SMITH NORMAL FORM CALCULATIONS

E1 Consider $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 2 & -1 \\ -1 & -1 & 5 \end{bmatrix}$ – ($A^T = A$ so $\lambda \in \mathbb{R}$ exist and A is orthonormally diagonalizable by Real Spectral Th^{m}) –

Find Smith Normal Form of A

$$xI - A = \begin{bmatrix} x-2 & -2 & 1 \\ -2 & x-2 & 1 \\ 1 & 1 & x-5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & x-5 \\ -2 & x-2 & 1 \\ x-2 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow[r_3-(x-2)r_1]{r_2+2r_1} \begin{bmatrix} 1 & 1 & x-5 \\ 0 & x & 2x-9 \\ 0 & -x & -x^2+7x-9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 2x-9 \\ 0 & -x & -x^2+7x-9 \end{bmatrix}$$

$$\xrightarrow{r_3+r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 2x-9 \\ 0 & 0 & -x^2+9x-18 \end{bmatrix} \xrightarrow{c_3-2c_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & -9 \\ 0 & 0 & -x^2+9x-18 \end{bmatrix}$$

$$c_2 \leftrightarrow c_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & -9 & x \\ 0 & -x^2+9x-18 & 0 \end{bmatrix} \xrightarrow{r_2/-9} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x/9 \\ 0 & -x^2+9x-18 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3+(x^2-9x+18)r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -x/9 \\ 0 & 0 & \frac{-x}{9}(x^2-9x+18) \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x(x-3)(x-6) \end{bmatrix}$$

E2

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x-2 & 0 \\ 0 & x-3 \end{bmatrix} \xrightarrow{c_1 - c_2} \begin{bmatrix} x-2 & 0 \\ 3-x & x-3 \end{bmatrix}$$

$$\xrightarrow{r_1 + r_2} \begin{bmatrix} 1 & x-3 \\ 3-x & x-3 \end{bmatrix} \xrightarrow{r_2 + (x-3)r_1} \begin{bmatrix} 1 & x-3 \\ 0 & \underbrace{x-3+(x-3)^2}_{(x-3)[1+x-3]} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & (x-2)(x-3) \end{bmatrix}$$

E3

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x-3 & -1 \\ 0 & x-3 \end{bmatrix} \xrightarrow{c_1 \leftrightarrow c_2} \begin{bmatrix} -1 & x-3 \\ x-3 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2 + (x-3)r_1} \begin{bmatrix} -1 & x-3 \\ 0 & (x-3)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & (x-3)^2 \end{bmatrix}$$

E4

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x & 1 \\ -1 & x \end{bmatrix} \sim \begin{bmatrix} 1 & x \\ x & -1 \end{bmatrix} \xrightarrow{r_2 - xr_1} \begin{bmatrix} 1 & x \\ 0 & -1-x^2 \end{bmatrix}$$

$$\xrightarrow{c_2 - xc_1} \begin{bmatrix} 1 & 0 \\ 0 & -x^2-1 \end{bmatrix} \xrightarrow{-r_2} \begin{bmatrix} 1 & 0 \\ 0 & x^2+1 \end{bmatrix}.$$

$\boxed{E5}$

$$A = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-3 \end{bmatrix} \sim \begin{bmatrix} x-2 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 3-x & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-3 \end{bmatrix}$$

$$\xrightarrow{r_1 + r_3} \begin{bmatrix} 1 & 0 & x-3 & 0 \\ 0 & x-2 & 0 & 0 \\ 3-x & 0 & x-3 & 0 \\ 0 & 0 & 0 & x-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & x-3 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & (x-3)+(x-3)^2 & 0 \\ 0 & 0 & 0 & x-3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & (x-2)(x-3) & 0 \\ 0 & 0 & 0 & x-3 \end{bmatrix}$$

$\cancel{x3}$

$(x-3)[1+x-3]$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 & \\ 0 & 0 & 0 & (x-2)(x-3) & \\ 0 & 0 & x-3 & 0 & \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-2 & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & (x-2)(x-3) \end{bmatrix}}$$

NOT, SMITH NORMAL FORM!

$(x-2) \nmid (x-3)$.

