TAYLOR SERIES EXPANSIONS & THE BINOMIAL SERIES

This is why we talk about series at all. In previous sections we have employed some rather indirect reasoning to connect a fnct & it's corresponding series representation.

Thm/(Taylor) If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$ then the coefficients c_n are given by $c_n = \frac{f^{(n)}(a)}{n!}$. Moreover

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

That is, noting $f^{(0)}(x) \equiv f(x)$,

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \cdots$$

<u>Proof</u>: We'll simply use the term-by-term diff. Thm multiple times.
Assume that $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \Rightarrow f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \Rightarrow f'(a) = c_1$$

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots \Rightarrow f''(a) = 2c_2$$

$$f'''(x) = 6c_3 + \cdots \Rightarrow f'''(a) = 6c_3$$

$$\vdots$$

$$f^{(n)}(x) = \left(\frac{d}{dx}\right)^n \left[\sum_{k=0}^{\infty} c_k (x-a)^k\right]$$

$$= \sum_{k=n}^{\infty} c_k \cdot k(k-1)(k-2)\cdots(k-(n-1))(x-a)^{k-n}$$

Then $f^{(n)}(a) = c_n \cdot n(n-1)(n-2)\cdots(2)(1)(a-a)^0 \Rightarrow f^{(n)}(a) = c_n \cdot n!$

Therefore $\boxed{c_n = \frac{f^{(n)}(a)}{n!}}$ as claimed and substituting back into $f(x)$ we find

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

(<u>Assuming</u> $f(x)$ has a power series expansion!)

Remark: WHEN $a=0$ the TAYLOR SERIES EXPANSION BECOMES

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In which case we can call it the $\boxed{\text{MACLAURIN SERIES}}$

DISCUSSION: For any ~~differentiable~~ smooth fnct. it's fairly clear that we can construct a taylor series for that fnct, we simply differentiate and evaluate then construct $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. In fact by the last Th^m we know that if $f(x)$ has a power series expansion then it matches the TAYLOR EXPANSION. A question we should answer then is given a differentiable function when does it have a TAYLOR SERIES EXPANSION? DOES IT ALLWAYS HAVE ONE? WELL NO, see next example after the th^m.

Th^m (TAYLOR'S TH^m) If f is differentiable at least $n+1$ times on an open interval I containing a then for each $x \in I$ $\exists c$ between x and a such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

When $R_n(x) \longrightarrow 0$ $\forall x \in I$ as $n \longrightarrow \infty$ we say the Taylor series generated by f converges to f on I, in which case

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$\boxed{\text{E1}}$
$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

~~differentiable~~ smooth $\not\Rightarrow$ analytic

this function is ~~differentiable~~ smooth at zero but its not equal to its taylor series expanded about zero.

$$f(x) = f(0) + f'(0)\cdot x + \cdots = 0 + 0\cdot x = 0 \quad \text{yet } f(x) \neq 0 \text{ !}$$

Remark: TAYLOR'S Th^{m} is a generalization of the mean-value th^{m}, and much like that th^{m} it only tells us $\exists$ some c in I, but how to find c such that $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$? Well please tell me if you know! What we can say is

> Th^{m} (TAYLOR'S INEQUALITY). If $|f^{(n+1)}(x)| \leq M$ for $|x-a| < R$ then the remainder $R_n(x)$ of the TAYLOR SERIES is bounded by
>
> $$0 \leq |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| < R$$

Pf: (See Text).

