Solution for TEST II, MA141 FALL 2004

① Begin by recalling that $-1 \le \sin(x) \le 1$ then multiply this inequality by $\frac{1}{x^2}$ to get $\frac{-1}{x^2} \le \frac{\sin(x)}{x^2} \le \frac{1}{x^2}$.

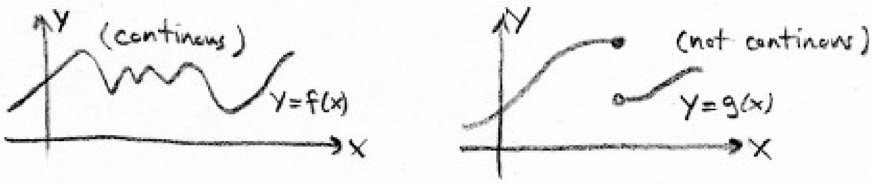
$$\lim_{x\to\infty}\left(\frac{1}{x^2}\right) = 0 \quad \text{and} \quad \lim_{x\to\infty}\left(\frac{-1}{x^2}\right) = 0$$

Thus by the squeeze theorem we can conclude that

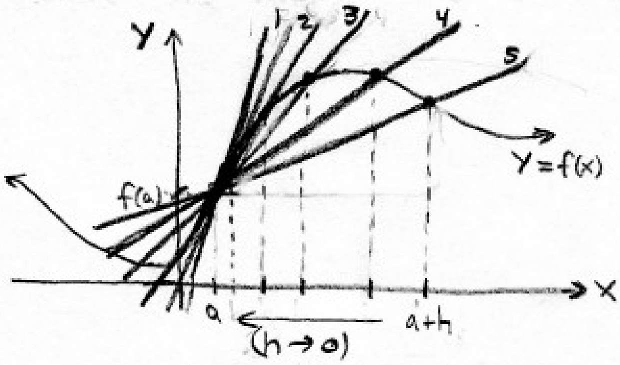
$$\lim_{x\to\infty}\left(\frac{\sin(x)}{x^2}\right) = 0$$

② The graphs below are one answer, there are others;

i) We defined f to be continous if $\lim_{x\to a} f(x) = f(a)$ for each x in the domain of f. Graphically this is equivalent to being able to draw the graph of f without lifting the pencil.



ii) Graphically we saw that the tangent line came from taking secant lines of points closer & closer to the point,

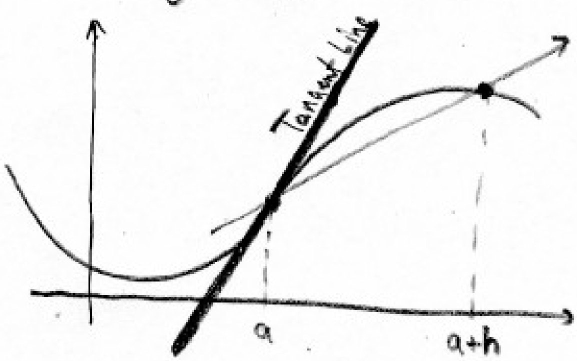


we defined the derivative of f(x) at x = a to be f'(a) which is given by:

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

this limit gives the slope of the tangent line.

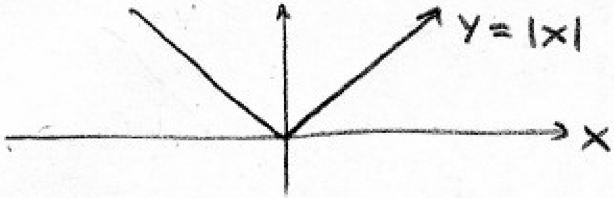
iii) the derivative of f at a is the slope of the tangent line through (a, f(a)) of the graph y = f(x).



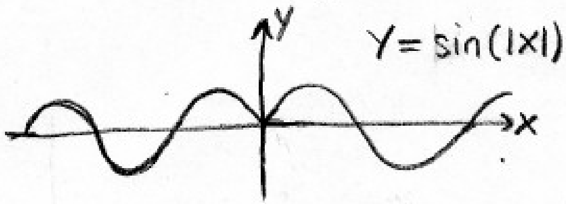
Same picture as above, I'm trying to show how the secant lines converge to the tangent line as we let h get smaller & smaller.

(It was reasonable to use one picture for ii & iii)

(2) iv) A graph is required, a formula would be nice, I'll give both,



there are many other possibilities, but all of have a pointy spot or a kink, corner whatever you want to call it, here's a few more examples,



at x=0 the graph is not smooth. We also refer to differentiable functions as "smooth functions".

Of course all this is not req^d for full-credit on (2); I include these comments to hopefully help your intuition some.

