MATH 131, Test 1, September 11, 2008 Name:

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. If proper notation is not present in your solution you will likely lose at least 3pts for each offense. This test has 105 points, 5 are bonus points. Make sure to at least attempt each part.

1) [10pts] State the following:

a.) Precise definition of the limit. (involves ϵ, δ)

$\lim_{x \to a} f(x) = L \iff$ [For each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$]

b.) Definition of continuity at a point for some function f.

f is continuous at $a \in dom(f)$ if $\lim_{x \to a} f(x) = L$

2) [10pts] Prove that $\lim_{x \to 4}(3x + 2) = 14$. By "prove" I mean for you to use the precise definition of the limit which involves the ϵ,δ arguments.

Let $\epsilon > 0$ choose $\delta = \epsilon/3$. Suppose $0 < |x - 4| < \delta$ and consider,

$$
\begin{aligned}
|3x + 2 - 14| &= |3x - 12| \\
&= |3(x - 4)| \\
&= 3|x - 4| \\
&< 3\delta = 3\left(\frac{\epsilon}{3}\right) = \epsilon.
\end{aligned}
$$

Thus, $0 < |x - 4| < \delta \Rightarrow |3x + 2 - 14| < \epsilon$. Hence

$$\lim_{x \to 4} (3x + 2) = 14$$

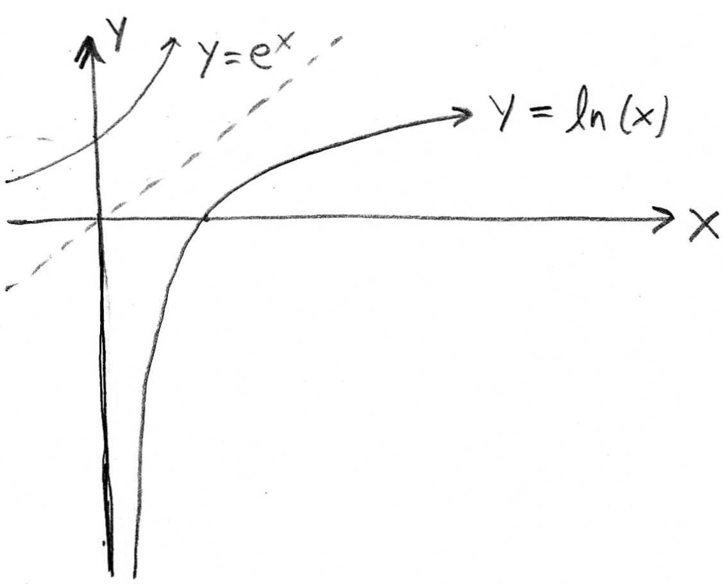
By the definition of the limit.

3) [28pts] Calculate the following limits. Use proper notation. If a limiting value does not exist then explain why in a short sentence. If it is possible to algebraically reduce the limit then make those reductions and analyze the reduced form. This means if you can factor and cancel things inside the limit then you should do such before concluding your analysis of the problem. Assume a, b, c, d are constants.

a.) $\lim_{x\to 0}\left(ax^2 + 4bx + 2ce^x\right) = a \underbrace{\lim_{x\to 0}(x^2)}_{0} + 4b \underbrace{\lim_{x\to 0}(x)}_{0} + 2c \underbrace{\lim_{x\to 0}(e^x)}_{e^0 = 1}$

$= \underline{2c}.$

b.) $\lim_{x\to 0^+}\left(\ln(x)\right) = \underline{-\infty}.$

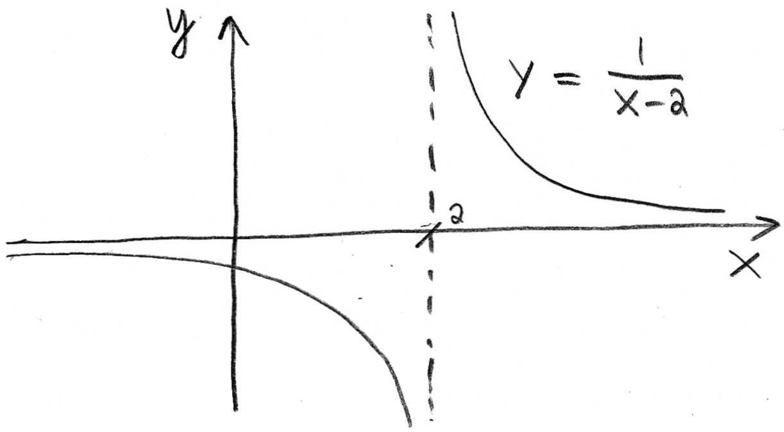


c.) $\lim_{x\to 2}\left(\frac{x}{x^2-2x}\right) = \lim_{x\to 2}\left(\frac{x}{x(x-2)}\right)$

