+1}\\&=1-\frac{1}{5}+\frac{1}{5}-\frac{1}{9}+\frac{1}{9}-\frac{1}{13}+\cdots+\frac{1}{4n-11}-\frac{1}{4n-7}+\frac{1}{4n-7}-\frac{1}{4n-3}+\frac{1}{4n-3}-\frac{1}{4n+1}\\&=1-\frac{1}{4n+1}\end{aligned}$$

Therefore, $S_n \to 1$ as $n \to \infty$ and we have shown the series is summable with sum 1. In other words, the series converges and $\displaystyle\sum_{k=1}^{\infty} \frac{4}{(4k-3)(4k+1)} = 1$.

3.2 geometric series

Definition 3.8. *geometric series*

Let $c, r \in \mathbb{R}$ then $\displaystyle\sum_{k=0}^{\infty} cr^k = c + cr + cr^2 + \cdots$ is a **geometric series**.

Geometric series are everywhere if you look for them. It is very simple to decide whether a given geometric series is convergent or divergent.

Theorem 3.9. *geometric series*

The geometric series $c + cr + cr^2 + \cdots$ is summable with sum $\frac{c}{1-r}$ if and only if $|r| < 1$. If $|r| \geq 1$ then the geometric series is divergent.

Proof: let $S_n = c + cr + cr^2 + \cdots + cr^{n-1} + cr^n$ be the n-th partial sum of the geometric series. Observe $rS_n = r(c + cr + cr^2 + \cdots + cr^{n-1} + cr^n) = cr + cr^2 + cr^3 + \cdots + cr^n + cr^{n+1}$. Therefore,

$$\begin{aligned} S_n - rS_n &= cr + cr^2 + cr^3 + \cdots + cr^n + cr^{n+1} - \left(c + cr + cr^2 + \cdots + cr^{n-1} + cr^n\right) \\ &= cr^{n+1} - c \end{aligned}$$

Algebra yields $(1-r)S_n = c(r^{n+1} - 1)$. Hence, $S_n = \dfrac{c(r^{n+1}-1)}{1-r}$. If $|r| < 1$ then $S_n \to \frac{c}{1-r}$ since $r^{n+1} \to 0$ as $n \to \infty$. If $|r| \geq 1$ then $\lim_{n\to\infty}(cr^n) \neq 0$ thus the geometric series diverges by the n-th term test. □

Example 3.10. *Whenever we have a number with a repeating decimal expansion we can use the geometric series to convert the number to an explicit fraction.*

$$2.577777\cdots = 2.5 + 0.07777\cdots = 2.5 + \underbrace{\frac{7}{100} + \frac{1}{10}\frac{7}{100} + \frac{1}{10^2}\frac{7}{100} + \cdots}_{\text{geometric with } c = 7/100 \text{ and } r = 1/10}$$

thus,

$$2.5777\cdots = 2.5 + \frac{7/100}{1-1/10} = \frac{5}{2} + \frac{7/100}{9/10} = \frac{5}{2} + \frac{7}{90} = \frac{5(90)+7(2)}{180} = \frac{464}{180} = \frac{116}{45}.$$

Example 3.11. *A possibly infinite food order*

Problem: *A man and infinitely many of his friends go to a hotdog stand. The first man says he wants a whole hotdog. Then the second man says he'll take a half a hotdog. The third man says he'll have half of half a hotdog. The fourth man asks for an eighth of a hotdog. If the first man is going to pay the bill for this infinite order than how many hot dogs does he need to buy ?*

Solution: *working in units of hotdogs, the order needs the following sum of hotdogs,*

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

by the geometric series with $c = 1$ and $r = 1/2$.

3.3 infinite series which correspond to definite integrals

Recall the definition of the Riemann integral:

$$\int_a^b f(x)\,dx = \lim_{n\to} \mathcal{R}_n = \lim_{n\to\infty}\left[\sum_{j=1}^{n} f(x_k^*)\Delta x\right].$$

where we partitioned $[a, b]$ into n-subintervals of width $\Delta x = \frac{b-a}{n}$ with endpoints given by $x_i = a + i\Delta x$ for $i = 0, 1, \dots, n$ and[6] $x_k^* \in [x_{k-1}, x_k]$ for each $k = 1, 2, \dots, n$. In retrospect, the definition of the integral itself is an infinite sum of a rather particular form. We can turn this idea around now. Since we know how to calculate integrals of reasonably uncomplicated functions, if we can identify a given infinite sum as a Riemann sum then we can calculate the series by using FTC II to calculate the integral.

Example 3.12. *Find the value of* $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n}$. *Apparently* $\Delta x = \frac{b-a}{n} = \frac{1}{n}$ *suggests we can set* $a = 0$ *and* $b = 1$. *Hence,* $x_i = a + i\Delta x = \frac{i}{n}$ *and we identify that* $1 + \frac{i}{n} = 1 + x_i$. *In fact,*

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n} &= \lim_{n\to\infty}\sum_{i=1}^{n}(1+x_i)^3\Delta x \\ &= \int_0^1 (1+x)^3\,dx \\ &= \frac{1}{4}(x+1)^4\bigg|_0^1 \\ &= \frac{1}{4}(16-1) = \frac{15}{4}.\end{aligned}$$

What is the n-th term of the series being summed in the example above ? That is not an entirely easy question. To answer it we use the idea we saw in the proof of the n-th term test. If S_n is the n-th partial sum then $S_n = a_n + S_{n-1}$ thus $a_n = S_n - S_{n-1}$. Since $S_n = \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n}$ we find that $S_{n-1} = \sum_{i=1}^{n-1}\left(1+\frac{i}{n-1}\right)^3\frac{1}{n-1}$ thus $a_n = \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\frac{1}{n} - \sum_{i=1}^{n-1}\left(1+\frac{i}{n-1}\right)^3\frac{1}{n-1}$.

Example 3.13. *Find the value of* $\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{\pi i}{2n}\right)\frac{1}{n}$. *We would like* $x_i = \frac{\pi i}{2n} = a + i\Delta x$ *thus identify* $a = 0$ *and* $\Delta x = \frac{\pi}{2n}$. *But,* $\Delta x = \frac{b-a}{n} = \frac{b}{n} = \frac{\pi}{2n}$ *so we want* $b = \frac{\pi}{2}$. *Notice that* $\frac{1}{n} = \frac{2}{\pi}\frac{\pi}{2n} = \frac{2}{\pi}\Delta x$ *thus:*

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\cos\left(\frac{\pi i}{2n}\right)\frac{1}{n} &= \lim_{n\to\infty}\sum_{i=1}^{n}\cos(x_i)\frac{2}{\pi}\Delta x \\ &= \frac{2}{\pi}\int_0^{\pi/2}\cos(x)\,dx \\ &= \frac{2}{\pi}(\sin(\pi/2)-\sin(0)) = \frac{2}{\pi}.\end{aligned}$$

[6] we said that the sample point could be taken in many different ways, but since the limit of $n \to \infty$ makes $\Delta x \to 0$ it follows our choice of x_k^* will not influence the end result of the calculation; we can use left, right, midpoint or even a more abstract choice to formulate the Riemann sum. Usually I use right-endpoint rule of $x_k^* = x_k$ as a first approach

Calculus instructors like the past two examples because they give a concrete set of problems to test student's understanding of the nuts and bolts of the Riemann integral, and they give us examples of series whose sums can be explicitly calculated. The downside of these examples is while the n-th partial sum is typically made explicit from the formulation of the problem, it is embarrassingly ugly to find the formula for a_n which are summed to form the series. Just look at the previous page and the hideous formula I gave for a_n (perhaps it could be simplified ? (bonus)). Typically such problems are a bit contrived and are, at least in my experience, not found in the *wild*. In contrast, the example below is an entirely natural approach to derive FTC II.

Example 3.14. *In this example we examine how FTC II can be derived via a telescoping series argument. Let me remind you what we already know:*
FTC II: *Suppose f is continuous on $[a,b]$ and has antiderivative F then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

The proof I gave in-class in the first week used a different argument, I think the one I share below is more interesting.

Proof: *We seek to calculate $\int_a^b f(x)\,dx$. Use the usual partition for the n-th Riemann sum of f on $[a,b]$; $x_o = a, x_1 = a + \Delta x, \dots, x_n = b$ where $\Delta x = \frac{b-a}{n}$. Suppose that f has an antiderivative F on $[a,b]$. Recall the Mean Value Theorem (MVT) for $y = F(x)$ on the interval $[x_o, x_1]$ tells us that there exists $x_1^* \in [x_o, x_1]$ such that*

$$F'(x_1^*) = \frac{F(x_1) - F(x_o)}{x_1 - x_o} = \frac{F(x_1) - F(x_o)}{\Delta x}$$

Notice that this tells us that $F'(x_1^*)\Delta x = F(x_1) - F(x_o)$. But, $F'(x) = f(x)$ so we have found that $f(x_1^*)\Delta x = F(x_1) - F(x_o)$. In other words, the area under $y = f(x)$ for $x_o \leq x \leq x_1$ is well approximated by the difference in the antiderivative at the endpoints. Thus we choose the sample points for the n-th Riemann sum by applying the MVT on each subinterval to select x_j^* such that $f(x_j^*)\Delta x = F(x_j) - F(x_{j-1})$. With this construction in mind calculate:

$$\begin{aligned}
\int_a^b f(x)\,dx &= \lim_{n\to\infty}\left(\sum_{j=1}^{n} f(x_j^*)\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{j=1}^{n}\big[F(x_j) - F(x_{j-1})\big]\right) \\
&= \lim_{n\to\infty}\big(F(x_1) - F(x_o) + F(x_2) - F(x_1) + \cdots + F(x_n) - F(x_{n-1})\big) \\
&= \lim_{n\to\infty}\big(F(x_n) - F(x_o)\big) \\
&= \lim_{n\to\infty}\big(F(b) - F(a)\big) \\
&= F(b) - F(a). \square
\end{aligned}$$

