No graphing calculators and show your work with proper notation. There are at least 140pts to earn. Box your answers and work problems in the white space provide and box your answers. Thanks and Enjoy! Please answer these survey questions to the best of your memory:

I have missed ________. unexcused classes to the best of my memory.
I estimate that I invested ________. hours per week studying outside of class on average to the best of my memory.

Problem 1 [5pts] Find parametric equations for the line segment from $(-1,-4)$ to the point $(3,6)$. Construct your equations such that the domain for the parameter is simply $[0,1]$.

$$\vec{r}(t) = \langle -1,-4\rangle + t\left[\langle 3,6\rangle - \langle -1,-4\rangle\right]$$
$$= \langle -1,-4\rangle + t\langle 4, 10\rangle$$
$$= \langle -1+4t, -4+10t\rangle$$

a.k.a

$$\boxed{\begin{aligned} x &= -1+4t \\ y &= -4+10t \end{aligned}}$$

Check: $\vec{r}(0) = \langle -1,-4\rangle$, $\vec{r}(1) = \langle -1+4, -4+10\rangle = \langle 3,6\rangle$ ✓

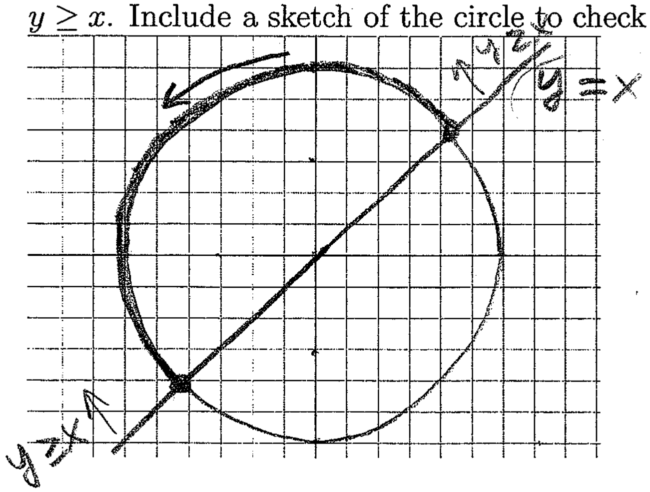
Problem 2 [2pts] Define C to be the set of all points $(x,y) \in \mathbb{R}^2$ such that $x^2 - y^2 = x$.Find a formula for $F(x,y)$ where F is a function $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that the inverse image $F^{-1}(\{0\}) = C$.

$$x^2 - y^2 = x \iff \underbrace{x^2 - y^2 - x}_{} = 0$$

$$\boxed{F(x,y) = x^2 - y^2 - x}$$

$$F^{-1}(\{0\}) = \{(x,y) \in \mathbb{R}^2 \mid F(x,y) = x^2 - y^2 - x = 0\} = C.$$

Problem 3 [10pts] Find a CCW parametrization of part of the circle $x^2 + y^2 = 4$ which has $y \geq x$. Include a sketch of the circle to check your logic on the inequality.



one choice is $\theta = t$

so clearly $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

and

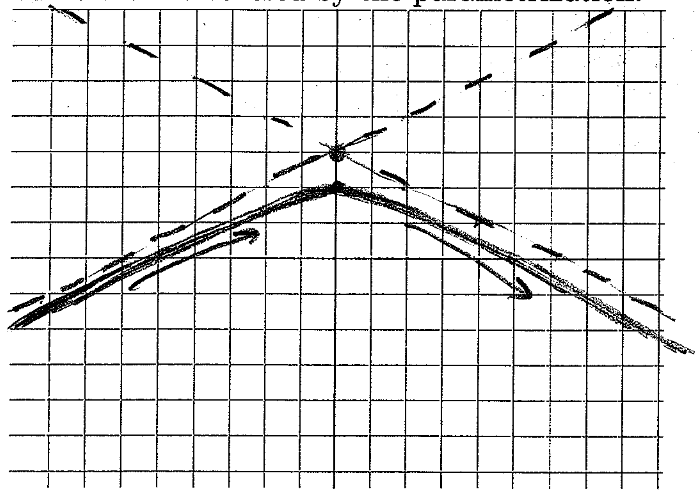
$$\boxed{\begin{aligned} &\vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta\rangle \\ &\theta \in [\pi/4, 5\pi/4] \end{aligned}}$$

a.k.a $x = 2\cos t$, $y = 2\sin t$ for $\frac{\pi}{4} \leq t \leq \frac{5\pi}{4}$.

