MA 141, Section 16, Fall 2004
Instructor : Mr. James Cook
Test 3

1. (10 points) Let $\sin(x) + \cos(y) = \sin(x)\cos(y)$,

find $\dfrac{dy}{dx}$.

2 . (15 points) Suppose that we have a parametric curve given by $x = 3\cos(t)$ and $y = 5\sin(t)$. Find the following

(a) The Cartesian equation of this curve,

(b) $\dfrac{dy}{dx}$,

(c) The equation of tangent line at $\left(\dfrac{3}{\sqrt{2}}, \dfrac{5}{\sqrt{2}}\right)$.

3. (10 points) A ladder 5 m long rests against a vertical wall if the bottom of the ladder slides away from the wall at 2 m / s. Then how fast is the top of the ladder sliding down the wall when the bottom is 3 m from the wall?

4. (10 points) Prove that $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$.

5. (35 points) Calculate the first derivatives with respect to x of the following

(a) $4\cos(bx)$, where b is a constant

(b) $\sqrt{4 + 3x}$

(c) $\csc(\log_{10}(2x))$

(d) 10^{1-x^2}

(e) $\sin^2(x)$

(f) $\ln(x^2 + 10)$

(g) $\ln(x^4 \sin^2(x))$

6. (20 points) Calculate $\dfrac{dy}{dx}$ of the following

(a) $y = \sqrt{x}\, e^{x^2} (x^2 + 1)^{10}$

(b) $y = (\sin(x))^x$

EXTRA CREDIT (5 points)

Calculate $\dfrac{d}{dx}(x^{x^x})$

SOLUTION TO TEST III FALL 2004

① $\sin(x) + \cos(y) = \sin(x)\cos(y)$ diff. with rspt. to x

$$\cos(x) - \sin y \frac{dy}{dx} = \cos(x)\cos(y) - \sin(x)\sin(y)\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\sin y - \sin x \sin y\right) = \cos x - \cos x \cos y$$

$$\frac{dy}{dx} = \frac{\cos(x)(1-\cos y)}{\sin(y)(1-\sin x)}$$

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② $X = 3\cos t$
$Y = 5\sin t$

a.) $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = \left(\frac{3\cos t}{3}\right)^2 + \left(\frac{5\sin t}{5}\right)^2 = \cos^2 t + \sin^2 t = 1$

thus $\boxed{\frac{x^2}{9} + \frac{y^2}{25} = 1}$ this is the Cartesian form of the curve.

b.) Differentiate a.) implicitly

$$\frac{2x}{9} + \frac{2y}{25}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{9}\frac{25}{2y} = \boxed{\frac{-25x}{9y} = \frac{dy}{dx}}$$

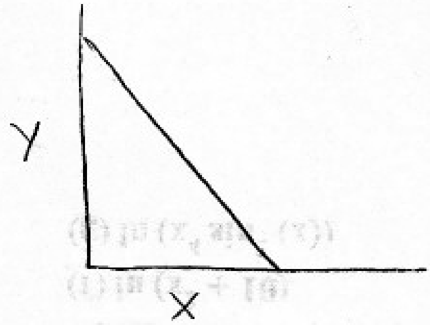
c.) The tangent line at $x = 3/\sqrt{2}$, $y = 5/\sqrt{2}$ is

$$Y = 5/\sqrt{2} - \frac{25(3/\sqrt{2})}{9(5/\sqrt{2})}(x - 3/\sqrt{2})$$

$$Y = 5/\sqrt{2} - \frac{5}{3}(x - 3/\sqrt{2}) = 5/\sqrt{2} - \frac{5}{3}x + 5/\sqrt{2}$$

$$\boxed{Y = 5\sqrt{2} - \frac{5}{3}x}$$

(3)



$$x^2+y^2 = 25$$

Treat x & y as functions of time t and differentiate with respect to t,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

We know that $\frac{dx}{dt} = 2$ when $x=3$. Notice that when $x=3$ we have $9+y^2=25$ $\therefore y=\pm 4$ $\therefore y=4$ for physical reasons. We wish to know $\frac{dy}{dt}$ so solve for it

$$\frac{dy}{dt} = \frac{-x\frac{dx}{dt}}{y} = \frac{-3(2)}{4} = -\frac{6}{4} = \boxed{-1.5\ m/s = \frac{dy}{dt}}$$

(4) Begin by setting $y=\sin^{-1}(x) \Rightarrow \sin(y) = x$ now diff. implicitly, to get $\cos(y)\frac{dy}{dx} = 1$. Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \boxed{\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}\left(\sin^{-1}(x)\right)}$$

Where we used $\sin^2 y + \cos^2 y = 1 \Rightarrow \cos y = \sqrt{1-\sin^2 y}$.

