Calculus III, Test 2, March 27, 2009 Name: .

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. This test has 100 points. Make sure to at least attempt each part.

1) [10pts] Calculate the following integral.

$$\int_0^1 \int_0^1 4yz \, dy \, dz = \int_0^1 \left(2z\, y^2 \Big|_0^1 \right) dz$$

$$= \int_0^1 2z \, dz$$

$$= z^2 \Big|_0^1$$

$$= \boxed{1}$$

2) [10pts] Calculate the following integral.

$$\int_0^{\frac{\pi}{2}} \int_0^{2x} \sin(x^2) \, dy \, dx = \int_0^{\pi/2} \left(\sin(x^2)\, y \Big|_0^{2x} \right) dx$$

$$= \int_0^{\pi/2} 2x \sin(x^2) \, dx$$

$$= \int_0^{\pi^2/4} \sin(u) \, du \qquad u = x^2, \quad du = 2x\,dx, \quad u(\pi/2) = \pi^2/4$$

$$= -\cos(u) \Big|_0^{\pi^2/4}$$

$$= \boxed{1 - \cos(\pi^2/4)}$$

3) [5pts] Give the Cartesian description of the surface defined by $\theta = \frac{\pi}{4}$. The "description" consists of an equation as well as an inequality.

$$\tan\theta = \frac{y}{x} \Rightarrow \tan(\pi/4) = 1 = \frac{y}{x} \quad \therefore \boxed{y = x}$$

However, by geometry of θ we need $x > 0$.

4) [5pts] Give the Cartesian description of the surface defined by $\phi = \frac{\pi}{3}$.

$$z = \rho\cos\phi \Rightarrow z = \rho\cos\left(\frac{\pi}{3}\right) = \rho/2$$

$$\Rightarrow z = \frac{1}{2}\sqrt{x^2+y^2+z^2}$$

$$\Rightarrow 4z^2 = x^2+y^2+z^2$$

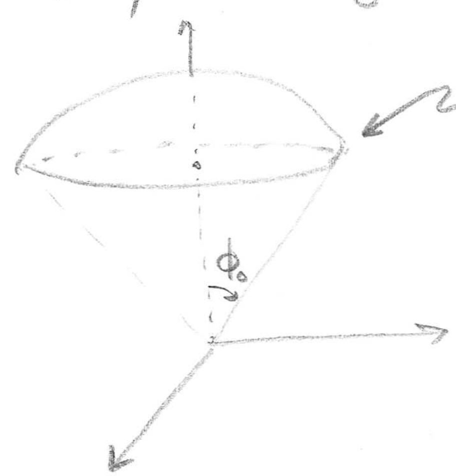
$$\Rightarrow \boxed{3z^2 = x^2+y^2} \quad \text{a cone}$$

we have $z > 0$ in this case.

5) [5pts] Let E be the solid region bounded by $x^2+y^2+z^2 = 4$ and $z \geq 1$. Describe this region in terms of explicit inequalities for spherical coordinates ρ, θ, ϕ. Drawing a picture might help.

Notice $x^2+y^2+z^2 = 4 \Leftrightarrow \rho = 2$, it's a sphere.

The plane $z = 1$ is the base of the solid region $z \geq 1$, this gives the cap of the sphere.



intersection of $\rho = 2$ and $z = 1$ has

$$x^2+y^2+z^2 = \underbrace{x^2+y^2+1 = 4}$$

$$x^2+y^2 = 3 \quad \therefore \underline{r = \sqrt{3}}$$

Recall $r = \rho\sin\phi$

thus $\sqrt{3} = 2\sin\phi_0 \Rightarrow \sin\phi_0 = \frac{\sqrt{3}}{2}$

$$\Rightarrow \boxed{\phi_0 = \pi/3}$$

We find that $\boxed{0 \le \theta \le 2\pi}$ from the symmetry of the shape. And from our analysis of the intersection it's clear $\boxed{0 \le \phi \le \pi/3.}$

Finally think about ρ,

ρ_{out} is given by $x^2+y^2+z^2=4$

ρ_{in} is given by $\underbrace{z=1}_{\rho\cos\phi} = 1$

Thus $\boxed{\frac{1}{\cos\phi} \le \rho \le 2}$

6) [10pts] Calculate the volume of a sphere of radius $A > 0$. Use calculus. While it is interesting and encouraging to me you have the formula memorized, I want a derivation of the formula.