Problem 4 [8pts] Find the Cartesian form of the curve described parametrically below:

$x = 2\sinh(t), \qquad y = 3 - \cosh(t)$

for $t \in \mathbb{R}$. Sketch the curve in the plot provided, do not draw more of the Cartesian curve than is covered by the parametrization.



$\cosh t = 3 - y \geq 1$

$\Rightarrow 2 \geq y$

$\Rightarrow y \leq 2$

$(3-y)^2 - \left(\frac{x}{2}\right)^2 =$

$= \cosh^2 t - \sinh^2 t$

$= 1$

$\therefore \boxed{(y-3)^2 - \frac{x^2}{4} = 1}$

hyperbola (just part of it with $y \leq 2$)

asymptotes

$(y-3)^2 = \frac{x^2}{4}$

$y = 3 \pm \frac{1}{2}x$

Problem 5 [12pts] Suppose $\vec{r} : [0,6] \to \mathbb{R}^2$ is the vector-parametric description of a curve C. In particular suppose:

$$\vec{r}(t) = \langle (t-4)^7, (t-1)(t-3)^2 \rangle$$

Find:

$x = (t-4)^7 \qquad y = (t-1)(t-3)^2$

(a) all horizontal tangents

$0 = \frac{dy}{dt} = (t-3)^2 + (t-1)2(t-3) = (t-3)[3t-5]$

$\Rightarrow \boxed{t = 3}$ or $\boxed{t = 5/3}$

(b) all vertical tangents

$0 = \frac{dx}{dt} = 7(t-4)^6 \rightarrow \boxed{t = 4}$

now we'd plug these into $\vec{r}(t)$ to find specific eq's for the tangents but I said the t-values sufficed in test.

(c) an integral representation for the arclength of C (DO NOT CALCULATE THE INTEGRAL !)

$$\boxed{s = \int_0^6 \sqrt{49(t-4)^{12} + (t-3)^2(3t-5)^2}\, dt}$$

obviously you might have written the radicand differently.

Problem 6 [12pts] Let $\vec{v}(t) = \langle \sin(t), t^3 \rangle$ denote the velocity of some particle at time t. You are given that the particle is at $(2, 1)$ at time zero. Derive the following:

(a) the acceleration as a function of time t

3

$$\vec{a} = \frac{d\vec{v}}{dt} = \langle \cos t, 3t^2 \rangle$$

(b) the position as a function of time t

54

$$\vec{r}(t) = \vec{c} + \int \langle \sin t, t^3 \rangle dt = \vec{c} + \langle -\cos t, \frac{t^4}{4} \rangle$$

$$\vec{r}(0) = (2,1) = \vec{c} + \langle -1, 0 \rangle \therefore \vec{c} = \langle 3, 1 \rangle$$

$$\boxed{\vec{r}(t) = \langle 3 - \cos t, 1 + t^4/4 \rangle}$$

(c) the speed as a function of time t

2

$$\frac{ds}{dt} = v = \sqrt{\sin^2(t) + t^6}$$

(d) an integral which represents the distance traveled during the duration $1 \leq t \leq 2$ (again set-up, but do NOT calculate the integral)

3

$$s = \int_{s(1)}^{s(2)} ds = \int_1^2 v\,dt = \boxed{\int_1^2 \sqrt{\sin^2 t + t^6}\, dt}$$