Remark: with Taylor's Inequality we no longer need to calculate $R_n(x)$ explicitly! We merely need to find an M. Then if we can show $\frac{M}{(n+1)!}|x-a|^{n+1} \longrightarrow 0$ we can apply squeeze th^{m} to to conclude $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

- Lets find the Maclaurin series rep. of e^x (Assume $R_n \longrightarrow 0$ as $n \to \infty$)

E4 Let $f(x) = e^x$ then $f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$. Thus the taylor series about zero for e^x is simple,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= \sum_{n=0}^{\infty} \frac{e^0}{n!}x^n$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}$$ ← Maclaurin series for the exponential fnct.

$$\boxed{e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots}$$

$$e = 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \cdots$$

Remark: to be more rigourous we ought to prove that the Taylor series for e^x does indeed converge to e^x. This amounts to showing $R_n \longrightarrow 0$. (Well avoid this for now)

$\boxed{E3}$ Prove $f(x)=\sin(x)$ is rep. by it's Maclaurin series $\forall x$,

$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f'''(x) = -\cos(x)$$
$$f''''(x) = \sin(x)$$

Thus $f^{(n)}(x) = \pm\sin(x)$ OR $\pm\cos(x)$ $\therefore$ $|f^{n}(x)| \leq 1 = M$.

By TAYLOR'S INEQ. Th^{m},

$$0 \leq |R_n(x)| \leq \frac{x^{n+1}}{(n+1)!} \longrightarrow 0 \quad \text{as } n \to \infty$$

Thus by squeeze th^{m} $R_n(x) \longrightarrow 0$ as $n \to \infty$ (for all x).

Moreover, the Maclaurin series is easily calculated.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$
$$= \sin(0) + \cos(0)\cdot x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 + \cdots$$
$$= \boxed{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin(x)}$$

$\boxed{E4}$ **Ok,** enough about existence, lets assume they exist, not an unreasonable step for most fncts. we can think of. Let $f(x) = \cos(x)$

$f(x) = \cos(x)$	$f(0) = 1$
$f'(x) = -\sin(x)$	$f'(0) = 0$
$f''(x) = -\cos(x)$	$f''(0) = -1$
$f'''(x) = \sin(x)$	$f'''(0) = 0$
$f''''(x) = \cos(x)$	$f''''(0) = 1$

Hence $\boxed{\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{2n!} = \cos(x)}$

$\boxed{E5}$ Geometric Series: Given $f(x) = \frac{1}{1-x}$ what do we find,

$$f(x) = \frac{1}{1-x} \qquad f(0) = 1$$
$$f'(x) = \frac{1}{(1-x)^2} \qquad f'(0) = 1$$
$$f''(x) = \frac{2}{(1-x)^3} \qquad f''(0) = 2!$$
$$f'''(x) = \frac{-3\cdot 2\cdot 1}{(1-x)^4} \qquad f'''(0) = 3!$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}} \qquad f^{(n)}(0) = n!$$

Checking consistency. we already knew the result here w/o work

Hence $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n = \sum_{n=0}^{\infty} x^n$

That is $\boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots}$

$\boxed{E6}$ $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \underset{d/dx}{\overset{d/dx}{\rightleftarrows}} \sinh(x) = \frac{1}{2}(e^x - e^{-x})$

Let us find Taylor expansion of $\cosh(x)$ about $x=0$,

$$f(x) = \cosh(x) \qquad f(0) = \tfrac{1}{2}(e^0 + e^{-0}) = 1$$
$$f'(x) = \sinh(x) \qquad f'(0) = \tfrac{1}{2}(e^0 - e^{-0}) = 0$$
$$f''(x) = \cosh(x) \qquad f''(0) = 1$$
$$f'''(x) = \sinh(x) \qquad f'''(0) = 0$$
$$f''''(x) = \cosh(x) \qquad f''''(0) = 1$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^x = \underbrace{\cosh x}_{\text{even}} + \underbrace{\sinh x}_{\text{odd}}$$

Hence $\boxed{\cosh(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$

Very similarly $\boxed{\sinh(x) = 1 + x + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}}$

__Remark:__ just like $\cos(x)$ and $\sin(x)$ just no alternating signs.

$\boxed{E7}$ $f(x) = \sin^2(x)$. One way is $\sin^2(x) = \frac{1}{2}(1 - \cos 2x)$

Using $\boxed{E4}$ with $2x$ in place of x we find

$$\cos(2x) = 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 - \frac{1}{6!}(2x)^6,$$