$\gcd(x-2, x-3) = 1$

$-(x-3) + (x-2) = 1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-2-(x-3) & 0 & 0 \\ 0 & 0 & x-3 & 0 \\ 0 & 0 & 0 & (x-2)(x-3) \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3-x & 0 \\ 0 & x-3 & x-3 & 0 \\ 0 & 0 & 0 & (x-2)(x-3) \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (x-2)(x-3) & 0 \\ 0 & 0 & 0 & (x-2)(x-3) \end{bmatrix}$$

$\leftarrow$ there we go. Now $(x-2)(x-3) \mid (x-2)(x-3)$ end of the road. Also, $m_A(x) = (x-2)(x-3)$.

$$(x-3) - (x-3)(3-x) = (x-3)[1-3+x]$$
$$= (x-3)(x-2)$$

E6

$$A = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{array}\right]$$

$$xI - A = \left[\begin{array}{cc|cc} x-2 & -3 & -1 & 0 \\ 3 & x-2 & 0 & -1 \\ \hline 0 & 0 & x-2 & -3 \\ 0 & 0 & 0 & x-2 \end{array}\right]$$

$$\sim \begin{bmatrix} -1 & -3 & x-2 & 0 \\ 0 & x-2 & 3 & -1 \\ x-2 & 0 & 0 & -3 \\ 0 & 0 & 0 & x-2 \end{bmatrix} \xrightarrow{r_3 + (x-2)r_1} \begin{bmatrix} -1 & -3 & x-2 & 0 \\ 0 & x-2 & 3 & -1 \\ 0 & -3(x-2) & (x-2)^2 & -3 \\ 0 & 0 & 0 & x-2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x-2 & 3 & -1 \\ 0 & -3(x-2) & (x-3)^2 & -3 \\ 0 & 0 & 0 & x-2 \end{bmatrix}$$

$$(x-2)^2+9 = x^2-4x+13$$

$$\vdots$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (x^2-4x+13)^2 & 0 \\ 0 & 0 & 0 & (x^2-4x+13)^2 \end{bmatrix}$$

← fixed. We need product of invariant factors to work out to $\text{char}_A(x)$. Here it's quartic.

E7

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \longrightarrow xI - A = \begin{bmatrix} x-3 & 0 \\ 0 & x-3 \end{bmatrix}$$

E8

$$A = \begin{bmatrix} 8 & 1 & 0 \\ 0 & 8 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

$$xI - A = \begin{bmatrix} x-8 & -1 & 0 \\ 0 & x-8 & -1 \\ 0 & 0 & x-8 \end{bmatrix} \sim \begin{bmatrix} -1 & x-8 & 0 \\ x-8 & 0 & -1 \\ 0 & 0 & x-8 \end{bmatrix}$$

$$\xrightarrow{r_2 + (x-8) r_1} \begin{bmatrix} -1 & x-8 & 0 \\ 0 & (x-8)^2 & -1 \\ 0 & 0 & x-8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x-8)^2 & -1 \\ 0 & 0 & x-8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & (x-8)^2 \\ 0 & x-8 & 0 \end{bmatrix}$$

$$\xrightarrow{r_3 + (x-8) r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & (x-8)^2 \\ 0 & 0 & (x-8)^3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (x-8)^3 \end{bmatrix}.$$

E9

$$A = \left[\begin{array}{cc|cc} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

$$xI - A = \left[\begin{array}{cc|cc} x-3 & -1 & 0 & 0 \\ 0 & x-3 & 0 & 0 \\ \hline 0 & 0 & x-3 & -1 \\ 0 & 0 & 0 & x-3 \end{array}\right] \sim \begin{bmatrix} 1 & x-3 & 0 & 0 \\ x-3 & 0 & 0 & 0 \\ 0 & 0 & x-3 & -1 \\ 0 & 0 & 0 & x-3 \end{bmatrix}$$

$$\xrightarrow{r_2 - (x-3) r_1} \begin{bmatrix} 1 & x-3 & 0 & 0 \\ 0 & (x-3)^2 & 0 & 0 \\ 0 & 0 & x-3 & -1 \\ 0 & 0 & 0 & x-3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (x-3)^2 & 0 & 0 \\ 0 & 0 & -1 & x-3 \\ 0 & 0 & x-3 & 0 \end{bmatrix} \xrightarrow{r_4 + (x-3) r_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (x-3)^2 & 0 & 0 \\ 0 & 0 & -1 & x-3 \\ 0 & 0 & 0 & (x-3)^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (x-3)^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (x-3)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & (x-3)^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (x-3)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & (x-3)^2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (x-3)^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (x-3)^2 & 0 \\ 0 & 0 & 0 & (x-3)^2 \end{bmatrix}$$