Problem 7 [16pts] Calculate the arclength of the curve described parametrically by:

$$x = 3^t \cos(t), \; y = 3^t \sin(t), \qquad 0 \leq t \leq 1.$$

$$\frac{dx}{dt} = \ln(3) 3^t \cos t - 3^t \sin t = 3^t [\ln(3) \cos t - \sin t]$$

$$\frac{dy}{dt} = \ln(3) 3^t \sin t + 3^t \cos t = 3^t [\ln(3) \sin t + \cos t]$$

$$ds^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3^t)^2 [\ln(3)^2 \cos^2 t - \cancel{2\ln(3) \cos t \sin t} + \sin^2 t + \ln(3)^2 \sin^2 t + \cancel{2\ln(3) \sin t \cos t} + \cos^2 t]$$

$$= (3^t)^2 [\ln(3)^2 + 1] \rightarrow ds = 3^t \sqrt{1 + (\ln(3))^2}\, dt$$

$$S = \int_0^1 3^t \sqrt{1 + (\ln(3))^2}\, dt = \frac{\sqrt{1 + (\ln(3))^2}}{\ln(3)} 3^t \Big|_0^1$$

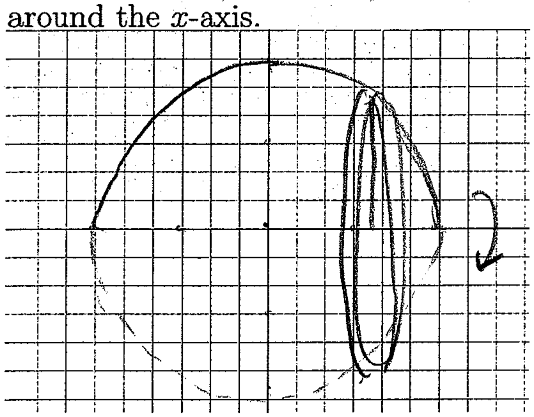
$$= \boxed{\frac{2\sqrt{1 + \ln(3)^2}}{\ln(3)}}$$

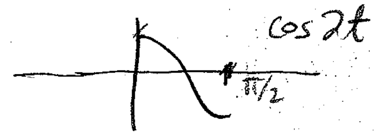
Problem 8 [15pts] Let $A > 0$ and define a curve C by the parametric equations

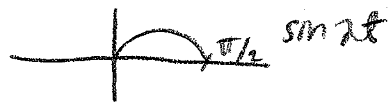
$$x = A\cos(2t), \quad y = A\sin(2t), \qquad 0 \le t \le \pi/2.$$

← half circle because

Calculate the surface area of the surface of revolution formed by rotating the curve C around the x-axis.





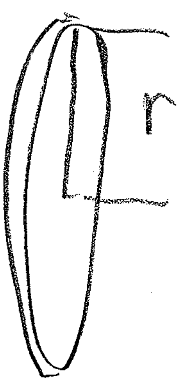


$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{(-2A\sin(2t))^2 + (2A\cos 2t)^2}\, dt$$

$$= \sqrt{4A^2(\sin^2 2t + \cos^2 2t)}\, dt$$

$$= 2A\, dt$$



$r = y = A\sin t$

$$A = \int_0^{\pi/2} 2\pi A \sin 2t \; 2A\,dt = 4\pi A^2 \left(-\frac{\cos(2t)}{2} \Big|_0^{\pi/2} \right) = \boxed{4\pi A^2}$$

Problem 9 [5pts] Find the polar form of the equation $xy^3 + \tan(y/x) = 3$.

$x = r\cos\theta, \quad y = r\sin\theta, \quad \tan\theta = y/x$

$$\boxed{(r\cos\theta)(r\sin\theta)^3 + \tan(\tan\theta) = 3}$$

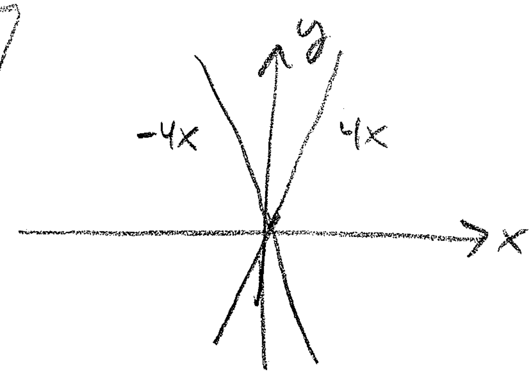
Problem 10 [5pts] Find the Cartesian form of the polar equation $3r\cos(\theta) = r$.