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where $0 \leq \rho \leq A$, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$

$$V = \int_0^A \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \int_0^A \int_0^{2\pi} \rho^2 \left(-\cos\phi \Big|_0^{\pi}\right) d\theta \, d\rho$$

$$= \int_0^A \int_0^{2\pi} 2\rho^2 \, d\theta \, d\rho$$

$$= \int_0^A \left(2\rho^2 \theta \Big|_0^{2\pi}\right) d\rho$$

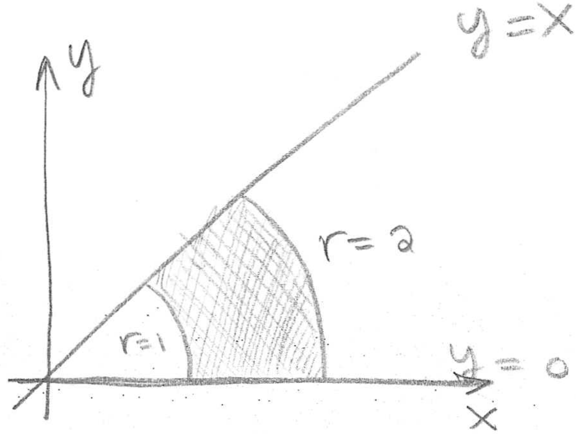
$$= \int_0^A 4\pi\rho^2 \, d\rho$$

$$= \frac{4}{3}\pi\rho^3 \Big|_0^A$$

$$= \boxed{\frac{4\pi A^3}{3}}$$

7) [20pts] Given that R is the subset of the xy-plane which satisfies the inequalities $1 \leq x^2 + y^2 \leq 4$ and $0 \leq y \leq x$. Calculate the integral below using polar coordinates.

$$\iint_R \tan^{-1}\left(\frac{y}{x}\right) dA$$



$1 \leq x^2 + y^2 \leq 4$

$1 \leq r^2 \leq 4$

$1 \leq r \leq 2$

$\tan\theta = \frac{y}{x} = \frac{x}{x} \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Thus R is (r, θ) such that $0 \leq \theta \leq \frac{\pi}{4}$ and $1 \leq r \leq 2$

$$\iint_R \tan^{-1}\left(\frac{y}{x}\right) dA = \int_1^2 \int_0^{\frac{\pi}{4}} \theta \, r \, d\theta \, dr$$

$$= \int_1^2 \left(r \left(\frac{1}{2}\theta^2 \Big|_0^{\pi/4} \right) \right) dr$$

$$= \int_1^2 \frac{\pi^2 r}{32} \, dr$$

$$= \frac{\pi^2 r^2}{64} \Big|_1^2$$

$$= \frac{\pi^2}{64}(4 - 1)$$

$$= \boxed{\frac{3\pi^2}{64}}$$

8) [20pts] Let $f(x,y) = x + e^y$

a.) Calculate ∇f.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \boxed{\langle 1, e^y \rangle = (\nabla f)(x,y)}$$

b.) Find the linearization of the function at $(1,0)$

$$\begin{aligned} L_f^{(1,0)}(x,y) &= f(1,0) + (\nabla f)(1,0) \cdot \langle x-1, y \rangle \\ &= 2 + \langle 1,1 \rangle \cdot \langle x-1, y \rangle \\ &= 2 + x - 1 + y \\ &= \underline{1 + x + y} \end{aligned}$$

c.) Find the equation of the tangent plane to $z = f(x,y)$ at $(1,0)$.

$$\boxed{z = 1 + x + y}$$

d.) In which direction(s) is the function increasing at the rate of 2 at $(1,0)$?

$(D_{\hat{u}} f)(1,0) = 2$ for what $\hat{u}$? Let $\hat{u} = \langle a,b \rangle$,

$(\nabla f)(1,0) \cdot \langle a,b \rangle = \langle 1,1 \rangle \cdot \langle a,b \rangle = a+b = 2$ and

we assume $\sqrt{a^2+b^2} = 1$. Notice $(D_{\hat{u}} f)(1,0) = |\nabla f(1,0)| \cos\theta$

thus $|(D_{\hat{u}} f)(1,0)| \leq |\nabla f(1,0)| = \sqrt{1^2+1^2} = \sqrt{2}$ $\therefore$ no such direction.

e.) In which direction(s) is the function increasing at a rate of 3 at $(1,0)$?.

Again the maximum rate of change is in the $(\nabla f)(1,0) = \langle 1,1 \rangle$ direction. Moreover, the max. rate is $\sqrt{2}$. There is no direction in which f increases at the rate of 3 at the point $(1,0)$.