$$= \lim_{x\to 2}\left(\frac{1}{x-2}\right) \leftarrow (*)$$

$$= \underline{\text{d.n.e.}}$$

Because



Or, we can argue that the form of the limit (*) is $\frac{\text{nonzero}}{\text{zero}}$ as $x \to 2$. Notice as $x \to 2^+$ we have the $x-2>0$ so $\lim_{x\to 2^+}\left(\frac{1}{x-2}\right) = \infty$ Whereas from the left $x\to 2^-$ we have $x-2<0$ so $\lim_{x\to 2^-}\left(\frac{1}{x-2}\right) = -\infty$

d.) $\lim_{x\to 2}\left(\frac{3x-6}{x^3-2x^2+x-2}\right)$

Observe that as $x \to 2$ both the numerator and denominator $\to 0$. In fact if $x=2$ then $3x-6 = 6-6=0$ and $x^3-2x^2+x-2 = 8-8+2-2=0$.

This indicates we can factor out $(x-2)$ in both the denominator and numerator. Clearly $3x-6 = 3(x-2)$.

On the other hand,

$$x^3-2x^2+x-2 = (x-2)\overbrace{(Ax^2+Bx+C)}^{\text{need to pin this down.}}$$

$$= Ax^3+Bx^2+Cx-2Ax^2-2Bx-2C$$

<u>Equate coefficients:</u>

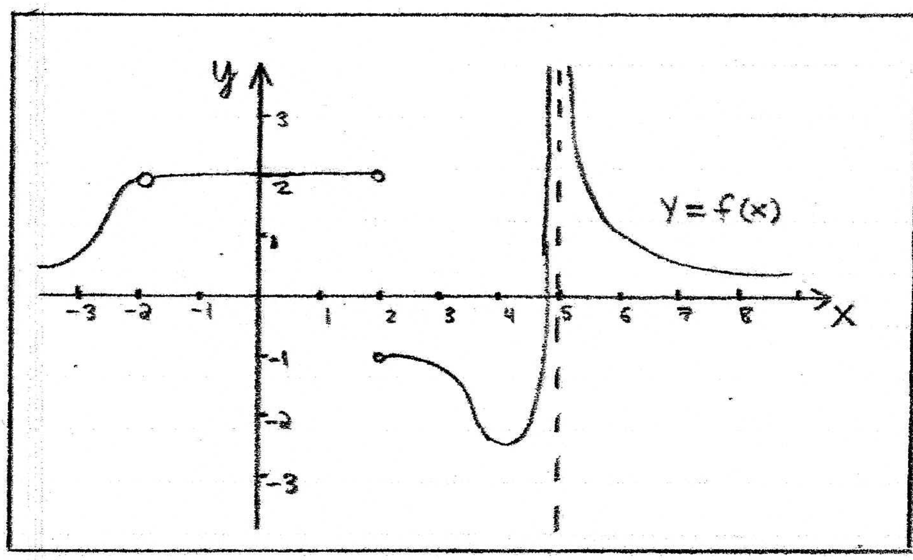
$$\left.\begin{array}{l|l} \underline{x^3} & 1 = A \\ \underline{x^2} & -2 = B-2A \\ \underline{x} & 1 = C-2B \\ \underline{x^0} & -2 = -2C \end{array}\right\} \quad \begin{array}{l} A=1 \\ B=-2+2=0 \\ \\ C=1 \end{array} \Rightarrow x^3-2x^2+x-2 = (x-2)(x^2+1).$$

Now use what we've learned,

$$\lim_{x\to 2}\left(\frac{3x-6}{x^3-2x^2+x-2}\right) = \lim_{x\to 2}\left(\frac{3\cancel{(x-2)}}{\cancel{(x-2)}(x^2+1)}\right) = \lim_{x\to 2}\left(\frac{3}{x^2+1}\right) = \boxed{\frac{3}{5}}$$

4) [22pts] Given the graph $y = f(x)$ pictured below, give the values of the following or explain why the quantity does not exist as a real number.

a.) $\lim_{x\to -2} f(x)$	b.) $f(-2)$	c.) $\lim_{x\to 2^-} f(x)$	d.) $\lim_{x\to 0} f(x)$	e.) $\lim_{x\to 5} f(x)$



(a.) $\lim_{x\to -2} f(x) = 2$: values of f(x) approach Y=2 near X = -2.