$$3x = \sqrt{x^2 + y^2}$$

$$9x^2 = x^2 + y^2$$

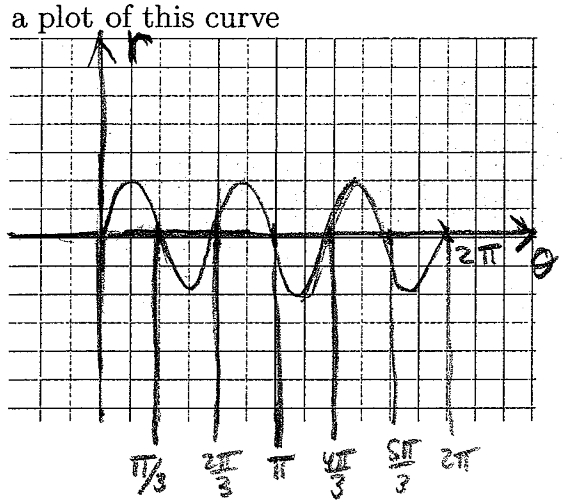
$$\boxed{8x^2 = y^2}$$

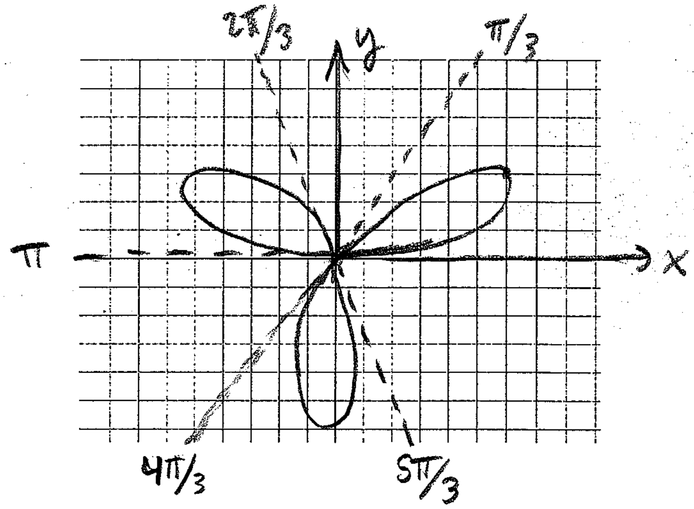
$$y = \pm 4x$$



Problem 11 [12pts] Consider the polar curve $r = \sin(3\theta)$. Find:

(a) a plot of this curve





(b) the area enclosed by the curve

$$A = 3\int_0^{\pi/3} \frac{1}{2}\left(\sin(3\theta)\right)^2 d\theta = \frac{3}{4}\int_0^{\pi/3}\left(1 - \overset{0}{\cancel{\cos(6\theta)}}\right)d\theta$$

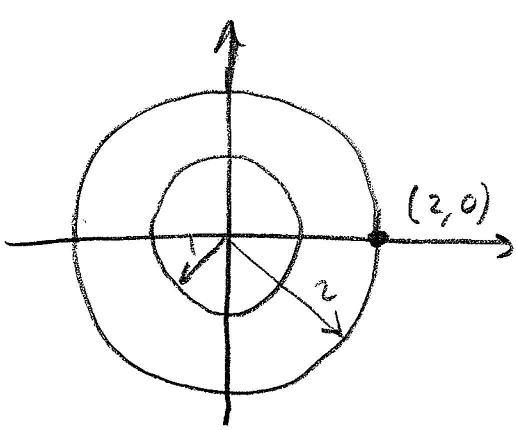
By symmetry!

$$= \left(\frac{3}{4}\right)\left(\frac{\pi}{3}\right)$$

$$= \boxed{\frac{\pi}{4}}$$

Problem 12 [3pts] Find the equation of the tangent line to the curve $r^2 - 3r + 2 = 0$ at the point $(2, 0)$.

