(1)

Calculus III, Take-home Test 3, April 29, 2009 Name: .

You may work with others, but do not search for solutions online.

10:30 **PROBLEM 1: [10pts]** parametrizations (no squareroot functions for full-credit, use trigonometric and hyperbolic trigonometric functions where appropriate):

(a.) Parametrize the surface S_1 defined by $x^2 + 4y^2 + 9z^2 = 1$, $y \geq 0$.

(b.) Parametrize the surface S_2 defined by $x^2 + y^2 - 2y + (z-1)^2 = 3$, $z \geq 1$.

(c.) Parametrize the surface S_3 formed by the intersection of $x^2 + y^2/4 = 1$ and $z = y + 3$.

(d.) Parametrize the surface K which is bounded by $x^2 + y^2 + z^2 = 4$ and $z^2 - x^2 - y^2 = 1$ for $z \geq 1$.

11:35 **PROBLEM 2: [10pts]** Find the normal vector field to S_2 from part 1b, find the equation for the tangent plane at ~~(2,0,1)~~. $(\sqrt{3}, 0, 1)$

11:55 **PROBLEM 3: [10pts]** Problem #2 of section 17.4

12:21 **PROBLEM 5: [10pts]** Problem #30 of section 17.7. Check answer with Divergence Theorem.

(Cookie break)

1:02 **PROBLEM 6: [20pts]** Let $\vec{F}(x,y) = < xy^2, x^2y + 2x >$. Let C be union of part of a circle and two line segments; $C = C_1 \cup C_2 \cup C_3$. The path C_1 is the part of the circle $x^2 + y^2 = 9$ that goes from $(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$ to the point $(3,0)$ in the counter-clockwise direction. C_2 is the line segment from $(3,0)$ to the origin $(0,0)$. C_3 is the line segment from the origin $(0,0)$ to $(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}})$. Calculate the following line integral:

(9:40-9:50 morning correction)

$$\int_C \vec{F} \cdot d\vec{r}$$

1:15 **PROBLEM 7: [20pts]** Let $S = S_1 \cup S_2 \cup S_3$ be the outward-oriented surface formed by the intersection of $z = 1$ and $x^2 + y^2 = 1$ and $z = 3$. Let $\vec{F}(x,y,z) =< x, y, z >$. For 15pts calculate the surface integral without using the divergence Theorem. For 5pts calculate the surface integral via the divergence Theorem. I hope you get the same answer.

$$\iint_S \vec{F} \cdot d\vec{S}$$

1:30 **PROBLEM 8: [20pts]** Section 17.8#10.

1:47, **Bonus[5pts]:** Solve the Euler Lagrange equations to obtain the equations for the geodesics on a cylinder of radius R.

PROBLEM 1 Parametrize the surfaces,

(a.) $x^2+4y^2+9z^2=1$, $y \geq 0$ was given. This is half of an ellipsoid. We can use

$$\boxed{\Phi(\phi,\theta) = \langle \cos\theta\sin\phi,\ \tfrac{1}{2}\sin\theta\sin\phi,\ \tfrac{1}{3}\cos\phi \rangle}$$

with $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi$.

(b.) $x^2+y^2-2y+(z-1)^2=3$, $z \geq 1$ was given. Complete the square to see $x^2+(y-1)^2+(z-1)^2=4$ which is a half-sphere of radius 2 centered at $(0,1,1)$.

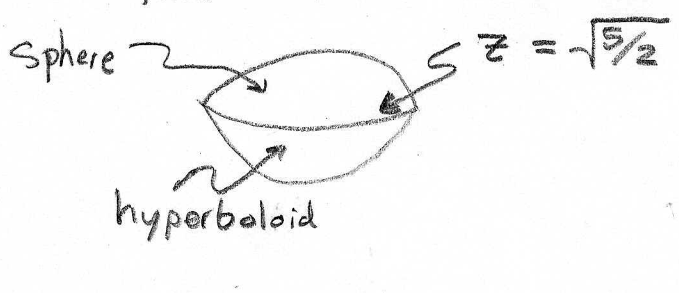
$$\boxed{\Phi(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 1+2\sin\theta\sin\phi,\ 1+2\cos\phi \rangle}$$

choose $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi/2$ to insure we only cover the top $(z \geq 1)$ half of the sphere.