Thus subst. this into identity above gives

$$\sin^2(x) = \frac{1}{2}\left(\cancel{1} - \left[\cancel{1} - 2x^2 + \frac{16}{24}x^4 - \frac{64}{720}x^6 + \right]\right)$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 - \dots$$

A second method is to multiply the series for $\sin(x)$

$$\sin^2(x) = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right)$$

$$= x^2 - \frac{1}{3!}x^4 + \frac{1}{5!}x^6 - \dots - \frac{1}{3!}x^4 + \frac{1}{(3!)^2}x^6 + \dots + \frac{1}{5!}x^6 + \dots$$

$$= x^2 - \frac{2}{3!}x^4 + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right)x^6 + \dots$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots$$

A third method is to simply taylor expand,

$$f(x) = \sin^2(x) \qquad f(0) = 0$$

$$f'(x) = 2\sin(x)\cos(x) \qquad f'(0) = 0$$

$$f''(x) = 2(\cos^2(x) - \sin^2(x)) \qquad f''(0) = 2$$

$$f'''(x) = -4\sin(x)\cos(x) - 4\sin(x)\cos(x) \qquad f'''(0) = 0$$

$$f''''(x) = -8(\cos^2(x) - \sin^2(x)) \qquad f''''(0) = -8$$

Thus $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \frac{2x^2}{2!} - \frac{8x^4}{4!} + \dots = x^2 - \frac{1}{3}x^4 \dots$

<u>Which method do you think is best?</u>

Obviously it depends on the example and what we're asked to find.

E8 Expand $f(x) = \sqrt{x}$ around $a = 4$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}} \qquad f(4) = 2$$

$$f''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)x^{-3/2} = -\left(\frac{1}{2}\right)^2\frac{1}{(\sqrt{x})^3} \qquad f'(4) = \frac{1}{4}$$

$$f'''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^{-5/2} = 3\left(\frac{1}{2}\right)^3\frac{1}{(\sqrt{x})^5} \qquad f''(4) = \frac{-1}{32}$$

$$f''''(x) = \left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)x^{-7/2} = -3\cdot 5\left(\frac{1}{2}\right)^4\frac{1}{(\sqrt{x})^7} \qquad f'''(4) = \frac{3}{2^8}$$

$$f''''(4) = \frac{-15}{2^{11}}$$

Thus we find

$$f(x) = \sum \frac{f^{(n)}(4)}{n!}(x-4)^n$$

$$= \boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 - \frac{5}{16{,}384}(x-4)^4 + \cdots = \sqrt{x}}$$

E9 Find taylor exp. of $f(x) = x^3 + 3x^2 + 3x + 1$ about $x = 0$ and $x = -1$. Notice $f(0) = 1$ and $f(-1) = 0$ and

$$f'(x) = 3x^2 + 6x + 3 \qquad f'(0) = 3 \qquad f'(-1) = 0$$

$$f''(x) = 6x + 6 \qquad f''(0) = 6 \qquad f''(-1) = 0$$

$$f'''(x) = 6 \qquad f'''(0) = 6 \qquad f'''(-1) = 6$$

$$f''''(x) = 0$$

about zero: $\rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = 1 + 3x + \frac{6}{2!}x^2 + \frac{6}{3!}x^3 = x^3 + 3x^2 + 3x + 1$

about $x = -1$ $\rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(x+1)^n$

$$= 0 + 0\cdot(x+1) + \frac{0}{2!}(x+1)^2 + \frac{6}{3!}(x+1)^3$$

$$= \boxed{(x+1)^3 = x^3 + 3x^2 + 3x + 1 = f(x)}$$

Remark: the best ∞-polynomial approximation to a polynomial is itself. Surprise, surprise.
Power series are basically polygnomials of infinite order.