Example 3.15. *I'll share one more example where I actually flesh out the calculation of the n-th term which I did not have the courage to complete in the earlier example. Let us begin with a reasonably simple integral:*

$$\int_0^1 x\,dx = \frac{x^2}{2}\bigg|_0^1 = \frac{1}{2}$$

I will write the integral in terms of a right-end-point rule where $\Delta x = 1/n$ *and* $x_i = i/n$ *hence*

$$\begin{aligned}
\int_0^1 x\,dx &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} x_i\Delta x\right) \\
&= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{i}{n^2}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\sum_{i=1}^{n} i\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{n^2}\cdot\frac{n(n+1)}{2}\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

Of course, we should expect the result above. Now to the somewhat sideways question I asked in the earlier example, what is a_i *for which*

$$\sum_{i=1}^{n}\frac{i}{n^2} = \sum_{i=1}^{n} a_i\,?$$

Let $S_n = \displaystyle\sum_{i=1}^{n}\frac{i}{n^2}$ *and notice we found previously that*

$$S_n = \frac{1}{2}+\frac{1}{2n}$$

Recall, $S_n - S_{n-1} = a_n$ *thus,*

$$\begin{aligned}
a_n &= \frac{1}{2}+\frac{1}{2n}-\frac{1}{2}+\frac{1}{2(n-1)} \\
&= \frac{1}{2n}-\frac{1}{2(n-1)} \\
&= \frac{-1}{2n(n-1)}
\end{aligned}$$

Notice $\frac{-1}{2n(n-1)} < 0$*. Yet, the sum of the terms works out to* $\frac{1}{2}$*.* **How can this be ?**.
Can you solve the riddle ? Be the first person to explain the resolution of this paradox in an email to me and it will earn you 10pts bonus. [7]

We could give more examples, but this section already illustrates the three major methods to explicitly calculate a given series. There is one additional method we learn, but we require power series for the remaining method. We'll come to that later. So, next we turn to indirect arguments to decide the summability or convergence of a given series.

[7] a paradox is a seeming contradiction, it is something which seems wrong, but in context is actually not wrong at all

4 Convergence and Divergence Theory

4.1 sum and scalar multiple of series

If we omit the explicit range of the index in a summation then please understand we mean the statement $\sum a_k$ indicates $\sum_{k=n_o}^{\infty} a_k$ in this section. Of course this abbreviation cannot be used when the range of the index is central to the calculation and especially when we are working with multiple series with different ranges.

Theorem 4.1. *addition and scalar multiplication of series*

Suppose $\sum a_k = A$ and $\sum b_k = B$ where $A, B \in \mathbb{R}$ and $c \in \mathbb{R}$ then

(1.) $\sum(a_k + b_k) = \sum a_k + \sum b_k$

(2.) $c \sum a_k = \sum(ca_k)$.

In other words, the sum and scalar multiple of summable series is summable. Similarly, if $\sum a_k$ diverges and $\sum_k b_k$ converges then for any $c \neq 0$, both $\sum ca_k$ and $\sum(a_k + b_k)$ diverge.

Proof: suppose $\sum_{k=n_o}^{\infty} a_k = A \in \mathbb{R}$ and $\sum_{k=n_o}^{\infty} b_k = B \in \mathbb{R}$ and $c \in \mathbb{R}$. Then, by definition, the sequence of partial sums for the given series converge to A and B respectively. Explicitly,

$$\lim_{n\to\infty} \sum_{k=n_o}^{n} a_k = A \qquad \& \qquad \lim_{n\to\infty} \sum_{k=n_o}^{n} b_k = B$$

By properties of finite sums we have

$$\sum_{k=n_o}^{n} a_k + \sum_{k=n_o}^{n} b_k = \sum_{k=n_o}^{n} (a_k + b_k) \qquad \& \qquad c \sum_{k=n_o}^{n} a_k = \sum_{k=n_o}^{n} ca_k$$

Applying sequential limit laws (see Theorem 1.22 parts **(1.)** and **(2.)**) we to find:

$$\lim_{n\to\infty} \sum_{k=n_o}^{n} (a_k + b_k) = \lim_{n\to\infty} \sum_{k=n_o}^{n} a_k + \lim_{n\to\infty} \sum_{k=n_o}^{n} b_k \qquad \& \qquad \lim_{n\to\infty} \sum_{k=n_o}^{n} ca_k = c \lim_{n\to\infty} \sum_{k=n_o}^{n} a_k.$$

The divergent case follows from the same formulas, except that the divergence of the partial sums for $\sum a_k$ imply divergence of the partial sums for $\sum ca_k$ and $\sum(a_k + b_k)$ in the case $c \neq 0$. $\square$

Example 4.2. *Consider* $\sum_{k=0}^{\infty} 4^{-k}(2^k + 3^k)$. *This series is the sum of two geometric series with different radii. To see why this claim is true we must see the following algebra:*

$$4^{-k}(2^k + 3^k) = 4^{-k}2^k + 4^{-k}2^k = \left(\frac{2}{4}\right)^k + \left(\frac{3}{4}\right)^k = \left(\frac{1}{2}\right)^k + \left(\frac{3}{4}\right)^k$$

Observe, the series below are convergent geometric series. Therefore,

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 \cdots = \frac{1}{1 - 1/2} = 2$$

$$\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = 1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots = \frac{1}{1-3/4} = 4$$

Thus the given series is the sum of convergent series and we conclude

$$\sum_{k=0}^{\infty} 4^{-k}(2^k+3^k) = \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k + \sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = 2+4 = 6.$$

4.2 integral test and the p-series test

Positive series are series where all the terms being summed are positive. The following theorem follows immediately from the bounded monotonic sequence theorem:

Theorem 4.3. *dichotomy theorem for positive series*

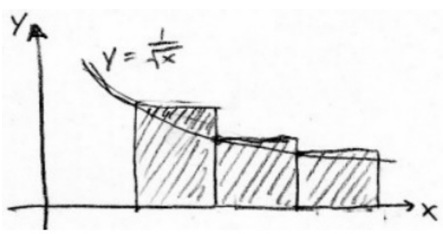
If $\sum_{k=n_o}^{\infty} a_k$ has $a_k > 0$ for all $k \geq n_o$ has n-th partial sum $S_n = \sum_{k=n_o}^{n} a_k$ then

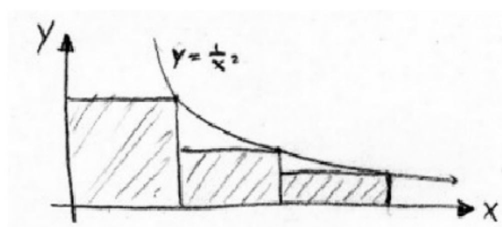
(1.) if S_n is bounded above then $\sum_{k=n_o}^{\infty} a_k < \infty$

(2.) if S_n are not bounded above then $\sum_{k=n_o}^{\infty} a_k = \infty$

A series of positive terms converges iff its sequence of partial sums is bounded above.

Let me share the intuition behind the integral test before I state it formally:





In view of the left picture, since $\int_1^{\infty}\frac{dx}{\sqrt{x}} = \infty$ it follows $\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}} = \infty$. Likewise, from the right picture, since $\int_1^{\infty}\frac{dx}{x^2} = \frac{1}{2}$ converges it follows $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges and $\sum_{k=1}^{\infty}\frac{1}{k^2} < \frac{1}{2}$.

Theorem 4.4. *integral test and error estimation*

Let $a_k = f(k)$, where $f(x)$ is a positive, decreasing, and continuous function for $x \geq 1$.

(1.) If $\displaystyle\int_1^{\infty} f(x)dx$ converges then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

(2.) If $\displaystyle\int_1^{\infty} f(x)dx$ diverges then $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.

Proof: the proof is partly by picture. Essentially the pictures given above the theorem suggest the following inequalities provided $f(x)$ is a positive, decreasing, continuous function for $x \geq 1$,

$$c_n = \int_1^n f(x)\,dx \leq a_1+a_2+\cdots+a_n \leq \int_1^{n+1} f(x)\,dx = b_n.$$

(1.) If $\displaystyle\int_1^\infty f(x)dx$ converges then $\lim_{t\to\infty}\int_1^t f(x)\,dx = L \in \mathbb{R}$. Thus $\lim_{n\to\infty}\int_1^{n+1} f(x)\,dx = L$. Hence $b_n = \int_1^{n+1} f(x)\,dx$ defines a convergent sequence. But, convergent sequences are bounded. If $b_n \leq B$ for all n then observe $a_1 + a_2 + \cdots a_n \leq B$ means the n-th partial sum of $\displaystyle\sum_{k=1}^\infty a_k$ is bounded. Apply the dichotomy theorem for positive series to see $\displaystyle\sum_{k=1}^\infty a_k$ converges.