(c.) Let S_3 be the part of the plane $z=y+3$ which is bounded by $x^2+y^2/4=1$. This is an oval-shaped region in the plane.

$$\boxed{\Phi(x,y) = \langle x, y, y+3 \rangle, \quad x^2+y^2/4 \leq 1}$$

(d.) Let K be formed by $x^2+y^2+z^2=4$ and $z^2-x^2-y^2=1$ for $z \geq 1$. This surface has two parts which meet at the intersection of the given sphere and hyperboloid; $x^2+y^2+z^2=4$ and $z^2-x^2-y^2=1$ solve for $x^2+y^2=4-z^2=z^2-1 \Rightarrow 2z^2=5$ thus the intersection is at $z=\pm\sqrt{5/2}$.



(d.) Continuing, $K = S \cup H$ where $S \cap H$ is the circle $z = \sqrt{5/2}$ and $x^2+y^2 = 4 - \left(\sqrt{5/2}\right)^2 = \frac{16-5}{4}$.

Notice, H has $x^2+y^2 \leq \frac{11}{4}$ and $1 \leq z \leq \sqrt{5/2}$.

The sphere S has $\sqrt{5/2} \leq z = 2\cos\phi \leq 2$

$$\boxed{X_S(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi \rangle}$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \cos^{-1}\left(\sqrt{5/8}\right)$.

Then the other half of K has

from $z=2$, $\rho = 2$

from $z = \sqrt{5/2}$, $\rho = 2$.

$$\boxed{X_H(\theta,\lambda) = \langle \cos\theta\sinh\lambda,\ \sin\theta\sinh\lambda,\ \cosh\lambda \rangle}$$

where $0 \leq \theta \leq 2\pi$ and $\cosh^{-1}(1) \leq \lambda \leq \cosh^{-1}\left(\sqrt{5/2}\right)$

which simplifies to $\underbrace{0 \leq \lambda \leq 1.0317}$.

(this insures $1 \leq z = \cosh\lambda \leq \sqrt{5/2}$)

<u>Remark</u>: there are many, many, ..., many correct answers to these questions. Another choice for part d. is to simply use cylindricals, this comes at a price, $x^2+y^2+z^2 = 4 \longrightarrow r^2+z^2=4$ thus $r = \sqrt{4-z^2}$ for S while H has $z^2-1 = r^2$ thus $r = \sqrt{z^2-1}$ on H. Consequently,

{the squareroots I cautioned against were not these ones.}

$$\boxed{\begin{aligned} X_S(\theta,z) &= \langle \sqrt{4-z^2}\cos\theta,\ \sqrt{4-z^2}\sin\theta,\ z \rangle \\ X_H(\theta,z) &= \langle \sqrt{z^2-1}\cos\theta,\ \sqrt{z^2-1}\sin\theta,\ z \rangle \end{aligned}}$$

and $\sqrt{5/2} \leq z \leq 2$ for $X_S(\theta,z)$ while $1 \leq z \leq \sqrt{5/2}$ for $X_H(\theta,z)$ and both have $0 \leq \theta \leq 2\pi$.

PROBLEM 2 From 1b,

$$\mathbf{X}(\phi,\theta) = \langle 2\cos\theta\sin\phi,\ 1+2\sin\theta\sin\phi,\ 1+2\cos\phi \rangle$$

$$\frac{\partial \mathbf{X}}{\partial\phi}(\phi,\theta) = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi \rangle$$