(3) We did this in class. Let $f(x) = e^x$ then

$$f'(x) \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{e^{x+h} - e^x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{e^x e^h - e^x}{h}\right)$$

$$= \lim_{h\to 0}\left(e^x\left(\frac{e^h-1}{h}\right)\right)$$

but e^x is constant with respect to h so we can pull it out of the limit.

$$= e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) \to 1$$

(I told you this in the problem statement.)

$$= \boxed{e^x = \frac{d}{dx}(x)}$$

Sol^n for TEST II

④ Let $s(t) = t^2 - 2t + 1$ be the position of a particle at time t,

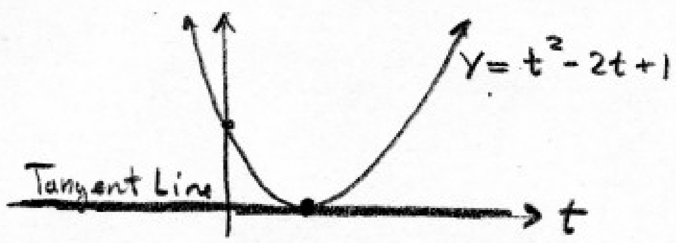
$$v(t) = \frac{ds}{dt}(t) = \frac{d}{dt}\left(t^2 - 2t + 1\right) = \boxed{2t - 2 = v(t) \quad \text{velocity at time } t}$$

$$a(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt}(t) = \frac{d}{dt}(2t-2) = \boxed{2 = a(t) \quad \text{acceleration at time } t}$$

Oops I wrote $(1,4)$ which isn't on the graph, as I told you in class find the eq^n of the tangent line thru $(1,0)$ instead

slope : $s'(1) = v(1) = 0$

tangent line has : $y - 0 = 0(t-1) \Rightarrow \boxed{y = 0 \text{ is tangent line}}$



We can factor $t^2 - 2t + 1 = (t-1)(t-1)$ then the graph is easy to construct its a parabola that bounces of t-axis at $t = 1$.

⑤ In each of the parts we'll follow the same strategy we used in class. ① Write the funny trig fnct. in terms of $\sin(x)$ and $\cos(x)$ ② use quotient rule to differentiate ③ simplify possibly with help of $\sin^2 x + \cos^2 x = 1$ and convert $\sin(x)$ & $\cos(x)$ back into the funny trig fncts (tan, cot, sec, csc)

a.) $$\frac{d}{dx}\left(\csc(x)\right) = \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) \underset{\text{quotient-rule}}{=} \frac{-\cos(x)}{\sin^2(x)} = \left(\frac{-1}{\sin(x)}\right)\frac{\cos(x)}{\sin(x)} = -\csc(x)\cot(x)$$

b.) $$\frac{d}{dx}\left(\sec(x)\right) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) \underset{\text{quotient rule}}{=} \frac{-(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

c.) $$\frac{d}{dx}\left(\tan(x)\right) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \underset{\text{quotient rule}}{=} \frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

d.) $$\frac{d}{dx}\left(\cot(x)\right) = \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \underset{\text{quotient rule}}{=} \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x).$$

⑥

a) $\frac{d}{db}(cb^2 + 2xd) = c\frac{d}{db}(b^2) + \frac{d}{db}(2xd)^{\nearrow 0}$ x & d don't depend on b. And c is a constant with respect to b I can pull it out

$= c(2b)$

$= \boxed{2cb}$

b) $\frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{d}{dx}\left((x^2)^{1/3}\right) = \frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{-1/3} = \boxed{\frac{2}{3\sqrt[3]{x}}}$

c.) $\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{d}{ds}(s^{-2}) = -2s^{-3} = \boxed{\frac{-2}{s^3}}$

d.) $\frac{d}{dt}\left(e^t \sec(t)\right) = \left(\frac{de^t}{dt}\right)\sec(t) + e^t\frac{d}{dt}(\sec(t))$ product rule!

$= e^t\sec(t) + e^t\sec(t)\tan(t)$ using problem 5b.

$= \boxed{e^t\sec t(1 + \tan(t))}$

e.) $\frac{d}{dx}\left(x\sin(x)\cos(x)\right) = \left[\frac{d}{dx}\left(x\sin(x)\right)\right]\cos(x) + x\sin x\frac{d}{dx}\cos(x)$: product rule

$= \left[\frac{dx}{dx}\sin x + x\frac{d}{dx}\sin x\right]\cos(x) + x\sin x(-\sin x)$: product rule again.

