Please work out the problems on seperate pieces of paper. Show your work and put your name on each page, thanks.

1 **20pts.** Given that $\lim_{x\to 2} f(x) = 10$ and $\lim_{x\to 2} g(x) = 5$ find the limits below if possible:

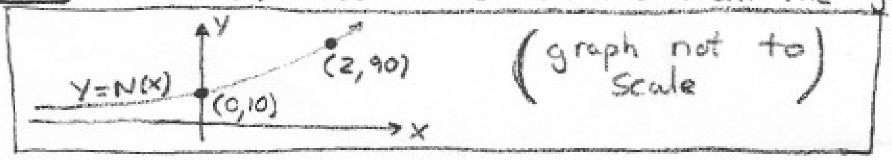
i) $\lim_{x\to 2}(f(x)+g(x))$	iv) $\lim_{x\to 2}\left(\frac{1}{f(x)-2g(x)}\right)$
ii) $\lim_{x\to 2}(f(x)g(x))$	v) $\lim_{x\to 2}\left(\frac{f(x)-2g(x)}{(f(x))^2 - f(x)g(x) - 2g(x)g(x)}\right)$
iii) $\lim_{x\to 2}(e^x f(x))$	

2 **10pts.** Given that $\lim_{x\to a} h(x) = \pi^2$ find $\lim_{x\to a} \cos\left(\sqrt{h(x)}\right)$

3 **21pts.** Calculate the limits below:

i) $\lim_{n\to 6}\left(\frac{n^2-36}{n^2+2n-48}\right)$	ii) $\lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$	iii) $\lim_{h\to 0}\frac{\sqrt{3+h}-3}{h}$

4 **15pts.** Let $N(x) = Ab^x$ find A and b from the graph $\lim \frac{\sqrt{3+h}-3}{h}$ (



5 **10pts.** Let $h(x) = \ln(x+3)$. Show that h is one to one and find the inverse function of h; find $h^{-1}(x)$.

6 **15pts.** Let $f(x) = x^2 - 4$ with $\mathrm{dom}(f) = \mathbb{R}$. Show that f is not one to one. Then let $g(x) = x^2-4$, what domain can you choose for g so that $g(x)$ will be one to one. Find such a domain for g and then find $g^{-1}(x)$. (Hint use graphs to guide your thinking)

7 **9pts.** Let $h(x) = \sqrt{9-x^2}$ find the domain of h.

Extra Credit: A well known theorem of algebra state that any n-degree polynomial with real coefficients can be factored into n-distinct linear factors over the complex numbers.

$$f(x) = A(x-r_1)(x-r_2)\cdots(x-r_n) \qquad r_i \in \mathbb{C} \text{ for } i=1,2,\dots,n$$

Use this Th^{m} along with the fact that complex roots always come in congugate pairs ($r_1 = a+ib$ means $r_2 = a-ib$) to argue that any polynomial with real coefficients can be expressed as

$$f(x) = \underbrace{A(x-r_1)(x-r_2)\cdots(x-r_m)}_{\text{linear factors.}}\underbrace{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)\cdots(a_dx^2+b_dx+c_d)}_{\text{irreducible quadratics.}}$$

Test One : MA141 FALL 2004

① Given $\lim_{x \to 2} f(x) = 10$ and $\lim_{x \to 2} g(x) = 5$ we have,

i.) $\lim_{x\to 2} (f(x) + g(x)) = \lim_{x\to 2}(f(x)) + \lim_{x\to 2} (g(x)) = 10 + 5 = 15$

ii) $\lim_{x\to 2} (f(x)\, g(x)) = \left(\lim_{x\to 2} f(x)\right)\left(\lim_{x\to 2} g(x)\right) = 10 \cdot 5 = 50$

iii) $\lim_{x\to 2} (e^x f(x)) = \left(\lim_{x\to 2} e^x\right)\left(\lim_{x\to 2} f(x)\right) = 10e^2$

iv) $\lim_{x\to 2} \left(\dfrac{1}{f(x) - 2g(x)}\right)$ does not exist, division by zero.