E10 Find Maclaurin series for $\tan(x) = f(x)$. Find 1^{st} 2 non-zero terms.

$$f(x) = \tan(x) \qquad f(0) = 0$$
$$f'(x) = \sec^2(x) \qquad f'(0) = 1$$
$$f''(x) = 2\sec^2(x)\tan(x) \qquad f''(0) = 0$$
$$f'''(x) = 4\sec^2(x)\tan^2(x) + 2\sec^4(x) \qquad f'''(0) = 2$$

Hence $\tan(x) = x + \frac{2}{3!}x^3 + \cdots = \boxed{x + \frac{1}{3}x^3 + \cdots = \tan(x)}$

We could differentiate further if we want to generate higher order terms.

Summary: The following Maclaurin series you should know

	I.O.C
$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = 1 + u + u^2 + u^3 + \cdots$	$(-1, 1)$
$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3$	$(-\infty, \infty)$
$\sin(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \cdots$	$(-\infty, \infty)$
$\cos(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n}}{(2n)!} = 1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 \cdots$	$(-\infty, \infty)$

Remark: for many problems you can use these basic series rather than explicitly generating the taylor exp.

E11 $$x\sin(x/2) = x \cdot \sum_{n=0}^{\infty} (-1)^n \frac{[\frac{1}{2}x]^{2n+1}}{(2n+1)!}$$

"sigma notation"

$$= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \frac{x^{2n+2}}{(2n+1)!}} = x\left(\frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 + \cdots\right)$$

$$= \boxed{\frac{1}{2}x^2 - \frac{1}{48}x^4 + \frac{1}{3840}x^6 - \cdots}$$

1^{st} three non-zero terms.

THE BINOMIAL SERIES

There is a neat trick for calculating $(a+b)^k$. It's called PASCAL'S Triangle, I use it occasionally. Below I write the triangle and what the line $\Rightarrow$ for $(a+b)^k$.

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & \longrightarrow & (a+b)^0 = 1 \\
 & & & & 1 & & 1 & & & & & \longrightarrow & (a+b)^1 = a+b \\
 & & & 1 & & 2 & & 1 & & & & \longrightarrow & (a+b)^2 = a^2 + 2ab + b^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & \longrightarrow & (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & \longrightarrow & (a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \longrightarrow & (a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{array}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

And so on, hopefully the pattern is clear. Well we can write this more compactly

$$\boxed{(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n}$$

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} = \frac{k!}{n!(k-n)!}$$

"k choose n", where $\binom{k}{0} = 1$

Binomial $Th^{\underline{m}}$ $(k \in \mathbb{N})$

This has been known for some time, however once k is allowed to be any real number we need an infinite series, to keep it simple we'll study $(1+x)^k$, once we know that we can easily find $(a+b)^k$ since $(a+b)^k = a^k(1+b/a)^k$.

$$\begin{aligned}
f(x) &= (1+x)^k & f(0) &= 1 \\
f'(x) &= k(1+x)^{k-1} & f'(0) &= k \\
&\vdots & &\vdots \\
f^{(n)}(x) &= k(k-1)\cdots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) &= k(k-1)\cdots(k-n+1)
\end{aligned}$$

Hence

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum \boxed{\frac{k(k-1)\cdots(k-n+1)}{n!}} x^n \text{ "}\equiv\text{" } \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Therefore, assuming $R_n \to 0$ as $n \to \infty$,

$$\boxed{(1+x)^k = \sum_{n=0}^{\infty} \boxed{\binom{k}{n}} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \cdots} \quad \text{Binomial Series}$$

Remark: $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$ converges for $|x| < 1$

While the I.O.C. depends on k, the result is

$$\text{I.O.C} = (-1, 1] \quad \text{if } -1 < k \le 0$$
$$\text{I.O.C} = [-1, 1] \quad \text{if } k \ge 0$$
$$\text{I.O.C} = (-1, 1) \quad \text{if } -1 > k$$

Proof left to reader. (I won't ask you to prove these)

$\boxed{E1}$ Expand $\frac{1}{(1+x)^2}$ using binomial series (this was $\boxed{E6}$ of §8.5)

