Calculus III, Test 1, February 13, 2008 Name: James Cook.

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. This test has 100 points. Try to at least attempt each part.

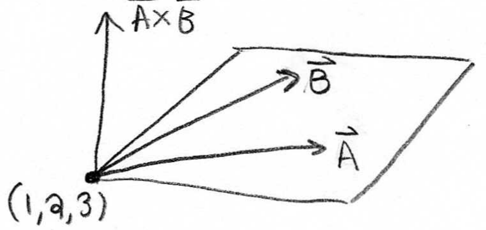
1) [5pts] If $\vec{v} = < 4, 3 >$ then find the vector in the same direction as $\vec{v}$ but with length 3.

$\vec{W} = k\vec{V}$ with $|\vec{W}| = 3$. We have $\vec{W} = <4k, 3k>$

hence $|\vec{W}| = \sqrt{16k^2 + 9k^2} = \sqrt{25k^2} = 5k = 3 \Rightarrow k = \frac{3}{5}$

Thus $\boxed{\vec{W} = \frac{3}{5}<4,3>}$

2) [15pts] Find the equation of a plane which contains the point $(1,2,3)$ and the vectors $\vec{A} = \hat{\imath}$ and $\vec{B} = \hat{\jmath} + 3\hat{k}$. (note these vectors lie entirely in the plane, they are not necessarily based at the origin)



$$\begin{aligned}\vec{A}\times\vec{B} &= \hat{\imath}\times(\hat{\jmath} + 3\hat{k}) \\ &= \hat{\imath}\times\hat{\jmath} + 3\hat{\imath}\times\hat{k} \\ &= \hat{k} - 3\hat{\jmath} \\ &= <0, -3, 1>\end{aligned}$$

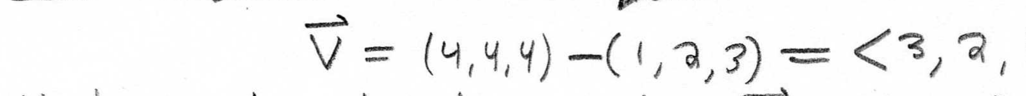
Thus, using (1,2,3) as the base-point, we find

$$\boxed{-3(y-2) + z - 3 = 0}$$

$\boxed{-3y + z = 3 - 6}$ or $\boxed{3y - z = 3}$

Lot's of ways to write the eq$^{\underline{n}}$.

3) [10pts] Find the equation of the line which passes through the points (1,2,3) and (4,4,4).



$$\vec{V} = (4,4,4) - (1,2,3) = <3,2,1>$$

We have the direction vector $\vec{V} = <3,2,1>$. I'll use (1,2,3) as my initial point (my choice, (4,4,4) also fine)

$\boxed{\vec{r}(t) = (1,2,3) + t<3,2,1>}$ or $\boxed{\begin{aligned} x &= 1 + 3t \\ y &= 2 + 2t \\ z &= 3 + t\end{aligned}}$

4) [10pts] What is the angle between the vectors $\vec{A} = <4, 3>$ and $\vec{B} = <12, 5>$? Please leave your answer in terms of an inverse trigonometric function. Also, what is the projection of $\vec{A}$ onto $\vec{B}$?

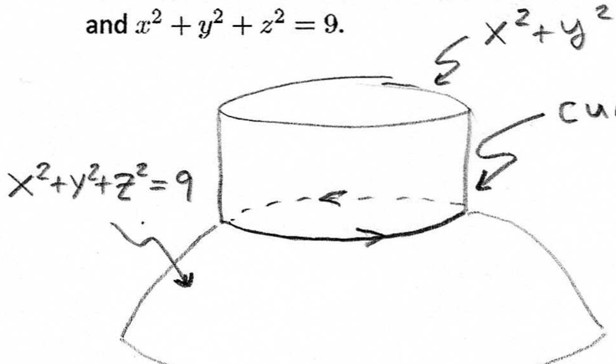
$$|\vec{A}| = \sqrt{16+9} = \sqrt{25} = 5$$
$$|\vec{B}| = \sqrt{144+25} = \sqrt{169} = 13$$

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta = 5(13)\cos\theta$$
$$\vec{A} \cdot \vec{B} = \langle 4,3\rangle \cdot \langle 12,5\rangle = 48 + 15 = 63$$

$$\Rightarrow 65\cos\theta = 63 \Rightarrow \cos\theta = \frac{63}{65} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{63}{65}\right)}$$

$$\text{proj}_{\vec{B}}(\vec{A}) = (\vec{A}\cdot\hat{B})\hat{B} = \frac{(\vec{A}\cdot\vec{B})}{B^2}\vec{B} = \frac{63}{169}\vec{B} = \boxed{\frac{63}{169}\langle 12,5\rangle = \text{proj}_{\vec{B}}(\vec{A})}$$

5) [10pts] Give parametric equations for a curve which is formed by the intersection of $x^2 + y^2 = 2$ and $x^2 + y^2 + z^2 = 9$.



Let $x = \sqrt{2}\cos\theta$

$y = \sqrt{2}\sin\theta$

This parametrizes $x^2+y^2=2$.