v) $$\lim_{x\to 2} \left(\frac{f(x) - 2g(x)}{(f(x))^2 - f(x)g(x) - 2g(x)g(x)}\right) = \lim_{x\to 2} \frac{\cancel{f(x) - 2g(x)}}{\cancel{(f(x) - 2g(x))}(f(x) + g(x))}$$

$$= \lim_{x\to 2} \frac{1}{f(x) + g(x)}$$

$$= \frac{1}{\lim_{x\to 2} f(x) + \lim_{x\to 2} g(x)}$$

$$= \frac{1}{15}$$

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② $\lim_{x\to a} h(x) = \pi^2$ find $\lim_{x\to a} \cos(\sqrt{h(x)})$ notice $\cos(x)$ is continuous and so is $\sqrt{x}$ for $x \geq 0$ thus

$$\lim_{x\to a} \cos(\sqrt{h(x)}) = \cos\left(\lim_{x\to a} \sqrt{h(x)}\right)$$

$$= \cos\left(\sqrt{\lim_{x\to a} h(x)}\right)$$

$$= \cos\left(\sqrt{\pi^2}\right)$$

$$= \cos(\pi) = \boxed{-1 = \lim_{x\to a} \cos\left(\sqrt{h(x)}\right)}$$

③ i) $$\lim_{n\to 6}\left(\frac{n^2-36}{n^2+2n-48}\right) = \lim_{n\to 6}\left(\frac{(n+6)\cancel{(n-6)}}{(n+8)\cancel{(n-6)}}\right)$$

$$= \lim_{n\to 6}\left(\frac{n+6}{n+8}\right)$$

$$= \frac{6+6}{6+8} = \frac{12}{14} = \boxed{\frac{6}{7} = \lim_{n\to 6}\left(\frac{n^2-36}{n^2+2n-48}\right)}$$

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ii) $$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0}\frac{2xh}{h} + \lim_{h\to 0}\frac{h^2}{h}$$

$$= \lim_{h\to 0} 2x + \cancel{\lim_{h\to 0} h} \to 0$$

$$= \boxed{2x = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}}$$

$2x$ is a constant with respect to h so $\lim_{h\to 0} 2x = 2x$

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iii) $$\lim_{h\to 0}\frac{\sqrt{3+h}-3}{h} = \frac{\sqrt{3}-3}{0}$$ does not exist, there is no way to cancel the zero

Remark: this was supposed to be the following:

$$\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h} = \lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right)$$

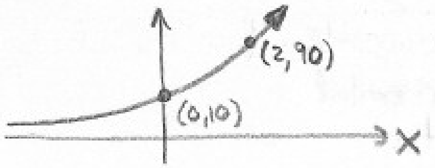
$$= \lim_{h\to 0}\frac{\cancel{9}+h-\cancel{9}}{h(\sqrt{9+h}+3)}$$

$$= \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

So I wrote 3 instead of 9 which made your life easier in principle, in practice I apologize for the confusion.

(4) Let $N(x) = Ab^x$ find A and b from the graph,



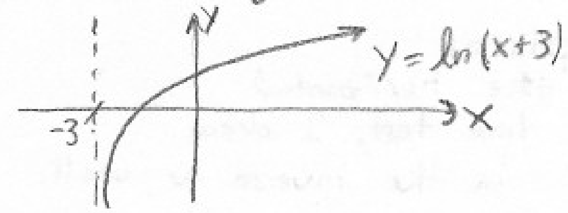
$N(0) = Ab^0 = \boxed{A = 10}$

$N(2) = 10b^2 = 90 \Rightarrow b^2 = 9 \Rightarrow b = \pm 3$

Now the graph looks like an exponential so must choose $\boxed{b = 3}$

$\boxed{N(x) = 10 \cdot 3^x}$

(5) Let $h(x) = \ln(x+3)$ then we can show h is one-one in several ways.



passes horizontal line test

$h(a) = h(b)$

$\ln(a+3) = \ln(b+3)$

$e^{\ln(a+3)} = e^{\ln(b+3)}$

$a+3 = b+3$

$a = b$

Thus $h(a) = h(b) \Rightarrow a = b$ which says that h is one-one.

Notice that once you find the inverse its easy to show h is one to one.

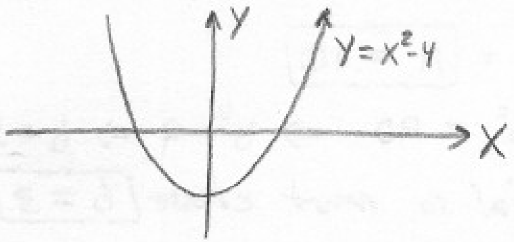
$h(a) = h(b) \Rightarrow h^{-1}(h(a)) = h^{-1}(h(b)) \Rightarrow a = b$

So if the inverse is well-defined then h is automatically one-one.