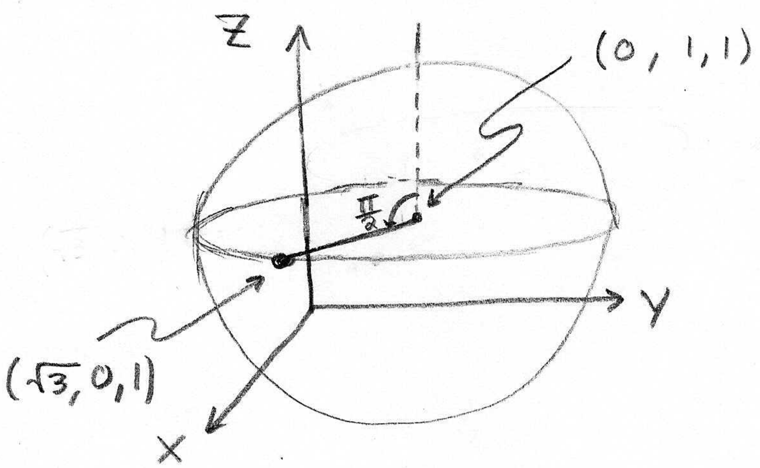
$$\frac{\partial \mathbf{X}}{\partial\theta}(\phi,\theta) = \langle -2\sin\theta\sin\phi,\ 2\cos\theta\sin\phi,\ 0 \rangle$$

$$\vec{N}(\phi,\theta) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \end{vmatrix}$$

$$= \langle 4\sin^2\phi\cos\theta,\ 4\sin^2\phi\sin\theta,\ 4\underline{\cos^2\theta}\sin\phi\cos\phi + 4\underline{\sin^2\theta}\sin\phi\cos\phi \rangle$$

$$= \boxed{4\sin\phi\langle \cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi \rangle = \vec{N}(\phi,\theta)}$$

The point $(\sqrt{3}, 0, 1)$ has $\sqrt{3} = 2\cos\theta\sin\pi/2$



$\therefore \cos\theta = \sqrt{3}/2$

$\Rightarrow \underline{\theta = \pm\pi/6}$

we choose $\theta = -\pi/6$ because we need

$\underbrace{1 + 2\sin\theta\sin\phi}_{1+2\sin\left(\frac{-\pi}{6}\right)\sin\left(\frac{\pi}{2}\right)\ \neq\ 0} = 0$

$$\boxed{\vec{N}(\pi/2, -\pi/6) = 4\langle \sqrt{3}/2,\ -1/2,\ 0 \rangle}$$

(this checks with my picture, and we can also note

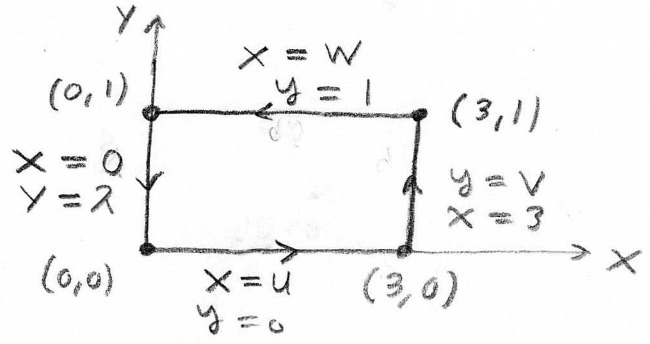
$\nabla(x^2+y^2-2y+(z-1)^2) = \langle 2x,\ 2y-2,\ 2(z-1) \rangle$

evaluated at $(\sqrt{3}, 0, 1)$ yields $\langle 2\sqrt{3}, -2, 0 \rangle$.

PROBLEM 3 ⑤

§17.4 #2 Calculate $\oint_C xy\,dx + x^2\,dy$ where C is rectangle with vertices $(0,0)$, $(3,0)$, $(3,1)$ and $(0,1)$.

(a.) direct calculation



$$\oint_C xy\,dx + x^2\,dy = \int_0^3 u(0)\,dx + \int_0^1 3^2\,dv + \int_3^0 w(1)\,dw + \int_1^0 0^2\,d\lambda$$

$$= 9v\Big|_0^1 + \frac{w^2}{2}\Big|_3^0$$

$$= 9 - \frac{9}{2}$$

$$= \boxed{9/2}$$

(b.) Green's Thm

$$\oint_C \underbrace{xy}_{P}\,dx + \underbrace{x^2}_{Q}\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^3 \int_0^1 [2x - x]\,dy\,dx$$