5+i

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \qquad xI - A = \begin{bmatrix} x-3 & 1 & & & & \\ 0 & x-3 & & & & \\ & & \boxed{x-3} & & & \\ & & & \boxed{x-4} & & \\ & & & & \boxed{x-4} & \\ & & & & & \boxed{x-5} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & x-3 & & & & \\ x-3 & 0 & & & & \\ & & \boxed{x-3} & & & \\ & & & \boxed{x-4} & & \\ & & & & \boxed{x-4} & \\ & & & & & \boxed{x-5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & & & & \\ 0 & (x-3)^2 & & & & \\ & & \boxed{x-3} & & & \\ & & & \boxed{x-4} & & \\ & & & & \boxed{x-4} & \\ & & & & & \boxed{x-5} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & & & \\ & x-5 & & & & \\ & & x-3 & & & \\ & & & x-4 & & \\ & & & & x-4 & \\ & & & & & (x-3)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & & & & & \\ & x-5 & 6-x & & & \\ & & x-3 & & & \\ & & & x-4 & & \\ & & & & x-4 & \\ & & & & & (x-3)^2 \end{bmatrix}$$

$$\xrightarrow[C_2+C_3]{\sim} \begin{bmatrix} 1 & & & & & \\ & 1 & 6-x & & & \\ & x-3 & x-3 & & & \\ & & & x-4 & & \\ & & & & x-4 & \\ & & & & & (x-3)^2 \end{bmatrix} \overset{r_3-(x-3)r_2}{\sim} \begin{bmatrix} \boxed{1} & & & & & \\ & \boxed{1} & 0 & & & \\ & 0 & \boxed{(x-3)-(x-3)(6-x)} & & & \\ & & 0 & \boxed{x-4} & & \\ & & & & \boxed{x-4} & \\ & & & & & \boxed{(x-3)^2} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -\boxed{(x-3)(x-5)} & x-4 & & \\ & & & \boxed{x-4} & & \\ & & & & \boxed{x-4} & \\ & & & & & \boxed{(x-3)^2} \end{bmatrix} \sim \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & (x-4) & & \\ & & (x-4)^2 & x-4 & & \\ & & & & x-4 & \\ & & & & & (x-3)^2 \end{bmatrix}$$

$c_2 + (x-4)c_3$

$\gcd(x^2-8x+15,\ x-4) = 1$ and $-1(x^2-8x+15) + (x-4)(x-4) = 1$

$5+2i$

$$\sim\begin{bmatrix}1&&&&&\\&\boxed{1}&&&&\\&&\boxed{1}&x-4&&\\&&0&x-4-(x-4)^3&&\\&&&&\boxed{x-4}&\\&&&&&\boxed{(x-3)^2}\end{bmatrix}$$

$$(x-4)\left(-(x-4)^2+1\right)$$
$$(x-4)\left[-x^2+8x-16+1\right]$$
$$-(x-4)(x^2-8x+15)$$
$$-(x-4)(x-3)(x-5)$$

$$\sim\begin{bmatrix}1&&&&&\\&\boxed{1}&&&&\\&&\boxed{1}&&&\\&&&\boxed{(x-3)(x-4)(x-5)}&&\\&&&&\boxed{x-4}&\\&&&&&\boxed{(x-3)^2}\end{bmatrix}\sim\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&(x-3)^2&&\\&&&&x-4&\\&&&&&(x-3)(x-4)(x-5)\end{bmatrix}$$

Notice $\underbrace{x^2-6x+9}_{(x-3)^2}+(-x+2)(x-4)=1$

$$\begin{bmatrix}1&&&&\\&\boxed{1}&&&\\&&\boxed{1}&&\\&&&\boxed{\begin{matrix}(x-3)^2&x-4\\&x-4\end{matrix}}&\\&&&&\boxed{(x-3)(x-4)(x-5)}\end{bmatrix}\xrightarrow{C_3+(2-x)C_4}\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&x-4&\\&&&(x-4)(2-x)&x-4&\\&&&&&(x-3)(x-4)(x-5)\end{bmatrix}$$

$$(x-4)(2-x)=-x^2+6x-8$$
$$=-(x-2)(x-4)$$

$$\begin{bmatrix}1&&&&&\\&\boxed{1}&&&&\\&&\boxed{1}&&&\\&&&\boxed{1}&x-4&\\&&&-x^2+6x-8&\boxed{x-4}&\\&&&&&\boxed{(x-3)(x-4)(x-5)}\end{bmatrix}\to\begin{bmatrix}1&&&&&\\&1&&&&\\&&1&&&\\&&&1&&\\&&&&(x-4)(x-3)^2&\\&&&&&(x-3)(x-4)(x-5)\end{bmatrix}$$

$$(x-4)+(x-4)(x^2-6x+8)$$
$$(x-4)\left[\underbrace{1+(x-2)(x-4)}_{x^2-6x+9}\right]=(x-4)(x-3)^2$$