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + \frac{(-2)(-2-1)}{2}x^2 + \frac{(-2)(-2-1)(-2-2)}{3!}x^3 + \cdots$$

$$= \boxed{1 - 2x + 3x^2 - 4x^3 + \cdots = \frac{1}{(1+x)^2}}$$

$\boxed{\text{for } x \in (-1, 1)}$

Same as $\boxed{E6}$ ☺

$\boxed{E2}$ $\frac{1}{\sqrt{1 - v^2/c^2}} = (1 - v^2/c^2)^{-1/2}$ let $u = -v^2/c^2$

$$= (1+u)^{-1/2}$$
$$= 1 - \tfrac{1}{2}u + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}u^2 + \cdots$$
$$= 1 - \tfrac{1}{2}u + \tfrac{3}{8}u^2 + \cdots$$
$$= 1 + \tfrac{1}{2}\left(\tfrac{v}{c}\right)^2 + \tfrac{3}{8}\left(\tfrac{v}{c}\right)^4 + \cdots$$

for $|\frac{v}{c}| < 1$ aka $-c < v < c$

$\boxed{E3}$ $\frac{3}{1-x^2} = 3(1-x^2)^{-1}$ $u = -x^2$

$$= 3(1+u)^{-1}$$
$$= 3\left[1 - u + \frac{(-1)(-1-1)}{2}u^2 + \cdots\right]$$
$$= \boxed{3\left[1 + x^2 + x^4 + \cdots\right] = \frac{3}{1-x^2}}$$

(with radius of convergence 1 $|x^2| < 1$)

- Alternatively you could have identified this to be a geometric series with $a = 3$ and $r = x^2$

HOMEWORK : CALCULUS II : §12.10 # 9, 11, 15, 25, 28, 29, 33, 36, 39, 41, 47, 50, 59

(STEWART 6th Ed.)

$\boxed{\text{§12.10 \#9}}$ $f(x) = e^{5x}$ find Maclaurin series from Taylor's Expansion directly

Note that $f'(x) = 5e^{5x}$ then $f''(x) = 5(5e^{5x}) = 5^2 e^{5x}$ then $f'''(x) = 5^3 e^{5x} \Rightarrow f^{(n)}(x) = 5^n e^{5x}$ $\therefore$ $f^{(n)}(0) = 5^n e^0 = 5^n$.

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n = 1 + 5x + \cdots}$$

Remark: if we already knew that $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ then we could just substitute $u = 5x$ and find the same answer quicker. However, Stewart asked us to use Taylor's Th^m directly.

Radius of Convergence? Examine ratio test,

$$L = \lim_{n\to\infty} \left| \frac{5^{n+1}(x)^{n+1}}{(n+1)!} \frac{n!}{5^n (x)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{5}{n+1}\right)|x|$$

$$= 0 \quad \therefore \quad \boxed{\text{R.O.C.} = \infty \quad \text{since I.O.C.} = \mathbb{R}}$$

$\boxed{\text{§12.10 \#11}}$ Find Maclaurin series for $f(x) = \sinh(x)$ and find the R.O.C.

$$f(x) = \sinh(x) \Rightarrow f(0) = 0$$
$$f'(x) = \cosh(x) \Rightarrow f'(0) = 1$$
$$f''(x) = \sinh(x) \Rightarrow f''(0) = 0$$
$$f'''(x) = \cosh(x) \Rightarrow f'''(0) = 1$$

Thus $f^{(2n+1)}(0) = 1$ whereas $f^{(2n)}(0) = 0$ for $n = 0, 1, 2, \ldots$

$$\therefore \quad \boxed{\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots}$$

The R.O.C. comes from examining ratio test,

$$L = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{[2(n+1)+1]!} \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!} \right|$$

$$= \lim_{n\to\infty} \left(|x^2| \left| \frac{1}{(2n+3)(2n+2)} \right| \right) = 0 \quad \therefore \quad \boxed{\text{R.O.C.} = \infty}$$

$\boxed{\S 12.10\ \#15}$ Find Taylor series centered at $a=3$ for $f(x) = e^x$

Notice $f^{(n)}(x) = e^x$ since $\frac{d}{dx}(e^x) = e^x \Rightarrow f^{(n)}(x) = f^{(n+1)}(x)$ etc...