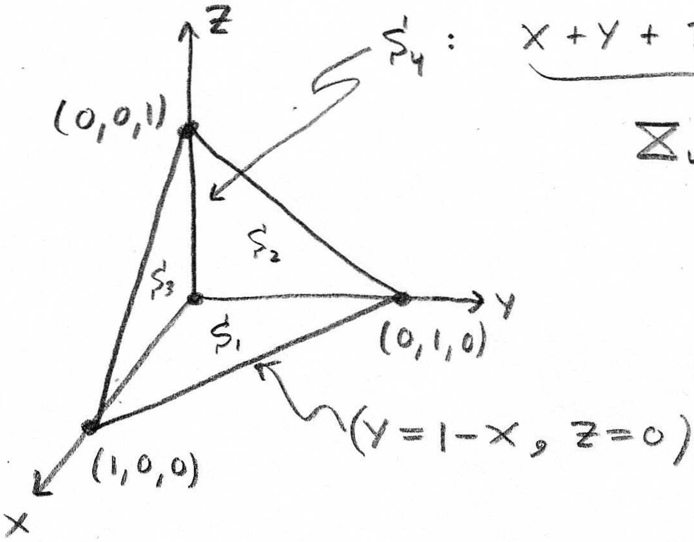
$$= \int_0^3 x\,dx$$

$$= \frac{1}{2}x^2\Big|_0^3$$

$$= \boxed{9/2}$$

well, that's a good sign.

PROBLEM 5 | #30 of §17.7 & Check answer via Div. Thm



S_4 : $\underbrace{x + y + z = 1 , \quad x, y, z \geq 0}$

$\Phi_4 (x, y) = \langle x, y, 1 - x - y \rangle$

$0 \leq x \leq 1 , \quad 0 \leq y \leq 1 - x$

$d\vec{S}_4 = \langle 1, 1, 1 \rangle \, dx\,dy$

S_1 : $z = 0$, $\Phi_1 (y, x) = \langle x, y, 0 \rangle$

$0 \leq x \leq 1 , \quad 0 \leq y \leq 1 - x$

$d\vec{S}_1 = \langle 0, 0, -1 \rangle \, dx\,dy$

S_2 : $x = 0$, $\Phi (z, y) = \langle 0, y, z \rangle$

$0 \leq y \leq 1 , \quad 0 \leq z \leq 1 - y$

$d\vec{S}_2 = \langle -1, 0, 0 \rangle \, dy\,dz$

S_3 : $y = 0$, $\Phi (x, z) = \langle x, 0, z \rangle$

$0 \leq x \leq 1 , \quad 0 \leq z \leq 1 - x$

$d\vec{S}_3 = \langle 0, -1, 0 \rangle \, dx\,dz$

PROBLEM 5 Continued, $T = S_1 \cup S_2 \cup S_3 \cup S_4$

$$\iint_T \langle y, z-y, x\rangle \cdot d\vec{S} = \int_0^1 \int_0^{1-x} \langle y, -y, x\rangle \cdot \langle 0,0,-1\rangle \, dy\, dx \; +$$

$$+ \int_0^1 \int_0^{1-y} \langle y, z-y, 0\rangle \cdot \langle -1,0,0\rangle \, dz\, dy \; +$$

$$+ \int_0^1 \int_0^{1-x} \langle 0, z, x\rangle \cdot \langle 0,-1,0\rangle \, dz\, dx \; +$$

$$+ \int_0^1 \int_0^{1-x} \langle y, 1-x-2y, x\rangle \cdot \langle 1,1,1\rangle \, dy\, dx \,.$$

$$= \int_0^1 \int_0^{1-x} -x \, dy\, dx + \int_0^1 \int_0^{1-y} -y \, dz\, dy$$

$$+ \int_0^1 \int_0^{1-x} -z \, dz\, dx + \int_0^1 \int_0^{1-x} (1-y) \, dy\, dx$$

$$= -4 \int_0^1 \int_0^{1-x} x \, dy\, dx + \int_0^1 \int_0^{1-x} dy\, dx$$

$$= -4 \int_0^1 (x - x^2)\, dx + \int_0^1 (1-x)\, dx$$

$$= -4\left(\tfrac{1}{2} - \tfrac{1}{3}\right) + 1 - \tfrac{1}{2}$$

$$= -4/6 + 3/6$$

$$= \boxed{-1/6}$$

Div. Th$^{\underline{m}}$ Check,

$$\iint_T \langle y, z-y, x\rangle \cdot d\vec{S} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (-1)\, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (x+y-1)\, dy\, dx$$

$$= \int_0^1 \left((1-x)(x-1) + \tfrac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \tfrac{-1}{2}(1-x)^2 dx = \tfrac{1}{6}(1-x)^3 \Big|_0^1 = \boxed{\tfrac{-1}{6}}$$