$x = \ln(y+3)$

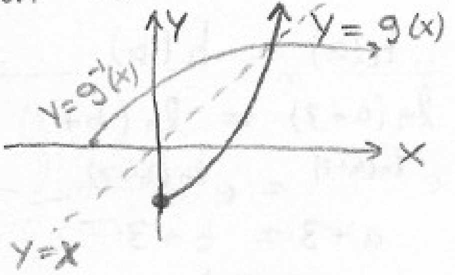
$e^x = e^{\ln(y+3)} = y+3 \Rightarrow y = e^x - 3 \therefore \boxed{h^{-1}(x) = e^x - 3}$

⑥ $f(x) = x^2 - 4$ with $\mathrm{dom}(f) = \mathbb{R}$



f(x) fails the horizontal line test

$g(x) = x^2 - 4$ well lets make $\mathrm{dom}(g) = [0, \infty)$
there are many other correct choices. ($(-\infty, 0]$, $[2,3]$, $(0,\infty)$, ...)
then for $\mathrm{dom}(g) = [0, \infty)$ we get only half the graph,



passes the horizontal line test, I drew in the inverse as well.

$$X = Y^2 - 4 \Rightarrow Y^2 = X + 4 \Rightarrow Y = \pm\sqrt{X+4}$$

But from the graph its clear we must choose $Y = +\sqrt{X+4}$

$$\therefore \boxed{g^{-1}(x) = \sqrt{X+4}}$$

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⑦ $h(x) = \sqrt{9 - x^2}$ need $9 - x^2 \geq 0 \Rightarrow -3 \leq x \leq 3$

thus $\boxed{\mathrm{dom}(h) = [-3, 3]}$

Extra Credit: The point of the extra credit is that no matter how complicated a polynomial may looks it can be decomposed into a product of linear & quadratic factors.

$$f(x) = A(x-r_1)(x-r_2)\cdots(x-r_n) \qquad r_i \in \mathbb{C} \equiv \text{Complex Numbers.}$$

Now lets put all the factors where $r_i \in \mathbb{R}$ first, we'll say they're are m of these.

$$f(x) = A(x-r_1)(x-r_2)\cdots(x-r_m)\Big[\underbrace{(x-r_{m+1})(x-r_{m+2})\cdots(x-r_n)}_{n-m \text{ factors.}}\Big]$$

Now the remaining $(n-m)$ factors have complex roots. But these come in congugate pairs. Lets see how each pair will collapse into an irreducible quadratic:

$$\begin{aligned}(x-r_i)(x-r_{i+1}) &= (x-(a+ib))(x-(a-ib)) \\ &= x^2 - x(a-ib) - x(a+ib) + (a+ib)(a-ib) \\ &= x^2 - 2xa + \cancel{ixb} - \cancel{ixb} + a^2 - \cancel{iab} + \cancel{iba} - i^2 b \\ &= x^2 - 2ax + a^2 + b^2\end{aligned}$$

Notice how requiring the roots to be conjugate pairs guarantees that the imaginary pieces will cancel, cool huh?

Anyway this cancellation will happen for each pair So just relable the constants and you'll get the expansion I gave ($x^2-2ax+a^2+b^2 = a_i x^2 + b_i x + c_i$ meaning $a_i = 1$, $b_i = -2a$, $c_i = a^2+b^2$ just lables)

$$f(x) = A\underbrace{(x-r_1)(x-r_2)\cdots(x-r_m)}_{m \text{ linear factors.}}\Big[\underbrace{(x^2+b_{m+1}x+c_{m+1})\cdots(x^2+b_{n-1}x+c_{n-1})}_{\frac{n-m}{2} \text{ quadratic factors}}\Big]$$

(By the way I assumed that all the real roots were in the 1st m factors, thats ok because we can always reorder the factors so this happens.)

Extra Credit

Cubic: These are the only possible forms.

$A(x-a)(x-b)(x-c)$

$A(x-a)(x^2+bx+c)$

(When I write a quadratic here I mean implicitly that it cannot be factored using real numbers.

Quartic

$A(x-a)(x-b)(x-c)(x-d)$

$A(x-a)(x-b)(x^2+cx+d)$

$A(x^2+ax+b)(x^2+cx+d)$

the polynomials for each order here are not equal. Rather they Illustrate the possible types you can get.

Quintic

$A(x-a)(x-b)(x-c)(x-d)(x-e)$

$A(x-a)(x-b)(x-c)(x^2+dx+e)$

$A(x-a)(x^2+bx+c)(x^2+dx+e)$

6th - Degree

$A(x-a)(x-b)(x-c)(x-d)(x-e)(x-f)$

$A(x-a)(x-b)(x-c)(x-d)(x^2+ex+f)$

$A(x-a)(x-b)(x^2+cx+d)(x^2+ex+f)$

$A(x^2+ax+b)(x^2+cx+d)(x^2+ex+f)$

And going back to quadratic

$A(x-a)(x-b)$

$A(x^2+ax+b)$

Its nice to know no matter how complicated a polynomial may look its just a product of these simple polynomials we know how to deal with. Of course you may complain, so what if this factorization exists, how do you find it? That's a much harder question.