(2.) If $\displaystyle\int_1^\infty f(x)dx$ diverges then $\lim_{t\to\infty}\int_1^t f(x)\,dx = \infty$. Observe $c_n = \int_1^n f(x)\,dx$ is a sequence which likewise diverges to ∞. Since $c_n \leq a_1 + a_2 + \cdots a_n$ we find the sequence of partial sums for $\displaystyle\sum_{k=1}^\infty a_k$ is unbounded hence the dichotomy theorem for positive series indicates $\displaystyle\sum_{k=1}^\infty a_k$ diverges. $\square$

Example 4.5. *Consider $\sum_{k=1}^\infty \frac{1}{1+k^2}$. Observe $a_k = \frac{1}{1+k^2}$ has $a_k = f(k)$ for $f(x) = \frac{1}{1+x^2}$. If $x \geq 1$ then clearly $f(x) > 0$ and f is continuous. To see f is a decreasing function notice that*

$$\frac{df}{dx} = \frac{-2x}{(1+x^2)^2} < 0.$$

Moreover,

$$\int_1^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty}\int_1^t \frac{dx}{1+x^2} = \lim_{t\to\infty}\left(\tan^{-1}(t) - \tan^{-1}(1)\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Therefore, $\sum_{k=1}^\infty \frac{1}{1+k^2}$ converges by the integral test.

Example 4.6. *Consider $\sum_{k=1}^\infty \frac{2}{2k+7}$. Observe $a_k = \frac{2}{2k+7}$ has $a_k = f(k)$ for $f(x) = \frac{2}{2x+7}$. If $x \geq 1$ then clearly $f(x) > 0$ and f is continuous. To see f is a decreasing function notice that*

$$\frac{df}{dx} = \frac{-2}{(2x+7)^2} < 0.$$

Moreover,

$$\int_1^\infty \frac{2dx}{2x+7} = \lim_{t\to\infty}\int_1^t \frac{2dx}{2x+7} = \lim_{t\to\infty}\left(\ln|2t+7| - \ln 9\right) = \infty.$$

Therefore, $\sum_{k=1}^\infty \frac{2}{2k+7}$ diverges by the integral test.

Just because we have a hammer, not everything has to be a nail. Use a fly swatter on a fly.

Example 4.7. *Consider $\sum_{k=1}^\infty \frac{2k}{2k+7}$. Observe $a_k = \frac{2k}{2k+7} \to 1$ as $k \to \infty$ thus $\sum_{k=1}^\infty \frac{2k}{2k+7}$ diverges by the k-th term test.*

Example 4.8. *If $p > 1$ then $\displaystyle\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\frac{1}{p-1}\left(\frac{-1}{t^{p-1}}+1\right) = \frac{1}{p-1}$. Notice $f(x) = \frac{1}{x^p}$ is a positive, continuous function for $x \geq 1$. Moreover,*

$$\frac{df}{dx} = \frac{-p}{x^{p+1}} < 0$$

thus f is decreasing on $[1,\infty)$. Therefore, the integral test gives $\sum_{k=1}^\infty \frac{1}{k^p}$ is convergent.

Example 4.9. *Observe* $\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty}(\ln t - \ln 1) = \infty$. *Notice* $f(x) = \frac{1}{x}$ *is a positive, continuous function for* $x \geq 1$. *Moreover,*

$$\frac{df}{dx} = \frac{-1}{x^2} < 0$$

thus f *is decreasing on* $[1,\infty)$. *Therefore, the integral test gives* $\sum_{k=1}^\infty \frac{1}{k}$ *is divergent. The series* $\sum_{k=1}^\infty \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ *is known as the* **harmonic series**.

Example 4.10. *If* $0 < p < 1$ *then* $\displaystyle\int_1^\infty \frac{dx}{x^p} = \lim_{t\to\infty}\int_1^t \frac{dx}{x^p} = \lim_{t\to\infty}\left(\frac{-1}{t^{p-1}} + \frac{1}{p-1}\right) = \infty$. *Notice* $f(x) = \frac{1}{x^p}$ *is a positive, continuous function for* $x \geq 1$. *Moreover,*

$$\frac{df}{dx} = \frac{-p}{x^{p+1}} < 0$$

thus f *is decreasing on* $[1,\infty)$. *Therefore, the integral test gives* $\sum_{k=1}^\infty \frac{1}{k^p}$ *is divergent.*

Theorem 4.11. *p-series*

$\sum_{k=1}^\infty \frac{1}{k^p}$ is convergent if and only if $p > 1$. If $p \leq 1$ then $\sum_{k=1}^\infty \frac{1}{k^p}$ diverges.

Proof: the preceding examples coverved the cases $p > 0$. If $p = 0$ then $\lim_{k\to\infty}\frac{1}{k^p} = \lim_{k\to\infty} 1 = 1 \neq 0$. Likeise, if $p < 0$ then $-p > 0$ hence $\lim_{k\to\infty}\frac{1}{k^p} = \lim_{k\to\infty} k^{-p} = \infty \neq 0$. Thus by the k-th term test $\sum_{k=1}^\infty \frac{1}{k^p}$ diverges in cases with $p \leq 0$. $\square$

The proof of the integral test discussed earlier in this section implies the following result:

Theorem 4.12. *error estimation courtesy of the integral test*

Let $a_k = f(k)$, where $f(x)$ is a positive, decreasing, and continuous function for $x \geq 1$ and suppose $\displaystyle\sum_{k=1}^\infty a_k = S \in \mathbb{R}$. If $S_n = \displaystyle\sum_{k=1}^n a_n$ then $S_n \to S$ as $n \to \infty$ and

$$\int_{n+1}^\infty f(x)\,dx \leq S - S_n \leq \int_n^\infty f(x)\,dx.$$

Notice the error in the n-th partial sum is given by $S_n - S$ and the inequality above can be used to calculate upper and lower bounds on the error.

Example 4.13. *The* $p = 2$ *series is convergent by the p-series test. Let* $S = \sum_{k=1}^\infty \frac{1}{k^2}$ *and define* $S_n = \sum_{k=1}^n \frac{1}{k^2} = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2}$. *How many terms to we need to sum in order to know* $S_n - S < 0.1$ *? Calculate*

$$\int_n^\infty \frac{dx}{x^2} = \lim_{t\to\infty}\int_n^t \frac{dx}{x^2} = \lim_{t\to\infty}\left(\frac{-1}{t} + \frac{1}{n}\right) = \frac{1}{n}$$

Thus by the error estimation theorem for the integral test we have $S - S_n \leq \frac{1}{n}$. *We desire* $\frac{1}{n} \leq 0.1$ *thus* $10 \leq n$. *Thus* $n = 10$ *should suffice. Calculate:*

$$S_{10} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100} = \frac{1968329}{1270080} \approx 1.5497$$

Direct calculation of the $p=2$*-series is beyond this course*[8]*, however, it was not beyond Euler in the 18-th century when he calculated that* $\sum_{k=1}^{n}\frac{1}{k^2}=\frac{\pi^2}{6}\approx 1.6449$*. Observe* $1.6449-1.5497\approx 0.0952<0.1$ *as advertised. For fun,*

$$S_{1000}=1+\frac{1}{4}+\cdots+\frac{1}{1000}=\frac{1968329}{1270080}\approx 1.64393$$

has $S-S_{1000}\approx 1.64493-1.64393\approx 0.001$*. The error bound given by the integral test estimation theorem is fairly tight to the actual error.*

4.3 re-indexing and the tail of a series

Given a convergent series we may add or subtract a finite number of terms and the resulting series will once again be convergent. Similarly, given a divergent series, if we add or subtract finitely many terms then the resulting series is likewise divergent. These observations are formalized in the saying that the convergence of a series is controlled by the convergence of the tail of the series. In other words, what matters is what happens in the limit as we add infinitely many terms. Let me be more precise:

Theorem 4.14. *convergence of tail*

Consider $\sum_{k=n_o}^{\infty}a_k$ where $m_o,n_o\in\mathbb{Z}$ with $m_o<n_o$ and $a_k\in\mathbb{R}$ for all $k\geq m_o$. Then

(1.) $\sum_{k=n_o}^{\infty}a_k$ converges if and only if $\sum_{k=m_o}^{\infty}a_k$ converges and

$$\sum_{k=m_o}^{\infty}a_k=a_{m_o}+\cdots+a_{n_o-1}+\sum_{k=n_o}^{\infty}a_k.$$

(2.) $\sum_{k=n_o}^{\infty}a_k$ diverges if and only if $\sum_{k=m_o}^{\infty}a_k$ diverges.

Proof: the theorem above follows immediately from the identity below for the partial sums:

$$\sum_{k=m_o}^{n}a_k=a_{m_o}+\cdots+a_{n_o-1}+\sum_{k=n_o}^{n}a_k$$

clearly the partial sums $\sum_{k=m_o}^{n}a_k$ and $\sum_{k=n_o}^{n}a_k$ share the same convergence or divergence as $n\to\infty$ since $a_{m_o}+\cdots+a_{n_o-1}$ is merely a finite constant. If the partial sum converges, we find:

$$\lim_{n\to\infty}\sum_{k=m_o}^{n}a_k=a_{m_o}+\cdots+a_{n_o-1}+\lim_{n\to\infty}\sum_{k=n_o}^{n}a_k\quad\Rightarrow\quad\sum_{k=m_o}^{\infty}a_k=a_{m_o}+\cdots+a_{n_o-1}+\sum_{k=n_o}^{\infty}a_k\ \square.$$

Example 4.15. $\displaystyle\sum_{k=4}^{\infty}\frac{1}{k^2}$ *is a tail of the* $p=2$ *series. Since the* $p=2$ *series converges it follows the* $\displaystyle\sum_{k=4}^{\infty}\frac{1}{k^2}$ *converges. Moreover, thanks to Euler,* $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ *thus*

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\sum_{k=4}^{\infty}\frac{1}{k^2}$$

[8] this is not entirely true, we could cover basic Fourier series in the last week of the course and thereby derive this result, if you are interested then by all means ask me to change the syllabus this term, we can take a vote

Therefore, $\sum_{k=4}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{49}{36}$.

Series for which we know closed-form expressions for the sum are in rare supply. The example above is quite special. Let us turn to the problem of re-indexing. Rather than attempt a general theorem here I will illustrate via example.