PROBLEM 6 Let $\vec{F}(x,y) = \langle xy^2,\ x^2y + 2x \rangle$ and $C = C_1 \cup C_2 \cup C_3$ where $C_1 : x^2+y^2 = 9$ is the part of the circle from $(3/\sqrt{2}, 3/\sqrt{2})$ to $(3,0)$ in CCW-direction and C_2 : line segment from $(3,0)$ to $(0,0)$ and C_3 : line segment from $(0,0)$ to $(\sqrt{3}/2, \sqrt{3}/2)$. Find $\int_C F \cdot d\vec{r}$

Let me settle the parametrizations to begin,

$$C_1 : \quad \vec{r}(\theta) = \langle 3\cos\theta,\ 3\sin\theta \rangle .$$
$$\vec{r}\,'(\theta) = \langle -3\sin\theta,\ 3\cos\theta \rangle$$
$$d\vec{r} = \langle -3\sin\theta,\ 3\cos\theta \rangle d\theta \ , \quad \pi/4 \le \theta \le 2\pi .$$

$$C_2 : \quad \vec{r}(t) = \langle 3,0 \rangle + t\langle -3, 0 \rangle = \langle 3-3t, 0 \rangle$$
$$d\vec{r} = \langle -3, 0 \rangle dt \ , \quad 0 \le t \le 1 .$$

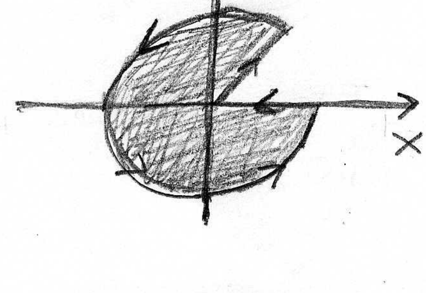
$$C_3 : \quad \vec{r}(\alpha) = \left\langle \frac{\alpha\sqrt{3}}{2}, \frac{\alpha\sqrt{3}}{2} \right\rangle$$
$$d\vec{r} = \langle \sqrt{3}/2,\ \sqrt{3}/2 \rangle d\alpha \ , \quad 0 \le \alpha \le 1 .$$

Calculate,

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\pi/4}^{2\pi} \langle 3\cos\theta\, 9\sin^2\theta,\ 27\cos^2\theta\sin\theta + 6\cos\theta \rangle \cdot \langle -3\sin\theta, 3\cos\theta \rangle d\theta$$
$$+ \int_0^1 \langle 0,\ 2(3-3t) \rangle \cdot \langle -3, 0 \rangle\, dt$$
$$+ \int_0^1 \left\langle \left(\frac{\alpha\sqrt{3}}{2}\right)^3, \left(\frac{\alpha\sqrt{3}}{2}\right)^3 + \frac{2\alpha\sqrt{3}}{2} \right\rangle \cdot \langle \sqrt{3}/2, \sqrt{3}/2 \rangle d\alpha$$

Just kidding, we should use GREENES TH^{m}

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$
$$= \iint_D (\cancel{2xy} + 2 - \cancel{2xy})\, dA$$



PROBLEM 6 continued : notice angle subtended in (c) is $\Delta\theta = \frac{7\pi}{4}$

$$\int_C P\,dx + Q\,dy = 2\iint_D dA = \frac{2}{2}\,3^2(\Delta\theta) = 9\left(\frac{7\pi}{4}\right) = \boxed{\frac{63\pi}{4}}$$

Remark: $\vec{F} = \langle xy^2, x^2y\rangle + \langle 0, 2x\rangle = \nabla f + \langle 0, 2x\rangle$ where $f(x,y) = \frac{1}{2}x^2y^2$. The conservative component of $\vec{F}$ integrated to zero around the closed loop while the $2x$ gave us the nontrivial contribution of $63\pi/4$.

