MA141, Section 16, Fall 2004

Instructor : Mr. James Cook

Test 4 (Show all your work on separate paper)

1. (28 points) Calculate the following limits.

(a) $\lim_{x \to 0} \left(\frac{\sin(x)}{x} \right)$

(b) $\lim_{t \to 0} \left(\frac{5^t - 2^t}{t} \right)$

(c) $\lim_{x \to 0} \left(x^{\sin(x)} \right)$

(d) $\lim_{x \to 0} \left(\frac{\tan(ax)}{\tan(bx)} \right)$

2. (42 points) $f(x) = x^4 - 4x^3$, find (state any theorem you use)

(a) Critical points

(b) The intervals in which f is increasing / decreasing

(c) The local maxima and minima

(d) The intervals in which f is concave up / down

(e) Inflection points

(f) x - intercepts

(g) Graph the function carefully

3. (15 points) What are the dimensions of the largest rectangle with a perimeter of 10 ft. State the dimensions and area of that rectangle.

4. (15 points) Calculate $\sqrt{9.06}$ using the linearization of $\sqrt{x} = f(x)$ near 9.

Extra Credit (5 points)

Given an arbitrary (2^{nd}, 3^{rd}, 4^{th}, 5^{th}) degree polynomial, what are the min / max number of critical points. Graph an example of each.

Extra Credit (2 points)

If there is an maximum error of 0.1 ft in the perimeter of the rectangle from question #3, then what is the maximum, relative and percentage errors in the area.

Test 4 Sol^{n}

① (a) $\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) \underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}} \lim_{x\to 0}\left(\frac{\cos(x)}{1}\right) = \cos(0) = \boxed{1}$

(b) $\lim_{t\to 0}\left(\frac{5^t-2^t}{t}\right) \underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}} \lim_{x\to 0}\left(\frac{\ln(5)5^t-\ln(2)2^t}{1}\right) = \boxed{\ln(5)-\ln(2)}$

(c) $\lim_{x\to 0}(x^{\sin(x)}) = \lim_{x\to 0} e^{\ln(x^{\sin(x)})}$

$= e^{\underbrace{\lim_{x\to 0}(\sin(x)\ln(x))}_{*}}$

$* = \lim_{x\to 0}\left(\frac{\ln(x)}{\frac{1}{\sin(x)}}\right) \underset{(\frac{\infty}{\infty})}{\overset{\mathcal{L}}{=}} \lim_{x\to 0}\left(\frac{\frac{1}{x}}{\frac{-\cos(x)}{\sin^2(x)}}\right)$

$= \lim_{x\to 0}\left(\frac{-\sin^2(x)}{x\cos(x)}\right)$

$\underset{(\frac{0}{0})}{\overset{\mathcal{L}}{=}} \lim_{x\to 0}\left(\frac{-2\sin(x)\cos(x)}{\cos(x)-x\sin(x)}\right)$

$= \frac{-2\sin(0)\cos(0)}{\cos(0)-0\sin(0)}$

$= 0$

$\therefore \lim_{x\to 0}(x^{\sin(x)}) = e^{\lim_{x\to 0}(\sin(x)\ln(x))} = e^0 = \boxed{1}$

(d) $\lim_{x\to 0}\left(\frac{\tan(ax)}{\tan(bx)}\right) = \lim_{x\to 0}\left(\frac{a\sec^2(ax)}{b\sec^2(bx)}\right)$

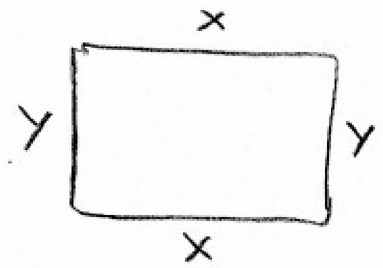
$= \frac{a}{b}\lim_{x\to 0}\left(\frac{\cos^2(bx)}{\cos^2(ax)}\right)$

$= \frac{a}{b}\,\frac{\cos^2(0)}{\cos^2(0)}$

$= \boxed{\frac{a}{b}}$

② See your textbook pgs. 284–285 Example 4

③



$$P = 2x + 2y = 10$$
$$\Rightarrow y = 5 - x$$
$$\Rightarrow A = xy = x(5-x) = 5x - x^2$$

$A'(x) = 5 - 2x = 0 \Rightarrow x = 5/2$ is only critical point

it is important to check that it is the max! It's not enough to just have $A'(x) = 0$

$A''(x) = -2 \Rightarrow A''(5/2) = -2 \therefore A(5/2)$ is the max area by 2^{nd} Derivative test. So the answer is

$\boxed{x = 5/2 \text{ ft}}$

$y = 5 - 5/2 = \boxed{5/2 \text{ ft} = y}$ $\quad \boxed{A = \frac{25}{4} \text{ ft}^2}$

④ Just like the in-class example for $\sqrt{4.02}$ we use $L_f(9.06)$ to approximate $\sqrt{9.06}$. Let $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$ so the linearization of f at $x = 9$ is

$$L_f(x) = f(9) + f'(9)(x-9)$$

geometrically this is the eq of the tangent line thru $(9,3)$ on $y = \sqrt{x}$

$$= \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9)$$
$$= 3 + \frac{1}{6}(x-9)$$

So evaluate this at 9.06 to get $\cong \sqrt{9.06}$,

$$L_f(9.06) = 3 + \frac{1}{6}(9.06 - 9) = 3 + \frac{1}{6}(0.06) = \boxed{3.01 \cong \sqrt{9.06}}$$

Extra Credit

For each case I start with an arbitrairy polynomial & differentiate to find the critical points which are the roots of the derivative. For each care we are faced with the question of how many roots or zeroes a polynomial has, you should look at the first test's extra credit to see how to list the possibilities for different orders.