Example 4.16. *Consider* $\sum_{k=1}^{\infty} \frac{1}{(k+4)^3}$. *Let* $j = k+4$ *and observe when* $k = 1$ *we find* $j = 1+4 = 5$ *whereas* $k \to \infty$ *implies* $j = k+4$ *likewise diverges to* ∞. *Consequently,*

$$\sum_{k=1}^{\infty} \frac{1}{(k+4)^3} = \sum_{j=5}^{\infty} \frac{1}{j^3}.$$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{(k+4)^3}$ *is a tail of the convergent* $p = 3$ *series and is thus a convergent series.*

Example 4.17. *Consider* $\sum_{k=1}^{\infty} \frac{1}{2k+4}$. *Let* $j = k+2$ *and observe when* $k = 1$ *we find* $j = 1+2 = 3$ *whereas* $k \to \infty$ *implies* $j = k+2$ *likewise diverges to* ∞. *Consequently,*

$$\sum_{k=1}^{\infty} \frac{1}{2k+4} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k+2} = \frac{1}{2}\sum_{j=3}^{\infty} \frac{1}{j}.$$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{(k+4)^3}$ *is a* $\frac{1}{2}$ *of the tail of the divergent* $p = 1$ *series and is thus a divergent series.*

Example 4.18. *Consider* $\sum_{k=10}^{\infty} \frac{1}{k^2 - 4k + 5}$. *Observe* $k^2 - 4k + 5 = (k-2)^2 + 1$ *so if we make a* $j = k - 2$ *substitution then* $k = 10$ *gives* $j = 10 - 2 = 8$ *and* $j = k - 2 \to \infty$ *when* $k \to \infty$ *thus*

$$\sum_{k=10}^{\infty} \frac{1}{k^2 - 4k + 5} = \sum_{j=8}^{\infty} \frac{1}{j^2 + 1}$$

hence the series converges as it is a tail of the convergent series studied in Example 4.5.

Example 4.19. *Consider* $\sum_{k=2}^{\infty} \frac{1}{k^3 - 3k^2 + 3k - 1}$. *Observe* $k^3 - 3k^2 + 3k - 1 = (k-1)^3$ *so if we make a* $j = k - 1$ *substitution then* $k = 3$ *gives* $j = 2 - 1 = 1$ *and* $j = k - 1 \to \infty$ *when* $k \to \infty$ *thus*

$$\sum_{k=2}^{\infty} \frac{1}{k^3 - 3k^2 + 3k - 1} = \sum_{j=1}^{\infty} \frac{1}{j^3}$$

hence the series converges as it the convergent $p = 3$ *series.*

The algebra we've applied in this section can at times be circumvented by more sophisticated tests we study in future sections of this article. We may return to these examples with other tools in later sections.

4.4 direct comparison test

I have seen proofs of the direct comparison test which are much shorter than the proof I offer here. If you want to make the proof simpler then just set $n_o = 1$ and assume $M = 1$ and it gets much less cluttered. I decided to attempt the cluttered full story proof and as such my arguments make good use of Theorem 4.14 on tails.

Theorem 4.20. *direct comparison test*

Suppose there exists $M > 0$ for which $n \geq M$ implies $0 \leq a_n \leq b_n$ then

- **(1.)** if $\sum b_k$ converges then, $\sum a_k$ converges,
- **(2.)** if $\sum a_k$ diverges then, $\sum b_k$ diverges.

Proof: suppose there exists $M > 0$ for which $n \geq M$ implies $0 \leq a_n \leq b_n$. I assume $n_o \in \mathbb{N}$ with $n_o < M$ in the interest of the broadest applicabilty of this proof. For **(1.)** assume $\sum_{k=n_o}^{\infty} b_k$ converges which means the sequence of partial sums $\sum_{k=n_o}^{n} b_k \to B$ as $n \to \infty$. Let $k, n_1 \in \mathbb{N}$ with $M < n_1 \leq k$ then $0 \leq a_k \leq b_k$ hence

$$0 \leq \sum_{k=n_1}^{n} a_k \leq \sum_{k=n_1}^{n} b_k \quad (\star).$$

Observe $\sum_{k=n_1}^{\infty} b_k$ is a tail of a convergent series hence the tail is summable. Therefore, the sequence of partial sums $\sum_{k=n_1}^{n} b_k$ is convergent and hence bounded. From $\star$ we see that the partial sum of $\sum_{k=n_1}^{n} a_k$ is also bounded. Since $a_k \geq 0$ we find $\sum_{k=n_1}^{n} a_k$ is an increasing sequence. Therefore, $\sum_{k=n_1}^{n} a_k$ converges since it is a bounded monotonic sequence. Thus the tail $\sum_{k=n_1}^{\infty} a_k$ is summable and hence $\sum_{k=n_o}^{n} a_k$ converges. The proof of **(2.)** is similar in that we can argue the unbounded partial sum for $\sum a_k$ implies the partial sum of $\sum b_k$ is likewise unbounded and hence the series $\sum b_k$ diverges. Once again, the complete proof would have to sort through the concept of the tail and use the equivalence of the convergence of the tail and the series to complete the thought. $\square$

Example 4.21. *Consider* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k\sqrt{k}}$. *Since* $\sqrt{k} \geq 1$ *for* $k \geq 1$ *we observe* $0 < \frac{1}{3^k\sqrt{k}} \leq \frac{1}{3^k}$. *Notice* $\sum_{k=1}^{\infty} \frac{1}{3^k} = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots = \frac{1/3}{1-1/3} = \frac{1}{2}$. *Thus* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3^k\sqrt{k}}$ *converges by the direct comparison test.*

Example 4.22. *Consider* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k^5+7)^{1/6}}$. *If* $2 \leq k$ *then* $32 \leq k^5$ *thus* $k^5 + 7 < k^5 + 32 \leq 2k^5$. *Notice* $f(x) = x^{1/6}$ *has* $f'(x) = \frac{1}{6x^{5/6}} > 0$ *for* $x > 0$ *hence* $f(x)$ *is an increasing function. Increasing functions preserve inequalities,* $k^5 + 7 < 2k^5$ *implies* $(k^5+7)^{1/6} < (2k^5)^{1/6} = 2^{1/6}k^{5/6}$. *Therefore, for* $k \geq 2$,

$$\frac{1}{(k^5+7)^{1/6}} > \frac{1}{2^{1/6}k^{5/6}}$$

But, notice $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2^{1/6}k^{5/6}} = \frac{1}{2^{1/6}}\sum_{k=2}^{\infty} \frac{1}{k^{5/6}}$ *is a multiple of a tail of the divergent* $p = 5/6$ *series. Therefore, by the direct comparison test,* $\displaystyle\sum_{k=2}^{\infty} \frac{1}{(k^5+7)^{1/6}}$ *diverges.*

4.5 limit comparison test

Theorem 4.23. *limit comparison test*

Suppose $\{a_k\}$ and $\{b_k\}$ are positive sequences and suppose $L = \lim_{k\to\infty} \dfrac{a_k}{b_k}$ is either finite or ∞. Then,

- **(1.)** if $L > 0$ then $\sum a_k$ converges if and only if $\sum b_k$ converges,
- **(2.)** if $L = \infty$ and $\sum a_k$ converges, then $\sum b_k$ converges,
- **(3.)** if $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges,

Proof: Let $a_k, b_k > 0$ for all k. Suppose $\lim_{k\to\infty} \frac{a_k}{b_k} = L \in \mathbb{R}$ and assume $\sum b_k$ converges. Choose $R > 0$ for which $R > L$. Let $\varepsilon = R - L > 0$ and choose $N \in \mathbb{N}$ for which $k > N$ implies $|a_k/b_k - L| < \varepsilon$. Hence, for $k > N$,

$$|a_k/b_k - L| < \varepsilon \quad \Rightarrow \quad L - R < a_k/b_k - L < R - L \quad \Rightarrow a_k/b_k < R$$

consequently, $0 < a_k < b_k R$ for all $k > N$. Observe $\sum b_k R = R \sum b_k$ is a convergent series thus by the direct comparison test $\sum a_k$ likewise converges. This proves the converse direction of **(1.)** as well as **(3.)**.

Next, suppose $\lim_{k\to\infty} \frac{a_k}{b_k} = L > 0$ and assume $\sum a_k$ converges. Notice $L^{-1} = \lim_{k\to\infty} \frac{b_k}{a_k}$. Hence the argument of the previous paragraph applies to show $\sum b_k$ converges. This proves **(1.)**.

Finally, suppose $\lim_{k\to\infty} \frac{a_k}{b_k} = \infty$ and assume $\sum a_k$ converges. Notice $\lim_{k\to\infty} \frac{b_k}{a_k} = 0$ hence by **(3.)** we find $\sum b_k$ converges which proves **(2.)** $\square$.