(5) Calculate the derivatives

a.) $\frac{d}{dx}\left(4\cos(bx)\right) = 4(-\sin bx)\frac{d}{dx}(bx) = \boxed{-4b\sin(bx)}$

b.) $\frac{d}{dx}\left(\sqrt{4+3x}\right) = \frac{1}{2\sqrt{4+3x}}\cdot\frac{d}{dx}(4+3x) = \boxed{\frac{3}{2\sqrt{4+3x}}}$

c.) $\frac{d}{dx}\Big(\csc(\underbrace{\log_{10}(2x)}_{u})\Big) = -\csc(u)\cot(u)\frac{d}{dx}\left(\log_{10}(2x)\right)$

$$= -\csc(u)\cot(u)\frac{1}{\ln(10)\cdot 2x}\frac{d}{dx}(2x)$$

$$= \boxed{-\frac{\csc(\log_{10}(2x))\cot(\log_{10}(2x))}{x\ln(10)}}$$

⑤ Continued

(d.) $\frac{d}{dx}\left(10^{1-x^2}\right) = \ln(10)\, 10^{1-x^2} \frac{d}{dx}(1-x^2)$

$= \boxed{-2x \ln(10)\, 10^{1-x^2}}$

(e.) $\frac{d}{dx}\left((\sin x)^2\right) = 2\sin(x) \frac{d}{dx}\sin(x) = \boxed{2\sin(x)\cos(x)}$

(f.) $\frac{d}{dx}\left(\ln(x^2+10)\right) = \frac{1}{x^2+10}\frac{d}{dx}(x^2+10) = \boxed{\frac{2x}{x^2+10}}$

(g) $\frac{d}{dx}\left(\ln(x^4 \sin^2 x)\right) = \frac{d}{dx}\left(4\ln(x) + 2\ln(\sin(x))\right)$

$= \frac{4}{x} + \frac{2}{\sin(x)}\frac{d}{dx}(\sin(x))$

$= \boxed{\frac{4}{x} + \frac{2\cos(x)}{\sin(x)}}$

⑥ Use log. diff.

a.) $\ln(y) = \ln(\sqrt{x}) + \ln(e^{x^2}) + \ln(x^2+1)^{10}$

$\ln(y) = \frac{1}{2}\ln(x) + x^2 + 10\ln(x^2+1)$

Differentiate implicitly remember y is function of x,

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + 2x + \frac{10}{x^2+1}\cdot\frac{d}{dx}(x^2+1)$

$$\boxed{\frac{dy}{dx} = \sqrt{x}\, e^{x^2}(x^2+1)^{10}\left\{\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right\}}$$

⑥

(b.) $y = (\sin(x))^x \Rightarrow \ln(y) = \ln\left[(\sin(x))^x\right] = x\ln(\sin(x))$

Differentiate with rspt. to x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{dx}{dx}\cdot\ln(\sin(x)) + x\frac{d}{dx}\left(\ln(\sin(x))\right)$$

$$= \ln(\sin(x)) + x\frac{1}{\sin(x)}\cdot\frac{d}{dx}(\sin(x))$$

Solving for $\frac{dy}{dx}$ then yields,

$$\boxed{\frac{dy}{dx} = (\sin(x))^x\left(\ln(\sin(x)) + \frac{x\cos(x)}{\sin(x)}\right)}$$

Extra Credit

$$y = x^{x^x}$$

$$\ln(y) = \ln(x^{x^x}) = x^x\ln(x)$$

$$\ln(\ln(y)) = \ln(x^x\ln(x))$$

$$= \ln(x^x) + \ln(\ln(x))$$

$$= x\ln(x) + \ln(\ln(x))$$

$$\frac{d}{dx}\left(\ln(\ln(y))\right) = \frac{1}{\ln(y)}\frac{1}{y}\frac{dy}{dx}$$

$$\frac{d}{dx}\left(x\ln(x)\right) = \ln(x) + 1$$

$$\frac{d}{dx}\left(\ln(\ln(x))\right) = \frac{1}{\ln(x)}\frac{1}{x}$$

$$\therefore \frac{1}{y\ln(y)}\frac{dy}{dx} = \ln(x) + 1 + \frac{1}{\ln(x)}\frac{1}{x}$$

$$\frac{dy}{dx} = y\ln(y)\left\{\ln(x) + 1 + \frac{1}{x\ln(x)}\right\}$$

$$\boxed{\frac{d}{dx}\left(x^{x^x}\right) = x^{x^x+x}\ln(x)\left\{\ln(x) + 1 + \frac{1}{x\ln(x)}\right\}}$$