(5+3i)

$$\sim \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & (x-4)(x-3)^2 & \\ & & & & & (x-3)(x-4)(x-5) \end{bmatrix}$$

$$\gcd\left((x-4)(x-3)^2,\ (x-3)(x-4)(x-5)\right) = (x-3)(x-4) = x^2-7x+12.$$

$$\sim \left[\begin{array}{c|c} \tfrac{1}{2}(x-4)(x-3)^2 & (x-3)(x-4)(x-5) \\ \hline & (x-3)(x-4)(x-5) \end{array}\right]$$

Clearly, $\frac{-1}{2}(x-3)(x-4)(x-5) + \frac{1}{2}(x-4)(x-3)^2 = (x-3)(x-4).$

$$\xrightarrow{C_5 - \frac{1}{2}C_6} \begin{bmatrix} (x-3)(x-4) & (x-3)(x-4)(x-5) \\ -\frac{1}{2}(x-3)(x-4)(x-5) & (x-3)(x-4)(x-5) \end{bmatrix}$$

$$\xrightarrow{r_6 + \frac{1}{2}(x-5)r_5} \begin{bmatrix} (x-3)(x-4) & (x-3)(x-4)(x-5) \\ 0 & \underbrace{(x-3)(x-4)(x-5) + \tfrac{1}{2}(x-5)^2(x-4)(x-3)} \end{bmatrix}$$

$$(x-3)(x-4)(x-5)\Big[\underbrace{1+\tfrac{1}{2}(x-5)}_{\frac{1}{2}(2+x-5)}\Big]$$

$$\sim \begin{bmatrix} (x-3)(x-4) & 0 \\ 0 & \frac{1}{2}(x-3)^2(x-4)(x-5) \end{bmatrix}$$

$$\therefore\quad xI - A \sim \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & (x-3)(x-4) & \\ & & & & & (x-3)^2(x-4)(x-5) \end{bmatrix}$$

Rational Canonical Form

Def^n/ If $p(x) = x^d - \alpha_{d-1}x^{d-1} - \cdots - \alpha_0$ for $\alpha_i \in \mathbb{F}$ is a polynomial in $\mathbb{F}[x]$ and $p(x)$ is an order of $\langle\langle v \rangle\rangle$ then the matrix $T_{\langle\langle v \rangle\rangle}$ w.r.t. basis $\{v, Tv, \ldots, T^{d-1}v\}$ of $\langle\langle v \rangle\rangle$ is:

$$A_{p(x)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & & 0 & \alpha_1 \\ 0 & 1 & & \vdots & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & 1 & \alpha_{d-1} \end{bmatrix}$$

Def^n/ The matrix $A_{p(x)}$ is called the <u>companion matrix</u> of a prime polynomial $p(x)$

$\boxed{E1}$ $p(x) = x^2+1$

$$A_{p(x)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\boxed{E2}$ $p(x) = x^2 - 2$

$$A_{p(x)} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$\boxed{E3}$ $p(x) = x^3 - 2$

$$A_{p(x)} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

<u>Remark</u>: even if $f(x)$ not prime, if $f(x)$ is the order of $\langle\langle x \rangle\rangle$ then $f(T)x = 0$ and if <u>monic</u> $f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ then $T_{\langle\langle x \rangle\rangle}$ is defined and w.r.t. $\beta_x = \{x, Tx, \ldots, T^{k-1}x\}$ we find

$$[T_{\langle\langle x \rangle\rangle}]_{\beta_x \beta_x} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & & \vdots & -a_1 \\ \vdots & 1 & & & \vdots \\ 0 & 0 & & 1 & -a_{k-1} \end{bmatrix}$$

<u>companion matrix</u>

Lemma 25.12 (CURTIS)