Example 4.24. *Consider* $\displaystyle\sum_{k=5}^{\infty} \frac{1}{k^{13} + k^5 - 7}$. *This is essentially the tail of the* $p = 13$ *series. Let's try to use the limit comparison test with the convergent series* $\sum \frac{1}{k^{13}}$. *Set* $a_k = \frac{1}{k^{13}+k^5-7}$ *and* $b_k = \frac{1}{k^{13}}$. *Notice* $k \geq 5$ *implies* $a_k > 0$ *and clearly* $b_k > 0$. *Observe,*

$$\frac{a_k}{b_k} = \frac{\frac{1}{k^{13}+k^5-7}}{\frac{1}{k^{13}}} = \frac{k^{13} + k^5 - 7}{k^{13}} = 1 + \frac{1}{k^8} - \frac{7}{k^{13}} \to 1 = L.$$

Thus $\displaystyle\sum_{k=5}^{\infty} \frac{1}{k^{13} + k^5 - 7}$ *converges by the limit comparison test.*

Example 4.25. *Consider* $\displaystyle\sum_{k=5}^{\infty} \frac{k^4 + 3k^2 + 1}{3k^5 + k^3 + 2}$. *Let* $a_k = \frac{k^4+3k^2+1}{3k^5+k^3+2}$ *and* $b_k = \frac{1}{k}$. *Clearly* $a_k, b_k > 0$. *Consider their quotient,*

$$\frac{a_k}{b_k} = \frac{k^4 + 3k^2 + 1}{3k^5 + k^3 + 2} \cdot \left(\frac{1}{k}\right)^{-1} = \frac{k^4 + 3k^2 + 1}{3k^5 + k^3 + 2} \cdot k = \frac{k^5 + 3k^3 + k}{3k^5 + k^3 + 2} = \frac{1 + 3/k^2 + 1/k^4}{3 + 1/k^2 + 2/k^5} \to \frac{1}{3}.$$

Apply the limit comparison test with $L = 1/3$ *to deduce* $\displaystyle\sum_{k=5}^{\infty} \frac{k^4 + 3k^2 + 1}{3k^5 + k^3 + 2}$ *is divergent since* $\sum \frac{1}{k}$ *is the divergent* $p = 1$ *series.*

Example 4.26. *Consider* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+\ln(k)}$. *Let* $a_k = \frac{1}{\sqrt{k}+\ln(k)}$ *and note* $a_k > 0$. *We suspect this series diverges. Let's study it with limit comparison test agains the* $p = 1/2$ *series. Consider* $b_k = \frac{1}{\sqrt{k}} > 0$. *Observe,*

$$\frac{a_k}{b_k} = \frac{\sqrt{k}}{\sqrt{k}+\ln(k)} \to \frac{\frac{1}{2\sqrt{k}}}{\frac{1}{2\sqrt{k}}+\frac{1}{k}} = \frac{1}{1+2\sqrt{k}} \to 0$$

where we extended k *to be a continuous variable as to apply L'Hopital's Rule to the type* ∞/∞ *limit. Unfortunately, this is not helpful. I'm leaving this argument here to help you see the actual process of the logic. Sometimes the first thing we try doesn't work. So, try again.*
Suppose $b_k = 1/k$ *then study the quotient as* $k \to \infty$,

$$\frac{a_k}{b_k} = \frac{k}{\sqrt{k}+\ln(k)} \to \frac{1}{\frac{1}{2\sqrt{k}}+\frac{1}{k}} \to \infty$$

Again, unhelpful. At this point we'll change tactics and work on a direct comparison argument. Question: which grows faster, the square root function or the natural log ? Consider:

$$\frac{d}{dx}\left[\sqrt{x}-\ln(x)\right] = \frac{1}{2\sqrt{x}} - \frac{1}{x} = \frac{1}{2\sqrt{x}}\left[1-\frac{2}{\sqrt{x}}\right] > 0$$

given $x > 4$. *Thus* $\sqrt{x}-\ln(x)$ *is an increasing function on* $[4,\infty)$. *Since* $\sqrt{4}-\ln(4) > 0$ *it follows* $\sqrt{x}-\ln(x) > 0$ *for* $x > 4$ *and hence* $\sqrt{k} > \ln k$ *for* $k = 4, 5, \dots$. *Therefore, for* $k \geq 4$,

$$\frac{1}{\sqrt{k}+\ln(k)} > \frac{1}{\sqrt{k}+\sqrt{k}} = \frac{1}{2\sqrt{k}}$$

Observe $\sum_{k=4}^{\infty}\frac{1}{2\sqrt{k}}$ *is divergent as it is* $1/2$ *of a tail of the* $p = 1/2$ *series. Thus by the direct comparison test* $\displaystyle\sum_{k=4}^{\infty}\frac{1}{\sqrt{k}+\ln(k)}$ *diverges and it follows that the given series is likewise divergent.*

Example 4.27. *Consider* $\sum_{k=0}^{\infty}\frac{1}{k!} = 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$ *the* k*-th term is* $a_k = 1/k!$. *The* $p = 2$ *series* $\sum_{k=1}^{\infty}\frac{1}{k^2}$ *is covergent with* k*-th term* $b_k = 1/k^2$. *Study the limit of the quotient as* $k \to \infty$,

$$\frac{a_k}{b_k} = \frac{k^2}{k!} = \frac{k^2}{k(k-1)(k-2)!} = \frac{1}{1-1/k}\frac{1}{(k-2)!} \to 0 = L.$$

Thus by **(3.)** *of the limit comparison we find* $\sum_{k=0}^{\infty}\frac{1}{k!}$ *converges.*

Example 4.28. *Consider* $\sum_{n=1}^{\infty}\frac{n!}{n^n+7}$. *Notice for* $n \geq 3$,

$$0 < \frac{n!}{n^n+7} < \frac{n!}{n^n} = \frac{n(n-1)\cdots 4\cdot 3\cdot 2\cdot 1}{n\cdot n\cdots n\cdot n\cdot n\cdot n} < \frac{n(n-1)\cdots 4\cdot 3\cdot 2\cdot 1}{n\cdot(n-1)\cdots 4\cdot 3\cdot n\cdot n} = \frac{2}{n^2}$$

thus the given series converges by direct comparison to the convergent $p = 2$ *series.*

Remark 4.29.

We have seen that the limit comparison and direct comparison tests each have their place. It is wise to remember both for best success. Later we learn the ratio test which is probably what I would first try on some of these examples. Keep in mind this article must be read as a whole for best results.

4.6 absolute convergence and alternating series

Absolute convergence is a very strong form of convergence. Absolutely convergent series allow the nicest calculations. For instance, an absolutely convergent series allows for *rearrangement*. This means an absolutely convergent series has terms which can be added in any order an still the resulting sum is the same. In contrast, there are other series which converge but if we rearrange the terms in the series then the value of the sum can be altered to any real value. This shocking result is known as **Riemann's Rearrangement Theorem**. Also, the product of an absolutely convergent series with a convergent series can be shown to exist. In contrast, the product of convergent series need not produce a convergent series.

Definition 4.30. *absolutely convergent series*

If $\sum_{k=n_o}^{\infty} |a_k|$ converges then $\sum_{k=n_o}^{\infty} a_k$ is said to be **absolutely convergent**.

Example 4.31. *If* $\sum_{k=n_o}^{\infty} a_k$ *converges and* $a_k \geq 0$ *for all* $k \geq n_o$ *then* $\sum_{k=n_o}^{\infty} |a_k|$ *is convergent.*

Theorem 4.32. *absolutely convergent series are summable*

If $\sum_{k=n_o}^{\infty} |a_k|$ is a convergent series then $\sum_{k=n_o}^{\infty} a_k$ is a convergent series.

Proof I: Suppose $\sum_{k=n_o}^{\infty} |a_k|$ is a converges. Define $b_k = a_k$ for $a_k \geq 0$ and $b_k = 0$ for $a_k < 0$. Likewise, define $c_k = a_k$ for $a_k < 0$ and $c_k = 0$ for $a_k \geq 0$. Then $a_k = b_k + c_k$. Similarly, for the n-th partial sum

$$\sum_{k=n_o}^{n} a_k = \sum_{k=n_o}^{n} b_k + \sum_{k=n_o}^{n} c_k = \sum_{k=n_o}^{n} b_k - \sum_{k=n_o}^{n} (-c_k).$$

By construction, $b_k = |a_k|$ for $k \geq n_o$ with $a_k \geq 0$. Also, $c_k = -|a_k|$ for $k \geq n_o$ with $a_k < 0$. Notice, $\sum_{k=n_o}^{n} |a_k|$ is a convergent sequence and is thus bounded above by some $M > 0$, further note:

$$0 \leq \sum_{k=n_o}^{n} b_k \leq \sum_{k=n_o}^{n} |a_k| \leq M \qquad \& \qquad 0 \leq \sum_{k=n_o}^{n} (-c_k) \leq \sum_{k=n_o}^{n} |a_k| \leq M$$

Consequently both $\sum_{k=n_o}^{n} b_k$ and $\sum_{k=n_o}^{n} (-c_k)$ are bounded increasing sequences. Therefore, by the bounded monotonic sequence theorem, there exist B, C for which $\sum_{k=n_o}^{n} b_k \to B$ and $\sum_{k=n_o}^{n} (-c_k) \to C$ as $n \to \infty$. Thus the series of positive terms and the series of negative terms of an absolutely convergent series must separately converge and we calculate:

$$\lim_{n\to\infty} \sum_{k=n_o}^{n} a_k = \lim_{n\to\infty} \sum_{k=n_o}^{n} b_k - \lim_{n\to\infty} \sum_{k=n_o}^{n} (-c_k) = B - C. \ \square$$

Proof II: Suppose $\sum |a_k|$ converges. Note $-|a_k| \leq a_k \leq |a_k|$ thus $0 \leq a_k + |a_k| \leq 2|a_k|$ hence $\sum (a_k + |a_k|)$ converges by direct comparison to $2\sum |a_k|$. Therefore, $\sum a_k$ converges since it is the difference of convergent series; $\sum a_k = \sum (a_k + |a_k|) - \sum |a_k|$. $\square$

Example 4.33. *Consider the series* $S = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \cdots$. *If we take the absolute value term-by-term we obtain* $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$ *which is the convergent* $p = 2$ *series. Thus* S *is absolutely convergent and hence* S *is a convergent series.*

Example 4.34. *Consider the series* $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{\sqrt{n+1}}$. *Observe* $\cos(0) = \cos(2\pi) = \cos(4\pi) = \cdots = 1$ *whereas* $\cos(\pi) = \cos(3\pi) = \cdots = -1$ *thus*[9] $\left|\frac{\cos(\pi n)}{\sqrt{n+1}}\right| = \frac{1}{\sqrt{n+1}}$ *and*

$$\sum_{n=0}^{\infty} \left|\frac{\cos(\pi n)}{\sqrt{n+1}}\right| = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which is a divergent $p = 1/2$ *series. Hence* $\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{\sqrt{n+1}}$ *is not absolutely convergent. However, it turns out*[10] *this series is convergent. This means this series is an example of a* **conditionally convergent series**.