Now $x^2+y^2+z^2=9$ intersected with $x^2+y^2=2$ yields $2+z^2=9 \Rightarrow z^2 = 7 \Rightarrow z = \pm\sqrt{7}$

So there are two choices. I only asked for one.

$$\boxed{\begin{array}{l} \vec{r}_+(\theta) = \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, \sqrt{7}\rangle \\ \hline \vec{r}_-(\theta) = \langle \sqrt{2}\cos\theta, \sqrt{2}\sin\theta, -\sqrt{7}\rangle \end{array}}$$

$0 \leq \theta \leq 2\pi$ will cover the curve once.

Other answer's possible: $\vec{r}_\pm(x) = \langle x, \pm\sqrt{2-x^2}, \sqrt{7}\rangle$

(need two parametrizations for one curve, for x-parametrization)

6) [20pts] Suppose that $\vec{v}(t) = < e^t, t^2, \cos(t) >$ represents the velocity of the Dwight. If the initial position of Dwight is the origin then find the position of Dwight at time t. Also find the acceleration of Dwight at time t.

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \boxed{\langle e^t, 2t, -\sin(t) \rangle = \vec{a}(t)}$$

$$\vec{V}(t) = \frac{d\vec{r}}{dt} \Rightarrow \int_0^t \vec{V}(u)\,du = \int_0^t \frac{d\vec{r}}{du}\,du = \vec{r}(t) - \vec{r}(0)$$

But, $\vec{r}(0) = \vec{0}$ hence,

$$\vec{r}(t) = \int_0^t \vec{V}(u)\,du = \int_0^t \langle e^u, u^2, \cos(u) \rangle\,du$$

$$= \left\langle e^u \Big|_0^t, \frac{1}{3}u^3 \Big|_0^t, \sin(u) \Big|_0^t \right\rangle$$

$$= \boxed{\langle e^t - 1, \frac{1}{3}t^3, \sin(t) \rangle = \vec{r}(t)}$$

7) [10pts] Suppose $f(x, y) = e^x + x^y$ calculate f_x and f_y.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^x + x^y\right] = \boxed{e^x + yx^{y-1} = f_x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\underset{0}{e^x} + x^y\right] = \boxed{\ln(x)\,x^y = f_y}$$

☆ (8) is easiest if you notice $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

thus $\frac{\partial f}{\partial r} = \frac{r}{\sqrt{x^2+y^2}} = \frac{r}{\sqrt{r^2}} = 1$, best answer was $\boxed{\frac{\partial f}{\partial r} = 1}$

8) [14pts] Suppose $f(x,y) = \sqrt{x^2+y^2}$ and $x = r\cos(\theta)$, y ~~=~~ $= r\sin(\theta)$. Calculate $\frac{\partial f}{\partial r}$ and $\frac{\partial^2 f}{\partial\theta\partial r}$.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta = \boxed{\frac{x\cos\theta}{\sqrt{x^2+y^2}} + \frac{y\sin\theta}{\sqrt{x^2+y^2}} = \frac{\partial f}{\partial r}}$$

Of course, $x = r\cos\theta$ and $y = r\sin\theta$ thus

we find $x\cos\theta = r\cos^2\theta$ and $y\sin\theta = r\sin^2\theta$

Consequently $\frac{\partial f}{\partial r} = \frac{r\cos^2\theta + r\sin^2\theta}{\sqrt{x^2+y^2}} = \boxed{\frac{r}{\sqrt{x^2+y^2}} = \frac{\partial f}{\partial r}}$

Hence,

$$\frac{\partial^2 f}{\partial\theta\partial r} = \frac{\partial}{\partial\theta}\left[\frac{r}{\sqrt{x^2+y^2}}\right] = r\left\{\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2}}\right]\frac{\partial x}{\partial\theta} + \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2}}\right]\frac{\partial y}{\partial\theta}\right\}$$

$$\therefore \frac{\partial^2 f}{\partial\theta\partial r} = r\left\{-x(x^2+y^2)^{-3/2}(-r\sin\theta) - y(x^2+y^2)^{-3/2}(r\cos\theta)\right\}$$

$$= r\left\{(x^2+y^2)^{-3/2}\left[r^2\cos\theta\sin\theta - r^2\sin\theta\cos\theta\right]\right\} = \boxed{0 = \frac{\partial^2 f}{\partial\theta\partial r}}$$

☆

9) [6pts] Let $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ be a space curve with $\frac{d\vec{r}}{dt} \neq 0$. If we define $T(t) = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt}$ then show that

a.) $|T(t)| = 1$

$$|T(t)| = \left|\frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\frac{d\vec{r}}{dt}\right| = \frac{1}{\left|\frac{d\vec{r}}{dt}\right|}\left|\frac{d\vec{r}}{dt}\right| = 1.$$

b.) $T(t)$ is orthogonal to $\frac{dT}{dt}$.

$$|T|^2 = T\cdot T = 1$$

$$\Rightarrow \underbrace{\frac{dT}{dt}\cdot T + T\cdot\frac{dT}{dt}}_{2\frac{dT}{dt}\cdot T} = \frac{d}{dt}(1) = 0$$

$$2\frac{dT}{dt}\cdot T = 0$$

$\therefore$ $\boxed{\frac{dT}{dt}\cdot T = 0 \quad \text{thus } T \text{ is } \perp \text{ to } \frac{dT}{dt}}$