2nd $f(x) = Ax^2 + Bx + C \implies f'(x) = 2Ax + B$

Thus the $f'(x)$ is a line which has one root; $\boxed{\text{1 – critical point always.}}$

3rd $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + C$

So $f'(x) = 0$ is a quadratic eqn which has $0 \to 2$ roots

$\therefore$ $\boxed{0 \to 2 \text{ critical points}}$

4th $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f'(x) = 0$ is a cubic eqn, it will have at least one root and upto 3.

$\boxed{1 \to 3 \text{ critical points}}$

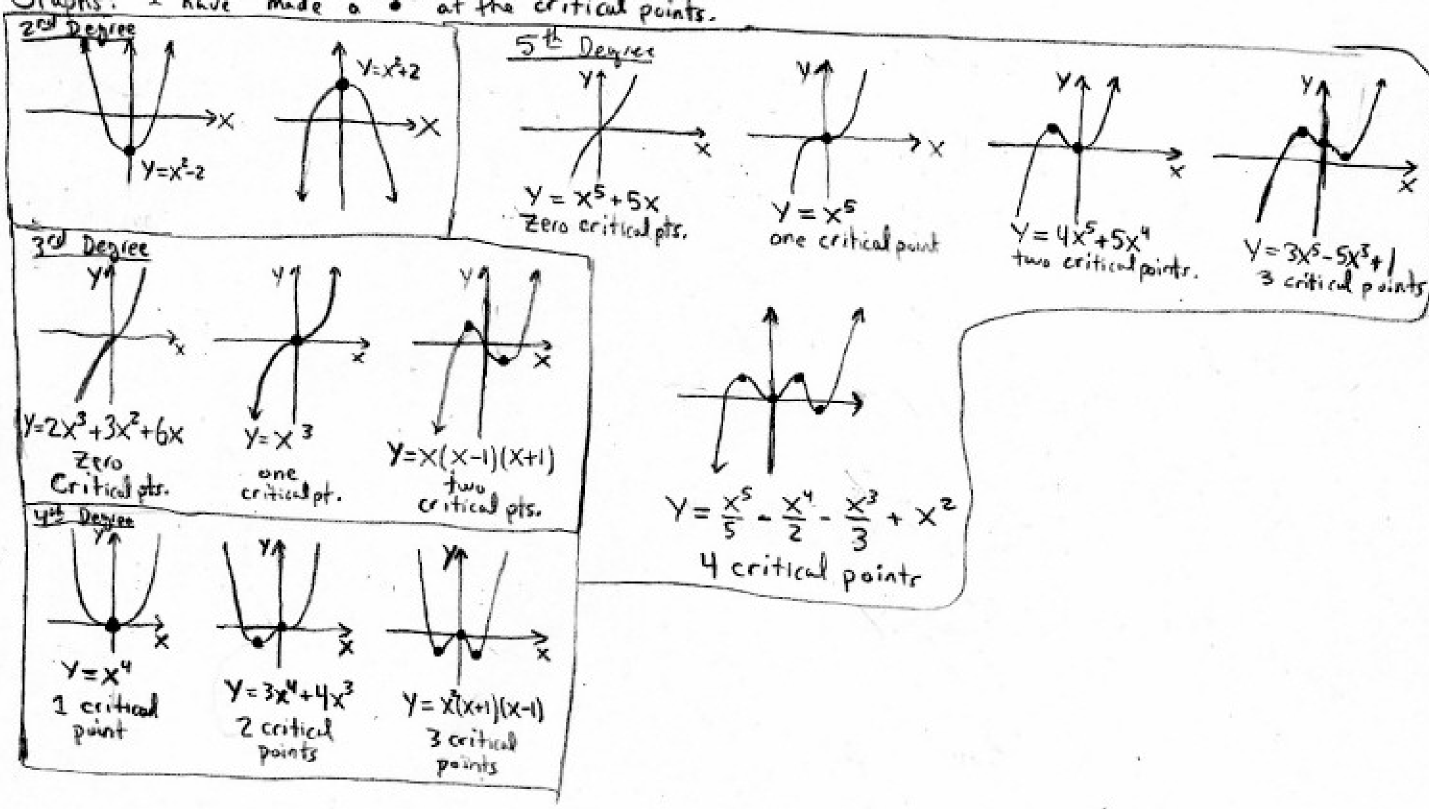
5th $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + g$

$f'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$

$f'(x) = 0$ is a quartic eqn, it will have 0 to 4 roots. Thus,

$\boxed{0 \to 4 \text{ critical points}}$

Graphs: I have made a "•" at the critical points.



- You can ask me about the 2nd Extra Credit if your interested.