Definition 4.35. *conditionally convergent series*

If $\sum_{k=n_o}^{\infty} |a_k|$ diverges and $\sum_{k=n_o}^{\infty} a_k$ converges then $\sum_{k=n_o}^{\infty} a_k$ said to be **conditionally convergent**. In other words, a series which is convergent but not absolutely convergent is conditionally convergent.

Perhaps the most famous example of a conditionally convergent series is the **alternating harmonic series**. I usually draw a picture in lecture to help understand why its partial sums necessarily converge.

Example 4.36. *Consider* $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. *Let us explicitly calculate the partial sums to gain intuition for why this is a convergent series: let* $S_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ *and calculate*

$$\begin{aligned}
S_1 &= 1 & S_2 &= 1 - \frac{1}{2} = 0.5 \\
S_3 &= S_2 + \frac{1}{3} \approx 0.8333 & S_4 &= S_3 - \frac{1}{4} \approx 0.5833 \\
S_5 &= S_4 + \frac{1}{5} \approx 0.7833 & S_6 &= S_5 - \frac{1}{6} \approx 0.6167 \\
S_7 &= S_6 + \frac{1}{7} \approx 0.7595 & S_8 &= S_7 - \frac{1}{8} \approx 0.6345 \\
S_9 &= S_8 + \frac{1}{9} \approx 0.7456 & S_{10} &= S_9 - \frac{1}{10} \approx 0.6456 \\
S_{100} &\approx 0.688172 & S_{1000} &\approx 0.692647 \\
S_{10000} &\approx 0.693097 & S_{100000} &\approx 0.693142.
\end{aligned}$$

You might recognize $\ln 2 \approx 0.693147$. *In fact, we can prove later that the alternating harmonic series has sum* $\ln 2$. *Notice the magnitude of the error* $|S_n - \ln 2| < \frac{1}{n+1}$ *in every case. This illustrates the alternating series estimation theorem I state later in this section.*

The numerical data in the above example should help demystify the proof of the theorem below.

[9] for your future reference it is helpful to see that $\cos(\pi n) = (-1)^n$.

[10] using Theorem 4.37, the alternating series test

Theorem 4.37. *alternating series test and estimation theorem*

Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k = b_1 - b_2 + b_3 - b_4 + \cdots$ where $b_k > 0$. If

(1.) $b_1 > b_2 > b_3 > \cdots > 0$

(2.) $b_k \to 0$ as $k \to \infty$

then $\sum_{k=1}^{\infty}(-1)^{k+1}b_k = S$ is a convergent series with partial sum S_n satisfying $0 < S < b_1$ and $S_{2n} < S < S_{2n+1}$ for $n \geq 1$. The magnitude of the error $|S_n - S| < b_{n+1}$ for $n = 0, 1, 2, \dots$.

Proof: let $a_k = (-1)^{k+1}b_k$ where $b_k > 0$ and b_k is a decreasing sequence with $b_k \to 0$ as $k \to \infty$. Define $S_n = \sum_{k=1}^{n}(-1)^{k+1}b_k$. Observe, $S_{2n} = -b_{2n} + S_{2n-1}$ for $n = 1, 2, \dots$ and $S_1 = b_1$ whereas $S_{2n-1} = b_{2n-1} + S_{2n-2}$ for $n \geq 2$. Therefore, for $n \geq 2$

$$S_{2n-2} = S_{2n-1} - b_{2n-1} = S_{2n} + b_{2n} - b_{2n-1} < S_{2n}$$

as $b_{2n} < b_{2n-1}$ since b_k is decreasing sequence. We find S_{2n} is an increasing sequence. Likewise, noting $S_{2n+1} = b_{2n+1} + S_{2n}$ yields $S_{2n} = S_{2n+1} - b_{2n+1}$ we calculate:

$$S_{2n-1} = S_{2n} + b_{2n} = S_{2n+1} - b_{2n+1} + b_{2n} > S_{2n+1}$$

as $b_{2n+1} < b_{2n}$ since b_k is decreasing sequence. Thus the subsequence of odd partial sums is a decreasing sequence. In summary:

$$0 < b_1 - b_2 = S_2 < S_4 < \cdots < S_{2n} < S_{2n} + b_{2n+1} = S_{2n+1} < \cdots < S_5 < S_3 < S_1 = b_1.$$

Therefore, S_{2n} and S_{2n-1} are bounded monotonic sequences which converge. Suppose $S_{2n} \to S$ and $S_{2n-1} \to T$ as $n \to \infty$. We find $S = T$ by the following calculation

$$\lim_{n\to\infty}(S_{2n+1} - S_{2n}) = \lim_{n\to\infty} b_{2n+1} \quad \Rightarrow \quad S - T = 0$$

Consequently both S_{2n} and S_{2n-1} converge to S and it follows[11] that the series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$ converges with sum S. Finally the error estimates claimed in the theorem are evident from the arguments given in this proof. $\square$

Remark 4.38.

If we have an alternating series of the form $\sum_{k=n_o}^{\infty}(-1)^{k+1}b_k$ where b_k is a decreasing sequence with $b_k \to 0$ as $k \to \infty$ then clearly the arguments given for the theorem above can be reasonably modified and we will reach similar conclusions. Alternatively, we can derive this result by changing the index to $j = k - n_o + 1$ so $j = 1$ /when $k = n_o$ generally $k = j + n_o - 1$ thus $\sum_{k=n_o}^{\infty}(-1)^{k+1}b_k = \sum_{j=1}^{\infty}(-1)^{j+n_o}b_{j+n_o-1}$ which converges by the alternating series test.

[11] technically I need a little lemma here that when both the even and odd subsequences of a given sequence converge to a common limit then the total sequence likewise converges. This is not hard to prove using ε-style arguments, I leave it to the reader, if interested I can show you in office hours.

Example 4.39. *Consider* $\sum_{k=1}^{\infty} \frac{(-1)^k}{ke^k}$ *notice* $b_k = \frac{1}{ke^k}$ *is positive with* $\lim_{k\to\infty} \frac{1}{ke^k} = 0$*. Define* $f(x) = \frac{1}{xe^x} = e^{-x}\cdot\frac{1}{x}$ *and differentiate to see*

$$f'(x) = -e^{-x}\cdot\frac{1}{x} + e^{-x}\left(\frac{-1}{x^2}\right) = -\frac{e^{-x}}{x^2}(x+1) < 0$$

for $x \geq 1$ *thus* $f(x)$ *is decreasing. Since* $f(k) = b_k$ *we find* b_k *is likewise decreasing. Thus* $\sum_{k=1}^{\infty} \frac{(-1)^k}{ke^k}$ *converges by the alternating series test.*

You probably could show $b_k = \frac{1}{ke^k}$ is decreasing without calculus. I included the calculus argument in the previous example to emphasize a potential technique you can use if in doubt.

Example 4.40. *Consider* $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ *notice* $b_k = \frac{1}{\ln k}$ *is positive with* $\lim_{k\to\infty} \frac{1}{\ln k} = 0$*. If* $f(x) = \frac{1}{\ln x}$ *then* $f'(x) = \frac{-1}{x(\ln x)^2} < 0$ *for* $x \geq 2$*. Since* $f(k) = b_k$ *we find* b_k *is likewise decreasing. Thus* $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln k}$ *converges by the alternating series test.*

Example 4.41. *The value of* $\sin\theta$ *can calculated by the following series in* θ

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \cdots$$

Given that the claim above is true (we'll explain this later in the course), how inaccurate is the approximation $\sin\theta = \theta$ *? For a fixed value* θ *the formula above is a convergent alternating series. Therefore, the error in truncating the series to the first term is no larger than the next term not included in the partial sum. In particular,*

$$|\sin\theta - \theta| < \frac{\theta^3}{6}$$

For example, in radians, $\sin 1 \approx 0.8415$ *thus* $|\sin 1 - 1| \approx 0.1585 < 0.166\cdots = \frac{1}{6}$*. If we to better approximate* $\sin 1$ *then we need to take more terms. For example, if we use* $\sin\theta \approx \theta - \theta^3/6$ *then the error is bounded by* $\theta^5/120$ *as* $5! = 120$*. Indeed,* $\sin 1 \approx 1 - 1/6 \approx 0.8333\ldots$ *gives* $|\sin 1 - 0.833\ldots| \approx 0.0082 < \frac{1}{120} = 0.00833\ldots$*. Alternatively, we can ask what range of* θ *makes the approximation* $\sin\theta \approx \theta$ *accurate to within a percent. Solving the inequality* $\theta^3/6 < 0.01$ *we find* $\theta < \sqrt[3]{0.06} \approx 0.39$*. This is in radians. If we convert to degrees,* $(0.39\ rad)\left(\frac{180^o}{\pi\ rad}\right) \approx 22.4^o$*. Replacing* $\sin\theta$ *with* θ *is known as the* **small angle approximation** *of sine. This is often used in engineering or physics to simplify an otherwise intractable algebra problem.*

Example 4.42. *A disturbing calculation:*

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots &= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \cdots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots\right) \end{aligned}$$

Why is this calculation false ? Clearly if the original series sums to S *then we cannot have* $S = \frac{1}{2}S$ *unless* $S = 0$*, but it is already clear from the previous example that the sum of the alternating harmonic series is nonzero. So... what have we done incorrectly ?*

4.7 ratio and root tests

The ratio and root tests are often useful for series involving factorials and powers. The proof of both of these tests rests on the geometric series result.