Let $p(x) = x^d - \alpha_{d-1}x^{d-1} - \cdots - \alpha_1 x - \alpha_0$ be prime and $\langle\langle v \rangle\rangle$ be cyclic subspace with order $p(x)^e$ for some positive integer e. Then w.r.t

$$\beta_v = \{ p(T)^{e-1}v, \ Tp(T)^{e-1}v, \ldots, T^{d-1}p(T)^{e-1}v, \ p(T)^{e-2}v, \ Tp(T)^{e-2}v, \ldots, T^{d-1}p(T)^{e-2}v, \ldots, v, Tv, \ldots, T^{d-1}v \}$$

we find

$$[T_{\langle\langle v\rangle\rangle}]_{\beta_v,\beta_v} = \underbrace{\begin{bmatrix} A & B & & \\ & A & B & \\ & & \ddots & B \\ & & & A \end{bmatrix}}_{(de)\times(de)}$$

"companion" matrix of $p(x)^e$

$A = A_{p(x)}$ $(d \times d)$

$B = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ & & & \\ 0 & & & 0 \end{bmatrix}$ $d\times d$

Remark: the above is from pg. 221 – 222 of CURTIS' LINEAR ALGEBRA, AN INTRODUCTORY APPROACH.

- we have more than one convention for the companion matrix for $p(x)^e$. Here's the difference: we assume $(x^2+1)^2$ is order of $\langle\langle v\rangle\rangle$ for $T: V \to V$.

$\boxed{E4}$ $(x^2+1)^2 \in \mathbb{Q}[x]$

$$* = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\beta_v = \{ (T^2+1)v, T(T^2+1)v, v, T(v) \}$

$[T_{\langle\langle v\rangle\rangle}]_{\beta_v,\beta_v} = *$

$\boxed{E5}$ $(x^2+1)^2 = x^4 + 2x^2 + 1$

$$** = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\beta_v = \{ v, Tv, T^2v, T^3v \}$

$[T_{\langle\langle v\rangle\rangle}]_{\beta_v,\beta_v} = **$

Rational Cannonical Form

For $T : V \longrightarrow V$ over $\mathbb{F}$ choose elementary divisors $\{ p_1(x)^{e_1}, \ldots, p_r(x)^{e_r} \}$ (repeats possible) and order of $p_j(x)^{e_j}$ has V_j for $1 \le j \le r$

Then

$$C = \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & & C_r \end{bmatrix} = C_1 \oplus C_2 \oplus \cdots \oplus C_r$$

is rational cannonical form for T. The basis β s.t. $[T]_{\beta,\beta} = C$ is a rational cannonical basis for T (perhaps, not in (CURTIS...)

Defn/ The Jordan normal form of $T : V \longrightarrow V$ is the rational cannonical form in the case the minimal polynomial is split (CURTIS ONLY!)

$\boxed{\text{EG}}$ if $m_T(x) = (x^2+1)^2 = (x-i)^2(x+i)^2$
$= (x^2 - 2ix - 1)(x^2 + 2ix - 1)$

$$\underbrace{\left[\begin{array}{cc|cc} i & 1 & & \\ 0 & i & & \\ \hline & & -i & 1 \\ & & 0 & -i \end{array}\right]}_{\text{CURTIS' Rational Cannonical Form.}} \qquad \underbrace{\left[\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 2i & & \\ \hline & & 0 & 1 \\ & & 1 & -2i \end{array}\right]}_{\text{typical rational cannonical form (D\&F or Insel Spence and Friedberg)}}$$

Remark: LEFT STEMS FROM choosing generalized e-basis for T, chains

$\boxed{E7}$

<u>Example</u> (4 from pg 496 - 497 of 3rd Ed. of Insel Spence & Friedberg's Linear Algebra)

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

has $\text{char}_A(x) = (x^2 - 2x + 5)^2$
$= [(x-1)^2 + 4]^2$

this matrix is similar to

$$C = \begin{bmatrix} 0 & 0 & 0 & -25 \\ 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$\boxed{E8}$

<u>Example</u>: $A = \begin{bmatrix} 0 & 2 & 0 & -6 & 2 \\ 1 & -2 & 0 & 0 & 2 \\ 1 & 0 & 1 & -3 & 2 \\ 1 & -2 & 1 & -1 & 2 \\ 1 & -4 & 3 & -3 & 4 \end{bmatrix}$

has $\text{char}_A(x) = (x^2+2)^2(x-2)$ and it turns out

$$C = \left[\begin{array}{cc|cc|c} 0 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 \end{array}\right] = C_1 \oplus C_2 \oplus C_3$$