$$(r-1)(r-2) = 0$$

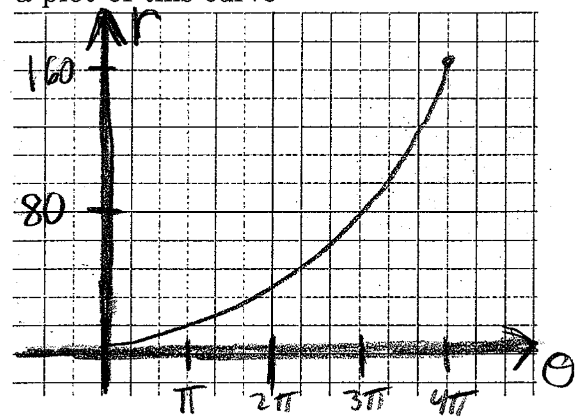


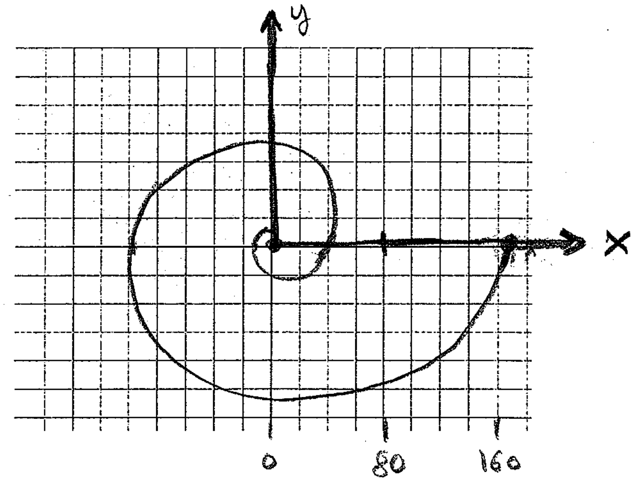
Eqⁿ of tangent line is clearly $\boxed{x = 2}$

$(4\pi)^2 \approx 16\pi^2 \approx 16(10) \approx 160 \qquad (3\pi)^2 \approx 9(10)$

Problem 13 [15pts] Consider the polar curve $r = \theta^2$ for $0 \le \theta \le 4\pi$. Find:

(a) a plot of this curve





~~(b) the area enclosed by the curve~~

(c.) the arclength of the curve

$$ds^2 = r^2 d\theta^2 + dr^2$$

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta$$

~~(c) the arclength of the curve.~~

$$= \sqrt{\theta^4 + 4\theta^2}\, d\theta$$

$$= \theta\sqrt{\theta^2 + 4}\, d\theta \qquad \text{since } \theta \ge 0$$

$$S = \int_0^{4\pi} \theta\sqrt{\theta^2+4}\, d\theta \qquad \text{let } u = \theta^2 + 4$$