Theorem 4.43. *ratio test*

Consider the series $\sum_{k=n_o}^{\infty} a_k$ with $a_k \neq 0$ for $k \geq n_o$. Let $\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ then

(1.) if $\rho < 1$ then $\sum_{k=n_o}^{\infty} a_k$ converges absolutely,

(2.) if $\rho > 1$ then $\sum_{k=n_o}^{\infty} a_k$ diverges,

Proof: (1.) suppose $\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ and $\rho < 1$. Choose R with $\rho < R < 1$ and set $\varepsilon = R - \rho > 0$ in the definition of the limit to see there exists $M \in \mathbb{N}$ for which $k \geq M$ implies

$$\left| \frac{|a_{k+1}|}{|a_k|} - \rho \right| < R - \rho \quad \Rightarrow \quad \frac{|a_{k+1}|}{|a_k|} - \rho < R - \rho \quad \Rightarrow \quad \frac{|a_{k+1}|}{|a_k|} < R$$

thus $|a_{k+1}| < R|a_k|$. Therefore,

$$|a_{M+1}| < R|a_M|, \quad |a_{M+2}| < R|a_{M+1}| < R^2|a_M|, \ \dots, \quad |a_{M+j}| < R^j|a_M|$$

Notice $0 < R < 1$ thus $\sum_{j=1}^{\infty} |a_M|R^j$ is a convergent geometric series. Therefore, the tail $\sum_{k=M+1}^{\infty} |a_k|$ converges by the direct comparison test. Thus $\sum_{k=n_o}^{\infty} a_k$ converges absolutely since the tail of a series converging absolutely implies the whole series converges absolutely.

(2.) Next, suppose $\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ and $\rho > 1$. Choose R such that $1 < R < \rho$. Since $\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \rho$ and $\rho - R > 0$ we may select $M \in \mathbb{N}$ for which $k \geq M$ implies

$$\left| \frac{|a_{k+1}|}{|a_k|} - \rho \right| < \rho - R \quad \Rightarrow \quad -(\rho - R) < \frac{|a_{k+1}|}{|a_k|} - \rho \quad \Rightarrow \quad R < \frac{|a_{k+1}|}{|a_k|}$$

thus $|a_{k+1}| > R|a_k|$ for $k \geq M$. Therefore,

$$|a_{M+1}| > R|a_M|, \quad |a_{M+2}| > R|a_{M+1}| > R^2|a_M|, \ \dots, \quad |a_{M+j}| > R^j|a_M|$$

Observe $\lim_{k\to\infty} a_k \neq 0$ since $|a_{M+j}| > R^j|a_M|$ implies a_k is not bounded since $R > 1$ implies $R^j \to \infty$ as $j \to \infty$. Thus $\sum_{k=n_o}^{\infty} a_k$ diverges by the k-th term test. $\square$

Sometimes this test is stated with a third case which declares that $\rho = 1$ is inconclusive. To see $\rho = 1$ tells you nothing consider that the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} k^2$ both give $\rho = 1$. Let's look at some interesting applications of the ratio test next:

Example 4.44. *Study* $\displaystyle\sum_{k=0}^{\infty} \frac{(-3)^k}{k!}$. *Observe*

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{|-3|^{k+1}}{(k+1)!} \cdot \frac{k!}{|-3|^k} = \frac{3^k 3}{(k+1)k!} \cdot \frac{k!}{3^k} = \frac{3}{k+1} \to 0$$

as $k \to \infty$*. Thus* $\rho = 0 < 1$ *for the ratio test and we conclude* $\displaystyle\sum_{k=0}^{\infty} \frac{(-3)^k}{k!}$ *converges absolutely.*

Fun fact you're not supposed to know yet, the series in the example above has sum $1/e^3$.

Example 4.45. *Study* $\displaystyle\sum_{k=0}^{\infty} \frac{k^k}{k!}$. *Observe*

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} = \frac{(k+1)^{k+1}}{(k+1)k!} \cdot \frac{k!}{k^k} = \frac{(k+1)^k}{k^k} = \left(\frac{k+1}{k}\right)^k = \left(1 + \frac{1}{k}\right)^k \to e$$

as $k \to \infty$*. I'm using the limit which was studied in Example 1.34 which is not immediately obvious, unless you happen to remember that the limit above is a possible definition for* $e \approx 2.71\ldots$*. Thus* $\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = e > 1$ *thus* $\displaystyle\sum_{k=0}^{\infty} \frac{k^k}{k!}$ *diverges by the ratio test. Similarly, we could show* $\displaystyle\sum_{k=0}^{\infty} \frac{k!}{k^k}$ *converges absolutely by the ratio test with* $\rho = 1/e$*.*

Series which diverge by the ratio test naturally give rise to absolutely convergent series formed by summing the reciprocals of the given divergent series.

Example 4.46. *Suppose you're given a series* $\sum_k a_k$ *for which* $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = M > 1$ *and* $a_k \neq 0$ *for each* k*. Then the given series diverges by the ratio test. However, it may be interesting to note* $\sum_k \frac{1}{a_k}$ *converges absolutely since the ratio test gives* $1/M$ *by the calculation below:*

$$\lim_{k\to\infty} \left|\frac{\frac{1}{a_{k+1}}}{\frac{1}{a_k}}\right| = \lim_{k\to\infty} \frac{1}{\left|\frac{a_{k+1}}{a_k}\right|} = \frac{1}{M} < 1.$$

Example 4.47. *Study* $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^2}{2^n}$. *Notice* $|(-1)^n| = 1$ *thus* $|a_n| = \frac{(n+1)^2}{2^n}$ *and observe*

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+2)^2}{2^{n+1}} \cdot \frac{2^n}{(n+1)^2} \\
&= \frac{(n+2)^2}{2^n 2} \cdot \frac{2^n}{(n+1)^2} \\
&= \frac{(n+2)^2}{2(n+1)^2} \\
&= \frac{1}{2}\left(\frac{n+2}{n+1}\right)^2 \\
&= \frac{1}{2}\left(\frac{1+2/n}{1+1/n}\right)^2 \to \frac{1}{2}.
\end{aligned}$$

Therefore, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (n+1)^2}{2^n}$ *converges absolutely by* $\rho = 1/2 < 1$ *ratio test.*

Theorem 4.48. *root test*

Suppose $L = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists.

(1.) If $L<1$ then $\sum_{k=n_o}^{\infty} a_k$ converges absolutely,

(2.) if $L>1$ then $\sum_{k=n_o}^{\infty} a_k$ diverges,

Proof: **(1.)** suppose $L = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ and $L<1$. Choose R with $L<R<1$ and note $R-L>0$ thus there exists $M \in \mathbb{N}$ for which $k \geq M$ implies $|\sqrt[k]{|a_k|} - L| < R-L$ hence $\sqrt[k]{|a_k|} - L < R - L$ which yields $\sqrt[k]{|a_k|} < R$. Therefore, for $k \geq M$ we have $|a_k| < R^k$. But, $\sum_{k=M} R^k$ is a convergent geometric series as $0<R<1$. Therefore, $\sum |a_k|$ converges by the direct comparison to the tail of the series.

(2.) suppose $L = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ and $L>1$. Choose R with $1<R<L$. Since $L-R>0$ we may select $M>0$ for which $k \geq M$ implies $|\sqrt[k]{|a_k|} - L| < L-R$ which gives $R - L < \sqrt[k]{|a_k|} - L$ hence $R < \sqrt[k]{|a_k|}$. Thus $R^k < |a_k|$ for $k \geq M$. Therefore a_k is not bounded and it follows $\lim_{k\to\infty} a_k \neq 0$. Thus $\sum a_k$ diverges by the k-th term test. $\square$.

Example 4.49. *Consider* $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{3k+5}\right)^k$. *Identify* $a_k = \left(\frac{k}{3k+5}\right)^k$ *thus*

$$L = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \frac{k}{3k+5} = \lim_{k\to\infty} \frac{1}{3+5/k} = \frac{1}{3} < 1.$$

Thus $\displaystyle\sum_{k=1}^{\infty}\left(\frac{k}{3k+5}\right)^k$ *converges absolutely by the root test.*

Example 4.50. *Consider* $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n}$. *Identify* $a_n = \frac{1}{(1+\frac{1}{n})^n}$ *and* $|a_n| = a_n$ *thus*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1.$$

The ratio test has nothing to say here. However, if we remember Example 1.34 then this series clearly diverges by the n-th term test since:

$$\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} \neq 0$$

as $n\to\infty$.

Example 4.51. *Consider* $\displaystyle\sum_{n=1}^{\infty}\frac{2^{n^2}}{n!}$. *Notice* $n! < n^n$ *for* $n \geq 1$ *and* $2^{n^2} = (2^n)^n$ *thus we may apply the root test to a series which we can directly compare to the given series. I'll work to show this series diverges in lecture.*

5 Problems

Determine if the given series converge or diverge. If possible, calculate the sum.