$= [\sin(x) + x\cos(x)]\cos(x) - x\sin^2 x$

$= \boxed{\sin(x)\cos(x) + x(\cos^2 x - \sin^2 x)}$

f.) $\frac{d}{dx}\left(\frac{x+3}{\cos(x)}\right) = \frac{\cos(x) + \sin(x)(x+3)}{\cos^2(x)} = \boxed{\sec(x) + \sec(x)\tan(x)(x+3)}$

Another way is $\frac{d}{dx}\left(\frac{1}{\cos(x)}\cdot(x+3)\right) = \frac{d}{dx}\left(\sec(x)\cdot(x+3)\right) = \sec(x) + \sec(x)\tan(x)(x+3)$

g.) $\frac{d}{dx}\left(ae^x(x^2+3)\right) = a\left[\frac{d}{dx}\left(e^x(x^2+3)\right)\right]$: a is constant, pull it out

$= a\left[e^x(x^2+3) + e^x(2x)\right]$: product rule

$= \boxed{ae^x(x^2 + 2x + 3)}$

If you completed the differentiation correctly you should have recieved full-credit, caution in future I'll require you to simplify your work.

h.) $\frac{d}{dx}\left(\frac{x+x^2}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{x}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right)$; think before you differentiate!

$= \frac{d}{dx}\left(x^{1/2} + x^{3/2}\right)$

$= \frac{1}{2}x^{-1/2} + \frac{3}{2}x^{1/2}$

$= \boxed{\frac{1}{2\sqrt{x}} + \frac{3}{2}\sqrt{x}}$

i.) $\frac{d}{du}\left(\sin(u)\csc(u)\right) = \frac{d}{du}\left(\sin(u)\frac{1}{\sin(u)}\right)$ again think first!

$= \frac{d}{du}(1) = \boxed{0}$

j) $\frac{d}{dx}\left(\frac{\sin(x)}{\tan(x)+e^x}\right) = \frac{\cos(x)(\tan(x)+e^x) - \frac{d}{dx}(\tan(x)+e^x)\sin(x)}{(\tan(x)+e^x)^2}$

$= \frac{\cos(x)(\tan(x)+e^x) - (\sec^2(x)+e^x)\sin(x)}{(\tan(x)+e^x)^2}$

$= \frac{\cos(x)\tan(x) + \cos(x)e^x - \sec^2(x)\sin(x) - e^x\sin(x)}{(\tan(x)+e^x)^2}$

$= \boxed{\frac{\sin(x) - \sec^2(x)\sin(x) + e^x(\cos(x)-\sin(x))}{(\tan(x)+e^x)^2}}$

Extra Credit

- Prove Product Rule using def^{n} of derivative, (Did this in class)

$$(fg)'(x) = \lim_{h\to 0}\left(\frac{(fg)(x+h)-(fg)(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)\,g(x+h) - f(x)g(x)}{h}\right)$$: product of functions is defined point wise ; $(fg)(x) \equiv f(x)\cdot g(x)$

$$= \lim_{h\to 0}\left(\frac{f(x+h)g(x+h) \overbrace{- f(x)\,g(x+h) + f(x)\,g(x+h)}^{\text{zero}} - f(x)g(x)}{h}\right)$$: added zero

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right) + \lim_{h\to 0}\left(f(x)\cdot \frac{g(x+h)-g(x)}{h}\right)$$: linearity of limit

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) \lim_{h\to 0}(g(x+h)) + \lim_{h\to 0} f(x) \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)$$: limit of product is product of limits.

$$= f'(x)\,g(x) + f(x)\,g'(x)$$: using the definitions of $f'(x)$, $g'(x)$ and the continuity of g.

As we discussed if g is differentiable it's also continous. And although we framed continuity by $\lim_{x\to a} g(x) = g(a)$ its equivalent to $\lim_{h\to 0} g(x+h) = g(x)$. think about it.

- Prove $\frac{d}{dx}(2^x) = \ln(2)\,2^x$. Of course once we can use the Chain-Rule this will be easier, but for this test it's extra credit because,

$$\frac{d}{dx}(2^x) = \lim_{h\to 0}\left(\frac{2^{x+h}-2^x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2^x 2^h - 2^x}{h}\right)$$

$$= 2^x \underbrace{\lim_{h\to 0}\left(\frac{2^h-1}{h}\right)}_{\text{prove this is } \ln(2) \text{ and were done.}}$$

upto now it's just like # 3 except the limit 2 instead of e so we don't know how to calculate it imediately, it takes some work,

$$\lim_{h\to 0}\frac{2^h-1}{h} \xrightarrow{\text{sneaky step}} = \lim_{h\to 0}\frac{e^{\ln(2^h)}-1}{h} = \lim_{h\to 0}\frac{e^{h\ln(2)}-1}{h}$$

Now factor out a $\ln(2)$; $\lim_{h\to 0}\left(\frac{2^h-1}{h}\right) = \ln(2)\lim_{h\to 0}\left(\frac{e^{h\ln(2)}-1}{h\ln(2)}\right)$. Think about this, if $h\to 0$ then so does $h\ln(2)\to 0$, call $h\ln(2) = u$ and the limit becomes; $\lim_{h\to 0}\left(\frac{2^h-1}{h}\right) = \ln(2)\lim_{u\to 0}\underbrace{\left(\frac{e^u-1}{u}\right)}_{1} = \ln(2)$. $\boxed{\text{QED}}$

$1 \leftarrow$ definition of e. as in # 3.