$$du = 2\theta\, d\theta$$

$$u(0) = 4$$

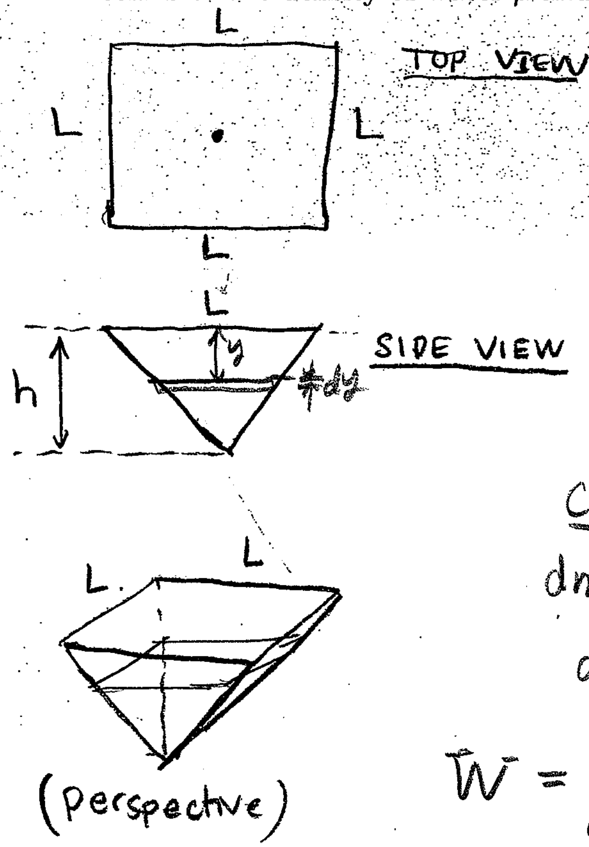
$$u(4\pi) = 4 + 16\pi^2$$

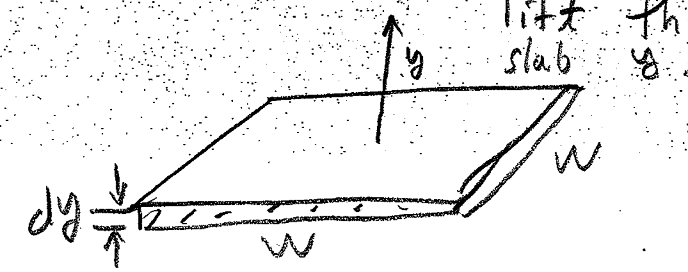
$$= \int_4^{4+16\pi^2} \frac{1}{2}\sqrt{u}\, du$$

$$= \frac{1}{3}(u)^{3/2}\Big|_4^{4+16\pi^2}, \qquad 4^{3/2} = (2^2)^{3/2} = 2^3 = 8$$

$$= \boxed{\frac{1}{3}\left((4+16\pi^2)^{3/2} - 8\right)}$$

Problem 14 [15pts] Suppose you wish to remove the water from a upside-down-square-pyramid shaped pool of side-length L and depth h. What is the work required to pump the water to the surface. Assume this pool is on earth and leave your answer in terms of the density of water ρ and the acceleration due to gravity g.





$$\underline{w(y) = L - \left(\frac{L}{h}\right) y = L(1 - y/h)}$$

$\underline{\text{Check:}}$ $w(0) = L$ and $w(h) = 0$

$$dm = \rho\, dV = \rho w^2 dy$$

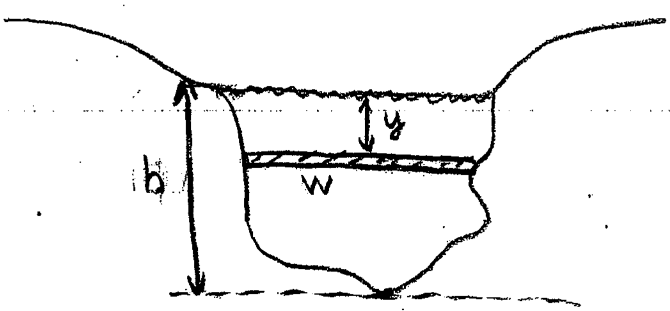
$$dW = [(dm)g][y] \quad (\text{force})(\text{distance})$$

$$W = \int_0^h \rho g y \left[L(1 - y/h)\right]^2 dy$$

$$= \rho g L^2 \int_0^h \left[y - \frac{2y^2}{h} + \frac{y^3}{h^2}\right] dy$$

$$= \rho g L^2 \left[h^2/2 - \frac{2h^3}{3h} + \frac{h^4}{4h^2}\right] = \boxed{\frac{\rho g L^2 h^2}{12}}$$

Problem 15 [5pts] Set-up an integral for the hydrostatic force on a dam pictured below. Note that y is the depth to the the horizontal strip with width $w = f(y)$.