Example 5.1. *(diverges)* $\sum_{n=1}^{\infty} \frac{n}{10n+12}$

Example 5.2. *(converges to* $8/7$ *)* $1 + \frac{1}{8} + \frac{1}{8^2} + \cdots$

Example 5.3. *(converges to* $1/(e-1)$ *)* $\sum_{n=1}^{\infty} e^{-n}$

Example 5.4. *(converges)* $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

Example 5.5. *(converges)* $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}+2^n}$

Example 5.6. $\sum_{n=1}^{\infty} (-1)^n n^2$

Example 5.7. *(converges)* $\sum_{m=1}^{\infty} \frac{4}{m!+4^m}$

Example 5.8. *(diverges)* $\sum_{n=3}^{\infty} \left(\frac{3}{11}\right)^{-n}$

Example 5.9. *(converges)* $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}(\ln n)^2}$

Example 5.10. *(converges)* $\sum_{k=1}^{\infty} \frac{\sin^2 k}{k^2}$

Example 5.11. $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)$

Example 5.12. *(converges)* $\sum_{n=1}^{\infty} \frac{2}{3^n + 3^{-n}}$

Example 5.13. *(converges)* $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Example 5.14. *(converges to* $7/15$ *)* $\frac{7}{8} - \frac{49}{64} + \frac{343}{512} - \frac{2401}{4096} + \cdots$

Example 5.15. *(converges)* $\sum_{n=1}^{\infty} \frac{(\ln n)^{100}}{n^{1.1}}$

Example 5.16. *(converges absolutely)* $\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

Example 5.17. *(converges)* $\sum_{n=1}^{\infty} \frac{n}{3^n}$

Example 5.18. *(converges)* $\sum_{k=0}^{\infty} \left(1 + \frac{1}{n}\right)^{-n^2}$

Example 5.19. *(converges)* $\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$

Example 5.20. *(diverges)* $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^3 + 1}}$

Example 5.21. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2 + 1}}$

Example 5.22. *(converges)* $\sum_{n=3}^{\infty} \frac{3n + 5}{n(n-1)(n-2)}$

Example 5.23. *(converges)* $\sum_{n=1}^{\infty} (1 - \cos(1/n))$

Example 5.24. *(converges)* $\sum_{n=1}^{\infty} \left(1 - 2^{-1/n}\right)$

Example 5.25. *(converges to $\frac{59049}{3328}$)* $\sum_{n=-4}^{\infty} \left(-\frac{4}{9}\right)^n$

Example 5.26. *(diverges)* $\sum_{n=1}^{\infty} \frac{2^n}{n^{100}}$

Example 5.27. *(diverges)* $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^4}$

Example 5.28. *(diverges)* $\sum_{n=1}^{\infty} \frac{1 + (-1)^n}{n}$

Example 5.29. *(diverges)* $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

Example 5.30. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n!}$

Example 5.31. *(converges)* $\sum_{n=1}^{\infty} (-1)^n n^2 e^{-n^3/3}$

Example 5.32. *(converges)* $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Example 5.33. *(converges)* $\sum_{n=1}^{\infty} \frac{10^n}{2^{n^2}}$

Example 5.34. *(converges)* $\sum_{n=1}^{\infty} \frac{e^n}{n^n}$

Example 5.35. *(diverges)* $\sum_{n=1}^{\infty} \frac{n!}{6^n}$

Example 5.36. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Example 5.37. *(converges)* $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

Example 5.38. *(converges) Given* $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3}$*, what can we say about* $\sum_{n=1}^{\infty} n^3 a_n$ *?*

Example 5.39. *(converges or diverges) Given* $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3}$*, what can we say about* $\sum_{n=1}^{\infty} 3^n a_n$

Example 5.40. *(converges) Given* $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3}$*, what can we say about* $\sum_{n=1}^{\infty} a_n^2$

Example 5.41. *(converges)* $\sum_{n=1}^{\infty} \frac{1}{n^n}$

Example 5.42. *(converges)* $\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^k$

Example 5.43. *(converges and you can find the sum)* $\sum_{n=1}^{\infty} 4^{-2n+1}$

Example 5.44. *(converges)* $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

Example 5.45. *(diverges)* $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}$

Example 5.46. *(diverges)* $\sum_{n=1}^{\infty} \left(\frac{n}{2}\right)^n \frac{1}{n!}$

Example 5.47. $\sum_{n=4}^{\infty} \frac{\ln n}{n^{3/2}}$

Example 5.48. $\displaystyle\sum_{n=1}^{\infty} \left(\cos\frac{1}{n}\right)^{n^3}$

Example 5.49. *(converges)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+\ln n}}$

Example 5.50. *(converges and you can find the sum)* $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

Example 5.51. *(converges)* $\displaystyle\sum_{n=2}^{\infty} n^{-\ln n}$

Example 5.52. *(converges to 47/180)* $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+3)}$

Example 5.53. *(converges)* $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}+10^n}{n^{11}+11^n}$

Example 5.54. *(converges)* $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+n}{3^n-2}$

Example 5.55. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+\pi n\right)}{\sqrt{n}}$

Example 5.56. *(diverges)* $\displaystyle\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{4}+2\pi n\right)}{\sqrt{n}}$

Example 5.57. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{n^8}$

6 Theorems on Convergence or Divergence of Series

Theorem: (K-TH TERM TEST)
If $\sum_{k=n_o}^{\infty} a_k$ converges then $\lim_{n\to\infty} a_n = 0$. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{k=n_o}^{\infty} a_k$ diverges.

Theorem: (GEOMETRIC SERIES)
The geometric series $c + cr + cr^2 + \cdots$ is summable with sum $\frac{c}{1-r}$ if and only if $|r| < 1$.
If $|r| \geq 1$ then the geometric series is divergent.

Theorem: (ON ADDING AND SCALAR MULTIPLYING SERIES)
Suppose $\sum a_k = A$ and $\sum b_k = B$ where $A, B \in \mathbb{R}$ and $c \in \mathbb{R}$ then

(1.) $\sum(a_k + b_k) = \sum a_k + \sum b_k$

(2.) $c \sum a_k = \sum(ca_k)$.

Similarly, if $\sum a_k$ diverges and $\sum_k b_k$ converges then for $c \neq 0$, both $\sum ca_k$ and $\sum(a_k + b_k)$ diverge.

Theorem: (INTEGRAL TEST)
Let $a_k = f(k)$, where $f(x)$ is a positive, decreasing, and continuous function for $x \geq 1$.

(1.) If $\int_1^{\infty} f(x)dx$ converges then $\sum_{k=1}^{\infty} a_k$ converges.

(2.) If $\int_1^{\infty} f(x)dx$ diverges then $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem: (p-SERIES TEST)
$\sum_{k=1}^{\infty} \frac{1}{k^p}$ is convergent if and only if $p > 1$. If $p \leq 1$ then $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges.

Theorem: (TAIL WAGS THE SERIES)
Consider $\sum_{k=n_o}^{\infty} a_k$ where $m_o, n_o \in \mathbb{Z}$ with $m_o < n_o$ and $a_k \in \mathbb{R}$ for all $k \geq m_o$. Then

(1.) $\sum_{k=n_o}^{\infty} a_k$ converges if and only if $\sum_{k=m_o}^{\infty} a_k$ converges and

$$\sum_{k=m_o}^{\infty} a_k = a_{m_o} + \cdots + a_{n_o - 1} + \sum_{k=n_o}^{\infty} a_k.$$

(2.) $\sum_{k=n_o}^{\infty} a_k$ diverges if and only if $\sum_{k=m_o}^{\infty} a_k$ diverges.

Theorem: (DIRECT COMPARISON TEST)(DCT)
Suppose there exists $M > 0$ for which $n \geq M$ implies $0 \leq a_n \leq b_n$ then

(1.) if $\sum b_k$ converges then, $\sum a_k$ converges,

(2.) if $\sum a_k$ diverges then, $\sum b_k$ diverges.

Theorem: (LIMIT COMPARISON TEST)(LCT)
Suppose $\{a_k\}$ and $\{b_k\}$ are positive sequences and suppose $L = \lim_{k\to\infty} \frac{a_k}{b_k}$ is either finite or ∞. Then,

(1.) if $L > 0$ then $\sum a_k$ converges if and only if $\sum b_k$ converges,

(2.) if $L = \infty$ and $\sum a_k$ converges, then $\sum b_k$ converges,

(3.) if $L = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges,

Theorem: (ABSOLUTE CONVERGENCE IMPLIES CONVERGENCE)
If $\sum_{k=n_o}^{\infty} |a_k|$ is a convergent series then $\sum_{k=n_o}^{\infty} a_k$ is a convergent series.

Theorem: (ALTERNATING SERIES TEST AND ESTIMATION THEOREM)
Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k = b_1 - b_2 + b_3 - b_4 + \cdots$ where $b_k > 0$. If

(1.) $b_1 > b_2 > b_3 > \cdots > 0$

(2.) $b_k \to 0$ as $k \to \infty$

then $\sum_{k=1}^{\infty}(-1)^{k+1}b_k = S$ is a convergent series with partial sum S_n satisfying $0 < S < b_1$ and $S_{2n} < S < S_{2n+1}$ for $n \geq 1$. The magnitude of the error $|S_n - S| < b_{n+1}$ for $n = 0, 1, 2, \ldots$.

Theorem: (RATIO TEST)
Consider the series $\sum_{k=n_o}^{\infty} a_k$ with $a_k \neq 0$ for $k \geq n_o$. Let $\rho = \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ then

(1.) if $\rho < 1$ then $\sum_{k=n_o}^{\infty} a_k$ converges absolutely,

(2.) if $\rho > 1$ then $\sum_{k=n_o}^{\infty} a_k$ diverges,

Theorem: (ROOT TEST)
Suppose $L = \lim_{k\to\infty} \sqrt[k]{|a_k|}$ exists.

(1.) If $L < 1$ then $\sum_{k=n_o}^{\infty} a_k$ converges absolutely,

(2.) if $L > 1$ then $\sum_{k=n_o}^{\infty} a_k$ diverges,