Thus $f^{(n)}(3) = e^3$ thus,

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!}(x-3)^n = e^3\left(1 + (x-3) + \tfrac{1}{2}(x-3)^2 + \dots\right)}$$

Remark: we could also obtain this in a sneaky way,

$$e^x = e^{x-3+3} = e^3 e^{x-3} = e^3 \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$$

$\boxed{\S 12.10\ \#25}$ Use the binomial series to expand $\sqrt{1+x}$ as a power series. State the R.O.C.

$$\begin{aligned}
\sqrt{1+x} &= (1+x)^{1/2} \\
&= 1 + \tfrac{1}{2}x + \tfrac{1}{2}\left(\tfrac{1}{2}-1\right)x^2 + \tfrac{1}{3!}\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)\left(\tfrac{1}{2}-2\right)x^3 + \cdots \\
&= 1 + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{3\cdot 2}\left(\tfrac{1}{2}\right)\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)x^3 + \cdots \\
&= \boxed{1 + \tfrac{1}{2}x - \tfrac{1}{4}x^2 + \tfrac{1}{16}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n}
\end{aligned}$$

(debateable notation)

the R.O.C. $= 1$ by the "Binomial Series th$^{\text{m}}$"

(we'll discuss in lecture why R.O.C. $= 1$)

$\boxed{\S 12.10\ \#28}$ Use binomial series to expand $(1-x)^{2/3}$ as power series, state R.O.C.

Let $u = -x$ then

$$\begin{aligned}
(1-x)^{2/3} &= (1+u)^{2/3} \\
&= 1 + \tfrac{2}{3}u + \tfrac{1}{2}\left(\tfrac{2}{3}\right)\left(\tfrac{-1}{3}\right)u^2 + \tfrac{1}{3!}\left(\tfrac{2}{3}\right)\left(\tfrac{-1}{3}\right)\left(\tfrac{-4}{3}\right)u^3 + \cdots \\
&= 1 + \tfrac{2}{3}(-x) - \tfrac{1}{9}(-x)^2 + \tfrac{8}{3^4\cdot 2}(-x)^3 + \cdots \\
&= \boxed{1 - \tfrac{2}{3}x - \tfrac{1}{9}x^2 - \tfrac{4}{81}x^3 - \cdots, \quad \text{R.O.C.} = 1}
\end{aligned}$$

$\boxed{\S 12.10\ \#29}$ Use known Maclaurin series to expand $f(x) = \sin(\pi x)$

$$\sin(\pi x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\pi x)^{2n+1} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!} x^{2n+1}}$$

Remark: Problems 25 & 28 don't allow for a nice clean answer like this.

§12.10 #33 Expand $f(x) = x\cos\left(\frac{1}{2}x^2\right)$ via the known cosine expansion

$$\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} u^{2n}$$

setting $u = x^2/2$

$$\begin{aligned} f(x) &= x\cos\left(\tfrac{1}{2}x^2\right) \\ &= x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\tfrac{1}{2}x^2\right)^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{2}\right)^{2n} x \cdot x^{4n} \\ &= \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n)!} x^{4n+1} = x - \frac{1}{8}x^9 + \cdots} \end{aligned}$$

§12.10 #36 Use known expansions to obtain power series for $f(x) = \dfrac{x^2}{\sqrt{2+x}}$