$$dA = w\,dy = f(y)\,dy$$

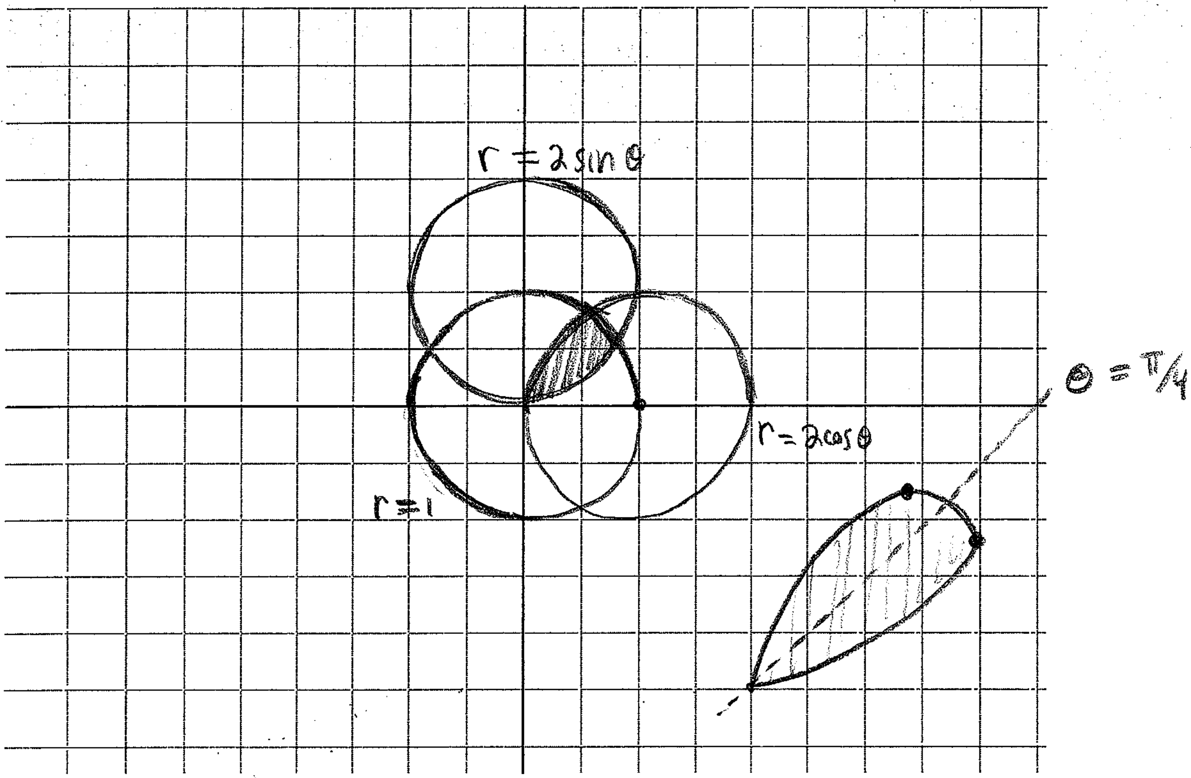
$$P = \rho g y = \frac{dF}{dA}$$

$$dF = \rho g y\, dA = \rho g y f(y)\, dy$$

$$\underline{F_{net} = \int_0^h \rho g y f(y)\, dy}$$

Problem 16 [15pts] [*this problem is intentionally difficult and is meant as a challenge.*]

Calculate the area of the intersection of the areas bounded by $r = 2\sin(\theta)$, $r = 2\cos(\theta)$ and $r = 1$.



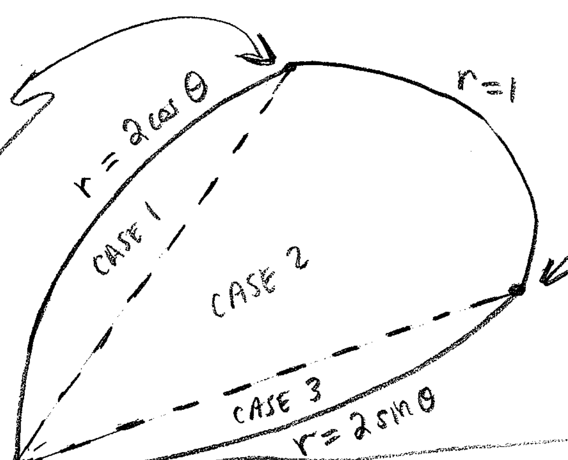
$$r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2+y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$$

$$r = 2\cos\theta \Rightarrow r^2 = 2r\cos\theta \Rightarrow x^2+y^2 = 2x \Rightarrow (x-1)^2 + y^2 = 1$$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \pi/6$



$1 = 2\sin\theta$

$\sin\theta = \frac{1}{2} \therefore \theta = \pi/6$

AREA CASE 1 = AREA CASE 3 By symmetry

$$A = 2\int_0^{\pi/6} \frac{1}{2}(2\sin\theta)^2\,d\theta + \int_{\pi/6}^{\pi/3} \frac{1}{2}(1)^2\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/6}(1-\cos 2\theta)\,d\theta + \frac{\pi}{12} = 2\left(\frac{\pi}{6} - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/6}\right) + \frac{\pi}{12}$$

$$= \frac{\pi}{3} + \frac{\pi}{12} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \boxed{\frac{5\pi}{12} - \frac{\sqrt{3}}{4}}$$