PROBLEM 7 Let $S = S_1 \cup S_2 \cup S_3$ be the cylinder $x^2+y^2=1$ capped at $z=1$ & $z=3$. Let $\vec{F}(x,y,z) = \langle x, y, z\rangle$. Calculate the flux through S with & w/o the Divergence Theorem's assistance

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \langle r\cos\theta, r\sin\theta, 1\rangle \cdot \langle 0,0,-1\rangle\, r\,dr\,d\theta$$

$$+ \int_0^{2\pi}\int_1^3 \langle \cos\theta, \sin\theta, z\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle\, dz\,d\theta$$

$$+ \int_0^{2\pi}\int_0^1 \langle r\cos\theta, r\sin\theta, 3\rangle \cdot \langle 0,0,1\rangle\, r\,dr\,d\theta$$

$$= (3-1)\int_0^{2\pi}\int_0^1 r\,dr\,d\theta + \int_0^{2\pi}\int_1^3 (\cos^2\theta + \sin^2\theta)\,dz\,d\theta$$

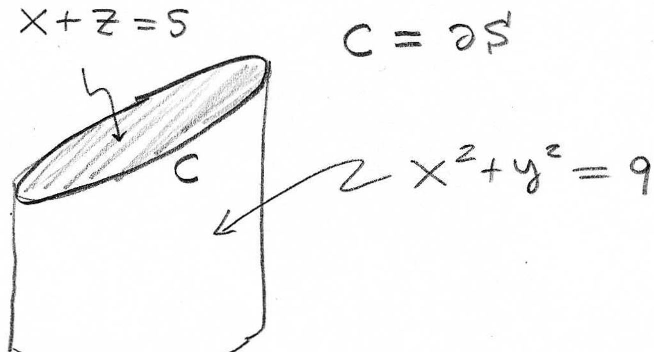
$$= \boxed{6\pi}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_{\substack{x^2+y^2\le 1\\ 1\le z\le 3}} 3\,dV = (3\pi)(3-1) = \boxed{6\pi.}$$

$\nabla\cdot\vec{F} = 3.$

PROBLEM 8 §17.8 #10. Use Stoke's Th$^{\underline{m}}$ to calculate $\int_C \vec{F}\cdot d\vec{r}$ where C is oriented CCW from above and C is the curve of intersection of the plane $x+z=5$ and $x^2+y^2=9$ and

$\vec{F}(x,y,z) = \langle xy,\ 2z,\ 3y \rangle$



The top has parametrization

$$\Sigma(x,y) = \langle x, y, 5-x \rangle$$

for $0 \le x^2+y^2 \le 9$.

$$\Sigma_x \times \Sigma_y = \langle 1,0,-1\rangle \times \langle 0,1,0\rangle = \langle 1,0,1\rangle$$

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial y}(3y) - \frac{\partial}{\partial z}(2z),\ \frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(3y),\ \frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial y}(xy) \right\rangle$$

$$\underline{\nabla \times \vec{F} = \langle 1, 0, -x \rangle}.$$

Stoke's Th$^{\underline{m}}$:

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\cdot d\vec{S}$$

$$= \iint_{0\le x^2+y^2\le 9} \langle 1,0,-x\rangle \cdot \langle 1,0,1\rangle\, dx\,dy$$

$$= \int_0^{2\pi}\int_0^3 (1 - r\cos\theta)\, r\,dr\,d\theta \quad : \text{switching to polar's.}$$