Idea: this is almost $(1+u)^k$ with $k = -1/2$ but the x^2 rides along and the 2 is annoying. Let's make it go away,

$$f(x) = \frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}} \underbrace{\left(1 + \frac{x}{2}\right)^{-1/2}}_{\text{apply binomial series to this part, } u = \frac{x}{2}}$$

$$\begin{aligned} \Rightarrow f(x) &= \frac{x^2}{\sqrt{2}}\left(1 - \tfrac{1}{2}u + \tfrac{1}{2!}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)u^2 + \tfrac{1}{3!}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)\left(\tfrac{-5}{2}\right)u^3 + \cdots\right) \\ &= \frac{x^2}{\sqrt{2}}\left(1 - \tfrac{1}{2}\left(\tfrac{x}{2}\right) + \tfrac{3}{8}\left(\tfrac{x}{2}\right)^2 - \tfrac{1}{3\cdot 2}\left(\tfrac{3\cdot 5}{8}\right)\left(\tfrac{x}{2}\right)^3 + \cdots\right) \\ &= \boxed{\frac{1}{\sqrt{2}}\left(x^2 - \frac{1}{4}x^3 + \frac{3}{32}x^4 - \frac{5}{128}x^5 + \cdots\right)} \end{aligned}$$

§12.10 #39 Find Maclaurin series for $f(x) = \cos(x^2)$

$$\cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x^2)^{2n} = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}}$$

(If you look at $y = \cos(x^2)$ and compare to $y = 1 - \frac{1}{2}x^4$ and then $y = 1 - \frac{1}{2}x^4 + \frac{1}{24}x^8$ you should see that the graphs of $y = T_4(x)$ then $y = T_8(x)$ get closer & closer to $y = \cos(x^2)$.)

§12.10 #41 Find Maclaurin series $f(x) = xe^{-x}$

$$f(x) = xe^{-x} = x\sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = x\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+1}}$$

(similar comments to #39 apply here.)

$\boxed{\S 12.10\ \#47}$ Find power series sol$^{\underline{n}}$ of the integral below:

$$\int x\cos(x^3)\,dx$$

Notice $x\cos(x^3) = x\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(x^3)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{6n+1}$. Thus,

$$\int x\cos(x^3)\,dx = \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{6n+1}\right)dx$$

$$= \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n)!}\,\frac{x^{6n+2}}{6n+2}\right] + C$$

$$= \left(\frac{1}{2}x^2 - \frac{1}{16}x^8 + \frac{1}{(24)(14)}x^{14} + \cdots\right) + C$$

$\boxed{\S 12.10\ \#50}$ Find series sol$^{\underline{n}}$ for $\int \tan^{-1}(x^2)\,dx$

Note $f(x) = \tan^{-1}(x^2) \Rightarrow \frac{df}{dx} = \frac{2x}{1+x^2} = \sum_{n=0}^{\infty}2x(-x^2)^n = \sum_{n=0}^{\infty}2(-1)^n x^{2n+1}$

Hence $f(x) = C + \sum_{n=0}^{\infty}\left(\frac{2(-1)^n}{2n+2}\right)x^{2n+2} = \tan^{-1}(x^2)$. However, we know $f(0) = \tan^{-1}(0) = 0 = C + 0 \therefore C = 0$. Hence,

$$\int \tan^{-1}(x^2)\,dx = \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{2n+2}\right)dx$$

$$= \boxed{C + \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)(2n+3)}x^{2n+3}}$$

$\boxed{\S 12.10\ \#59}$ Find first 3 nontrivial terms for $e^{-x^2}\cos(x)$ via multiplication of known series

$e^u = 1 + u + \frac{1}{2}u^2 + \cdots$ setting $u = -x^2$

$$e^{-x^2}\cos(x) = \left(1 - x^2 + \tfrac{1}{2}(-x^2)^2 + \cdots\right)\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{4!}x^4 + \cdots\right)$$

$$= \left(1 - x^2 + \tfrac{1}{2}x^4 + \cdots\right)\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 + \cdots\right)$$

$$= 1 - \tfrac{1}{2}x^2 - x^2 + \tfrac{1}{2}x^4 + \tfrac{1}{2}x^4 + \tfrac{1}{24}x^4 + \cdots$$

$$= \boxed{1 - \tfrac{3}{2}x^2 + \tfrac{25}{24}x^4 + \cdots}$$