CALCULUS 132:	TEST IV, SPRING 2011

No graphing calculators and show your work with proper notation. There are at least 140pts to earn. Box your answers and work problems in the white space provide and box your answers. Thanks and Enjoy! Please answer these survey questions to the best of your memory:

I have missed _________. unexcused classes to the best of my memory.
I estimate that I invested _________. hours per week studying outside of class on average to the best of my memory.

9:48

Problem 1 [15pts] Find the first 3 nontrivial terms in the power series expansions centered at zero for the following functions:

(a.) $f(x) = e^{-2x} = 1 - 2x + \frac{1}{2}(-2x)^2 + \cdots$

$= \boxed{1 - 2x + 2x^2 + \cdots,}$

(b.) $g(x) = x^2 3^x = x^2 e^{x\ln(3)}$

$= x^2\left(1 + x\ln(3) + \frac{1}{2}(\ln(3)x)^2 + \cdots\right)$

$= \boxed{x^2 + x^3\ln(3) + \frac{1}{2}[\ln(3)]^2 x^4 + \cdots}$

(c.) $h(x) = 2\sin(x)\cos(x) = 2\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right)$

$= 2x - x^3 - \frac{2}{6}x^3 + \frac{2}{24}x^5 + \frac{1}{12}x^5 + \frac{1}{120}x^5 + \cdots$

$= \boxed{2x - \frac{8}{6}x^3 + \frac{32}{120}x^5 + \cdots}$

easier

$h(x) = \sin(2x) = 2x - \frac{1}{6}(2x)^3 + \frac{1}{120}(2x)^5 + \cdots$

$= 2x - \frac{8}{6}x^3 + \frac{32}{120}x^5 + \cdots$

Problem 2 [10pts] Calculate the complete power series solution centered at zero for the integral below:

$$\int x^{10}\sin(x^2)\,dx = \int x^{10}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(x^2)^{2n+1}\right)dx$$

$$= C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int x^{4n+12}\,dx \quad : \quad \text{Using } \int\text{-term by term Th}^{\underline{m}}.$$

$$= \boxed{C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\,(4n+13)}x^{4n+13}}$$

Problem 3 [15pts] Multiply the power series below to obtain the first 4 nontrivial terms in the series expansion for $h(x)$ centered at zero. Once the multiplication is completed then calculate $h(0)$, $h'(0)$, $h''(0)$ and $h^{(3)}(0)$ WITHOUT direct differentiation.

$$h(x) = (1 + x + 2x^2 + 3x^3 + \cdots)(4 - 5x - 6x^2 - 7x^3 - \cdots)$$

$$= 4 - 5x - 6x^2 - 7x^3 + \cdots$$

$$+ 4x - 5x^2 - 6x^3 + \cdots$$

$$+ 8x^2 - 10x^3 + \cdots$$

$$+ 12x^3 + \cdots$$

$$= \boxed{4 - x - 3x^2 - 11x^3 + \cdots}$$

But, $h(x) = h(0) + h'(0)\,x + \frac{1}{2}h''(0)x^2 + \frac{1}{6}h'''(0)x^3$

$$\Rightarrow \boxed{\begin{aligned} h(0) &= 4 \\ h'(0) &= -1 \\ h''(0) &= -6 \\ h^{(3)}(0) &= -66 \end{aligned}}$$

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Problem 4 [15pts] Find the first 3 nontrivial terms in the power series solution of

$$\frac{dy}{dx} + 3x^2 y = \frac{1}{1-x^2}.$$

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots + 3x^2(a_0 + a_1 x + \cdots) = 1 + x^2 + x^4 + \cdots$$

const	$a_1 = 1$	$\therefore \ a_1 = 1$.
x	$2a_2 = 0$	$\therefore \ a_2 = 0$.
x^2	$3a_3 + 3a_0 = 1$	$\therefore \ a_3 = 1/3 - a_0$
x^3	$4a_4 + 3a_1 = 0$	$\therefore \ a_4 = \frac{-3}{4} a_1 = \frac{-3}{4}$.

$$\begin{aligned} y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ &= a_0 + x + \left(\tfrac{1}{3} - a_0\right) x^3 + \cdots \\ &= \boxed{a_0(1 - x^3 + \cdots) + x - \tfrac{1}{3} x^3 + \cdots} \end{aligned}$$

$\leftarrow$ also ok for Math 132 students.

Problem 5 [10pts] Recall that $\sec x = \frac{1}{\cos(x)}$ and use division of series to calculate the first 3 nontrivial terms in the power series centered at $x_o = 0$ for $\sec x$. (or for 2/3 credit use some other argument)

$$\begin{array}{r|l} & 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots \\ \hline 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots & 1 \\ & 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots \\ \hline & \frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots \\ & \frac{1}{2}x^2 - \frac{1}{4}x^4 + \cdots \\ \hline & \frac{5}{24}x^4 + \cdots \\ & \frac{5}{24}x^4 + \cdots \\ \hline \end{array}$$

$$\boxed{\sec(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots}$$

Problem 6 [10pts] Write $f(x) = \sin(x)$ as a power series centered at $x_o = 2$.