(b.) f(-2) is undefined. (Hole in graph)

(c.) $\lim_{x\to 2^-} f(x) = 2$: values of f(x) approach Y=2 as X gets near 2 from the left (X<2 in the limit)

(d.) $\lim_{x\to 0} f(x) = 2$: function continuous at x=0 $\lim_{x\to 0} f(x) = f(0) = 2$

(e.) $\lim_{x\to 5} f(x) = \infty$: values of function become arbitrarily large as X gets arbitrarily close to X = 5.

5) [10pts] Is the following function continuous? (explain your answer using the definition of continuity)

$$f(x) = \begin{cases} \frac{x^2+x}{x^2+3x+2} & x < -1 \\ x^2+4 & x \geq -1 \end{cases}$$

Observe,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \left(\frac{x^2+x}{x^2+3x+2} \right)$$ ← form $\left(\frac{0}{0}\right)$ need to reduce by factoring.

$$= \lim_{x \to -1^-} \left(\frac{x\cancel{(x+1)}}{(x+2)\cancel{(x+1)}} \right)$$

$$= \lim_{x \to -1^-} \left(\frac{x}{x+2} \right)$$

$$= \frac{-1}{-1+2} = -1/2 = \lim_{x \to -1^-} f(x) \quad \text{I}$$

While from the right,

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2+4) = 1+4 = 5 = \lim_{x \to -1^+} f(x) \quad \text{II}.$$

Thus by I & II $\lim_{x \to -1} f(x)$ d.n.e. which shows f is <u>not</u> continuous at $x=-1$ since $\lim_{x \to -1} f(x) \neq f(-1)$. So the function is <u>not continuous</u>.

6) [10pts] Find the instantaneous velocity at time $t = 3$ for a function with position $s = t^2 + 6$. Show all work carefully, do not skip steps. (do not use power rule alone, you must calculate from the definition which involves a limit on this test)

$$V(3) = \lim_{h \to 0} \left[\frac{s(3+h) - s(3)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{(3+h)^2 + 6 - 3^2 - 6}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{\cancel{9} + 6h + h^2 + \cancel{6} - \cancel{9} - \cancel{6}}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{6h + h^2}{h} \right]$$

$$= \lim_{h \to 0} \left[6 + h \right]$$

$$= \boxed{6 = V(3)}$$

7) [10pts] Find the equation for the tangent line to $y = 3\sqrt{x}$ at $(1,3)$. (you should calculate the slope of the tangent line from a limit, you may use the power rule to check your work if you wish, but you may not replace the work with the power rule)

$$f'(1) = \lim_{x\to 1}\left(\frac{f(x)-f(1)}{x-1}\right)$$

$$= \lim_{x\to 1}\left(\frac{3\sqrt{x}-3}{x-1}\right)$$

$$= 3\lim_{x\to 1}\left(\frac{\sqrt{x}-1}{x-1}\right)$$

$$= 3\lim_{x\to 1}\left[\frac{\sqrt{x}-1}{x-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right)\right]$$

$$= 3\lim_{x\to 1}\left[\frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)}\right]$$

$$= 3\lim_{x\to 1}\left[\frac{1}{\sqrt{x}+1}\right]$$

$= \frac{3}{2}$ = slope of tangent line.

Thus as the tangent line goes through $(1,3)$

$$\boxed{Y = 3 + \frac{3}{2}(x-1)}$$ Eq^n of Tangent Line.

8) [3pts] How can we make $g(x) = \cos(ax)$ an even function? How can we make $g(x)$ an odd function? Is it possible that $g(x)$ could be neither even nor odd?

- If $a \in \mathbb{R}$ then $g(-x) = \cos(-ax) = \cos(ax) = g(x)$ then g is clearly even.
- Making $g(x) = \cos(ax)$ odd requires some thought, We could use $a = \cos^{-1}(\sin(x))\frac{1}{x}$ then

$$g(x) = \cos\left(\cos^{-1}(\sin(x))\frac{1}{x}\cdot x\right) = \underline{\sin(x).}$$

odd.

- Same trick let $a = \cos^{-1}(x+x^2)\frac{1}{x}$ then

$$g(x) = \underline{x+x^2}$$

neither even nor odd.

9) [2pts] Evaluate $\cosh(0), \sinh(0)$, and $\tanh(0)$.

$$\cosh(0) = \tfrac{1}{2}(e^0 + e^{-0}) = \tfrac{1}{2}(1+1) = 1.$$

$$\sinh(0) = \tfrac{1}{2}(e^0 - e^{-0}) = \tfrac{1}{2}(1-1) = 0.$$

$$\tanh(0) = \frac{\sinh(0)}{\cosh(0)} = \frac{0}{1} = 0.$$

<u>Remark</u>: there are several versions of this test. They are conceptually similar. Certain parts are harder on certain tests. For example, 8) seems easier when $g(x) = \sin(ax)$. I adjust partial credit to reflect the inherent difficulty of the test.