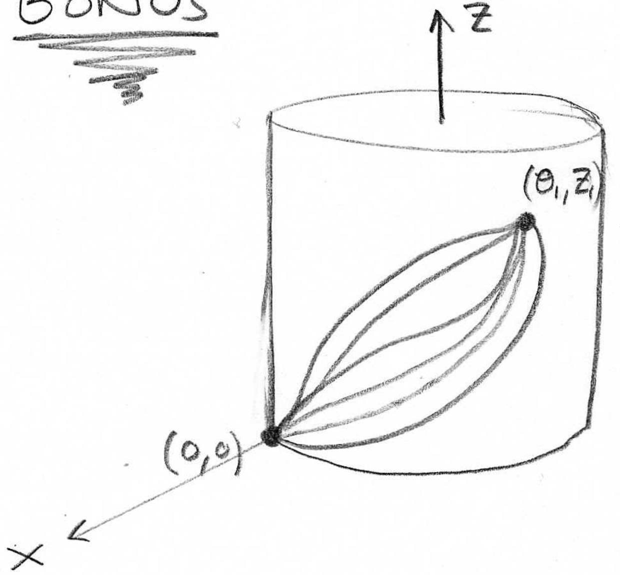
$$= \int_0^{2\pi}\int_0^3 (r - \underbrace{r^2\cos\theta}_{0})\,dr\,d\theta \quad \text{(cyclic fnct, whole period symmetry} \Rightarrow \text{vanishes)}$$

$$= \frac{r^2}{2}\Big|_0^3 \ \theta\Big|_0^{2\pi}$$

$$= \frac{9}{2}(2\pi)$$

$$= \boxed{9\pi}$$

BONUS



$$ds^2 = dx^2 + dy^2 + dz^2$$

cylinder has $x^2+y^2 = R^2$

thus $r^2 = R^2 \Rightarrow \underline{dr = 0}$

Use cylindricals,

$$ds^2 = r^2 d\theta^2 + dr^2 + dz^2$$

Hence on the cylinder,

$$ds = \sqrt{R^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

Integrate from $(0,0)$ to (z_1, θ_1) to obtain arclength integral <u>functional</u> (depends on the function $z = z(\theta)$ used to calculate $dz/d\theta$)

$$S = \int_{(0,0)}^{(\theta_1, z_1)} \sqrt{R^2 + \dot{z}^2}\, d\theta \quad , \quad L(z, \dot{z}, \theta) = \sqrt{R^2 + \dot{z}^2}$$

Yields Euler Lagrange Eq^{ns} of

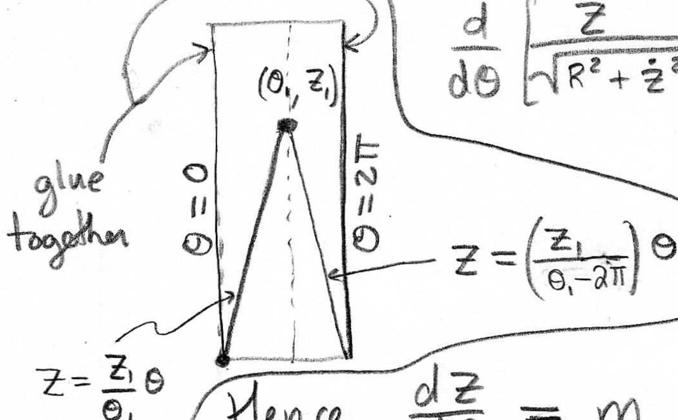
$$\frac{\partial L}{\partial z} + \frac{d}{d\theta}\left[\frac{\partial L}{\partial \dot{z}}\right] = 0$$

$$\frac{d}{d\theta}\left[\frac{\dot{z}}{\sqrt{R^2+\dot{z}^2}}\right] = 0 \Rightarrow \frac{\dot{z}}{\sqrt{R^2+\dot{z}^2}} = k$$

$$\Rightarrow \dot{z}^2 = k^2(R^2 + \dot{z}^2)$$

$$\Rightarrow \dot{z}^2(1-k^2) = k^2 R^2$$

$$\Rightarrow \dot{z} = \frac{\pm kR}{\sqrt{1-k^2}} = m$$



Hence, $\frac{dz}{d\theta} = m$ yielding $\underline{z = m\theta + b}$. However, this a bit trickier than first glance since θ and $\theta + 2\pi$ yield same point, this formula allows two solutions. Need to fit $(0,0)$ and $(z_1, \theta_1) \Rightarrow z = \left(\frac{z_1}{\theta_1}\right)\theta$ or $z = \left(\frac{z_1}{\theta_1 - 2\pi}\right)\theta$