$$f(x) = \sin(x - 2 + 2)$$
$$= \cos(2)\sin(x-2) + \sin(2)\cos(x-2)$$
$$= \sum_{n=0}^{\infty} \frac{\cos(2)(-1)^n}{(2n+1)!}(x-2)^{2n+1} + \sum_{j=0}^{\infty} \frac{\sin(2)(-1)^j}{(2j)!}(x-2)^{2j}$$

Problem 7 [10pts] Suppose we decide to approximate the function $f(x) = \frac{1}{1+x}$ by the first two terms in the binomial series; $f(x) \cong 1 - x$. If you need a precision of 0.1 then what is the largest set you should trust calculations where you have used $1 - x$ in place of $\frac{1}{1+x}$?

$$f(x) = \frac{1}{1+x} = 1 - x + x^2 + \cdots$$ by geometric series actually with $r = -x$

max error in $f(x) = 1 - x$ given by $|x^2|$ by the Alt. series Estimation Th^{m}.

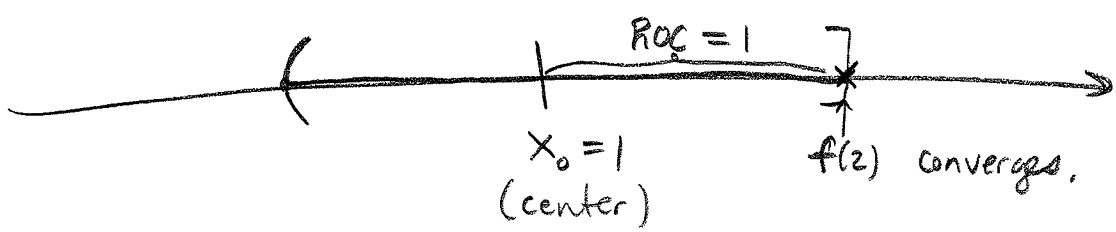
We want $|x^2| \leq 0.1 \Rightarrow |x| \leq \sqrt{0.1} \cong 0.316$

Thus $-0.316 < x < 0.316$ can use approximation for $\boxed{(-0.31,\ 0.31)}$

(slightly different answers still logical)

Problem 8 [10pts] Suppose the coefficients $c_n \in \mathbb{R}$ are chosen such that $f(x) = \sum_{n=0}^{\infty} c_n(3x-3)^n$ **converges** at $x = 2$. What is the smallest possible interval of convergence for $f(x)$?

$$f(x) = \sum_{n=0}^{\infty} c_n 3^n (x-1)^n$$



$\Rightarrow$ $\boxed{\text{I.O.C.} = (0, 2] \text{ is smallest IOC}}$

10:09

Problem 9 [15pts] Find the complete power series expansion for $f(x) = xe^{x^2} + \sin(x)$ and use this to calculate the 29-th and 30-th derivatives for f evaluated at zero.

$$f(x) = \sum_{n=0}^{\infty} x \frac{(x^2)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n!} + \frac{(-1)^n}{(2n+1)!} \right) x^{2n+1}$$

It follows $f^{(30)}(0) = 0$ since all even coefficients are zero in expansion for $f(x)$ above.
Set $29 = 2n+1 \implies n = 14 \therefore$ equating coeff. of x^{29},

$$\frac{f^{(29)}(0)}{(29)!} = \frac{1}{(14)!} + \frac{1}{(29)!} \implies \boxed{f^{(29)}(0) = 1 + \frac{(29)!}{(14)!}}$$

Problem 10 [15pts] Define $\gamma(v) = \frac{c}{\sqrt{c^2-v^2}}$ for $-c < v < c$ where c is a particular positive constant. Suppose that $|v| << c$ and derive the first three nontrivial terms in a power series approximation for $\gamma(v)$ centered at $v = 0$. (I would use binomial series here, do what you think is best)

$$\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$= \left(1 - v^2/c^2\right)^{-1/2}$$

$$= 1 - \frac{1}{2}\left(\frac{v^2}{c^2}\right) + \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-v^2}{c^2}\right)^2 + \cdots$$

$$= 1 - \frac{v^2}{2c^2} + \frac{3v^4}{8c^4} + \cdots$$

Bonus: due Today by 10pm.

Problem 13 [5pts] Suppose we are given that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ has interval of convergence $(-R, R)$. It can be shown that $\lim_{n\to\infty} |c_{n+1}/c_n| = 1/R$. (you do not have to show it here). Find the open IOC for the series $h(x) = f(g(x)) = \sum_{n=0}^{\infty} c_n \left[g(x)\right]^n$. You need $x \in dom(g)$ subject to a certain condition (you find the condition).

Can argue $|g(x)| < R$.

Problem 14 [5pts] The IOC could be some rather strange sets for the previous problem. For example, note that $f(x) = 1 + x + x^2 + \cdots$ has $IOC = (-1, 1)$ but $f(e^x) = 1 + e^x + e^{2x} + \cdots$ has an open IOC of $(-\infty, 0)$. Does this contradict the power series domain theorem we discussed in lecture? Explain.

$\text{Th}^{\underline{m}}$ not applicable.

$f(e^x) = 1 + e^x + e^{2x} + \cdots$

(not a power series).