(9.) find point on the ellipse $x^2+2y^2=1$ which is closest to $(3,4)$. Which point is furthest away?

$$d = \sqrt{(x-3)^2+(y-4)^2} \quad \text{distance from } (x,y) \text{ to } (3,4).$$

Easier to work with $f(x,y) = d^2 = (x-3)^2+(y-4)^2$.
We wish to max/min $f(x,y)$ subject to $g(x,y) = x^2+2y^2-1=0$.
The method of Lagrange Multipliers says to look for the point where the tangent lines match-up. This meas

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x-3), 2(y-4) \rangle = \lambda \langle 2x, 4y \rangle$$

$$2(x-3) = 2\lambda x \longrightarrow \lambda = \frac{x-3}{x} \quad (x \neq 0)$$

$$2(y-4) = 4\lambda y \longrightarrow \lambda = \frac{y-4}{2y} \quad (y \neq 0)$$

Then equating (assuming $x, y \neq 0$)

$$\frac{x-3}{x} = \frac{y-4}{2y}$$

$$(x-3)2y = x(y-4)$$

$$4(x-3)^2y^2 = x^2(y-4)^2 \quad \& \quad \underline{x^2 = 1-2y^2}$$

$$4(x-3)^2y^2 = (1-2y^2)(y-4)^2$$

$$(x^2-6x+9) = \frac{1}{4y^2}(1-2y^2)(y-4)^2$$

$$6x = 9+x^2-\frac{1}{4y^2}(1-2y^2)(y-4)^2$$

$$x = \frac{1}{6}\left[9+x^2-\frac{1}{4y^2}(1-2y^2)(y-4)^2\right]$$

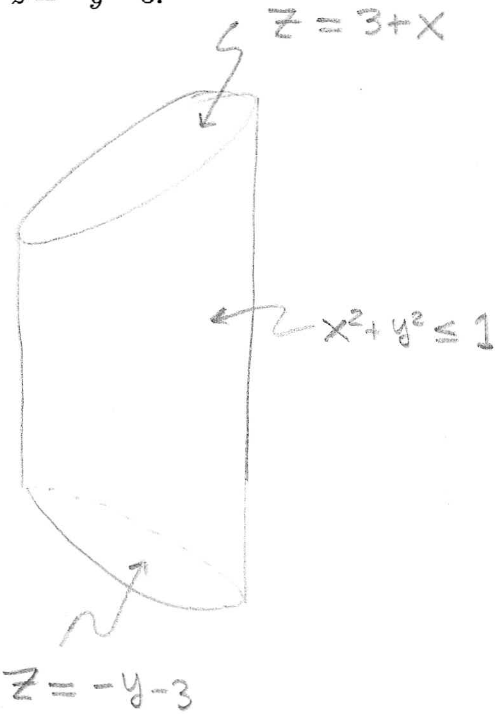
$$x^2 = \frac{1}{36}\left[10-2y^2-\frac{1}{4y^2}(1-2y^2)(y-4)^2\right]^2$$

$$36(4y^2)^2(1-2y^2) = \left[40y^2-8y^4-(1-2y^2)(y-4)^2\right]^2$$

$$\boxed{16(36)\left[y^4-2y^6\right] = \left[40y^2-8y^4-(1-2y^2)(y^2-8y+16)\right]^2}$$

need numerical method sol^{n} here, sorry folks.

10) [10pts] Find the volume of the solid bounded by cylinder $x^2 + y^2 = 1$ and the planes $z = 3 + x$ and $z = -y - 3$.



$$-y - 3 \le z \le 3 + x$$
$$\underbrace{x^2 + y^2 \le 1}$$

Cartesian Description of the solid.

This is nicer in cylindricals,

$$0 \le \theta \le 2\pi$$
$$0 \le r \le 1$$
$$-r\sin\theta - 3 \le z \le 3 + r\cos\theta$$

$$\text{Volume} = \int_0^{2\pi} \int_0^1 \int_{-r\sin\theta - 3}^{3 + r\cos\theta} r \, dz \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r(3 + r\cos\theta + r\sin\theta + 3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (6r + r(\cos\theta + \sin\theta)) \, dr \, d\theta$$

$$= \int_0^{2\pi} \left(3 + \frac{1}{3}(\underbrace{\cos\theta}_{0} + \underbrace{\sin\theta}_{0})\right) d\theta$$

By symmetry of sine, cosine over whole cycle.

$$= \boxed{6\pi}$$