The First Order

James S. Cook
Liberty University
Department of Mathematics

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Abstract

In this article we primarily study first order ordinary differential equations. We study three major methods of solution: (1.) separation of variables (2.) the integrating factor technique (3.) substitution. We study how differential equations arise from natural questions in science and engineering. Direct solution is not always possible, so we begin by examining methods to visualize differential equations directly using its direction field or isocline plot. Euler's Method uses a recursive algorithm to approximate solutions. Certainly there are more examples here than I can reasonably cover in lecture, and there is a fair amount of physical reasoning mixed into the later part of this article. Please understand this is a math class and I primarily test on math. I include the physical reasoning and analysis in the hope it is interesting to the physically motivated student.

1 Introduction to Differential Equations

What is a differential equation ? In short, it is an equation which involves derivatives. We will use the abbreviation DEqn for *differential equation* and we will use ODE for *ordinary differential equation*. I should mention, the abbreviation PDE means *partial differential equation* which is an equation which involves partial derivatives[1].

Definition 1.1. *Differential Equation*

A **first order ODE** can be expressed as $\frac{dy}{dx} = F(x,y)$ where F is an expression in x, y. If φ is a function for which $\frac{d\varphi}{dx} = F(x, \varphi(x))$ then we say φ is an **explicit solution**.

Example 1.2. *Consider* $\frac{dy}{dx} = x^2$ *then observe* $\int \frac{dy}{dx}\,dx = y = \int x^2\,dx = \frac{1}{3}x^3 + c$ *thus* $y = \frac{1}{3}x^3 + c$ *gives a family of explicit solutions for the given DEQn.*

Not every ODEqn is so simple to solve as the example above. Usually it is not possible to just directly integrate the problem. We'll see that some algebra, a clever multiplication, subsitution or both are needed to solve problems which naturally arise in applications.

There are also higher order ODEqns such as the **second order** $y'' + t^2 y = 0$ or $y'' + y^3 = 0$. We say $y'' + y = 0$ is a **linear** DEqn since its dependent variable y appears linearly. In contrast, $y'' + y^3 = 0$

[1] we cover partial derivatives in Calculus III, or my Physics 231. I suppose I should mention, most of the fundamental theorems of modern physics at a classical level are PDEs (Maxwell's Equation, Schrodinger's Equation, Einstein's Field Equations, etc...

is a **nonlinear** ODE due to the y^3-term. Notice the notation y'' hides the explicit notation for the independent variable. We might have $y'' = \frac{d^2y}{dx^2}$ or we might have $y'' = \frac{d^2y}{dt^2}$. The choice of notation depends on the setting. When the independent variable is time we tend to use t. However, I often give examples with x since students tend to be more comfortable with calculus done in the letter x. There are also interesting **systems of ODEs**. For example,

$$\frac{dx}{dt} = x + y \qquad \& \qquad \frac{dy}{dt} = -x + y$$

might model a predator-prey problem where x is the number of cute delicious bunnies and y is the number of dogs. Newton's law $\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$ is a system of second order ODEs. ODEs appear in numerous physical applications. I conclude this article with many applied examples ranging from population growth to electrical circuits to problems from Newton's classical mechanics.

Sometimes we cannot find an explicit solution so the following definition is very helpful.

Definition 1.3. *Differential Equation*

If implicit differentiation of $G(x, y) = C$ yields $\frac{dy}{dx} = F(x, y)$ then the curve defined by $G(x, y) = C$ is known as an **implicit solution** of the $\frac{dy}{dx} = F(x, y)$. The family of all explicit or implicit solutions to a DEqn is known as the **general solution**. A solution to the **intial value problem** $\frac{dy}{dx} = F(x, y)$ with (x_o, y_o) is a solution of the DEQn which includes the point (x_o, y_o).

Example 1.4. *The differential equation* $\frac{dy}{dx} = x^2$ *has general solution* $y = x^3/3 + c$. *We can select an appropriate value of* c *to solve a given initial value problem for the DEqn. For instance, to find the solution through* $(1, 4/3)$ *we set* $4/3 = 1/3 + c$ *and find* $c = 1$ *which means* $y = x^3/3 + 1$ *solves the given intial value problem.*

In retrospect, every integral we solved is equivalent to a first order differential equation:

$$\int f(x)dx = y \qquad \Leftrightarrow \qquad \frac{dy}{dx} = f$$

So, we've already solved a bunch of DEqns this term, you just didn't realize it. That said, we will need new methods to solve the DEqns found in this article.The main techniques we will study are:

(1.) separation of variables

(2.) integrating factor method

(3.) substitution

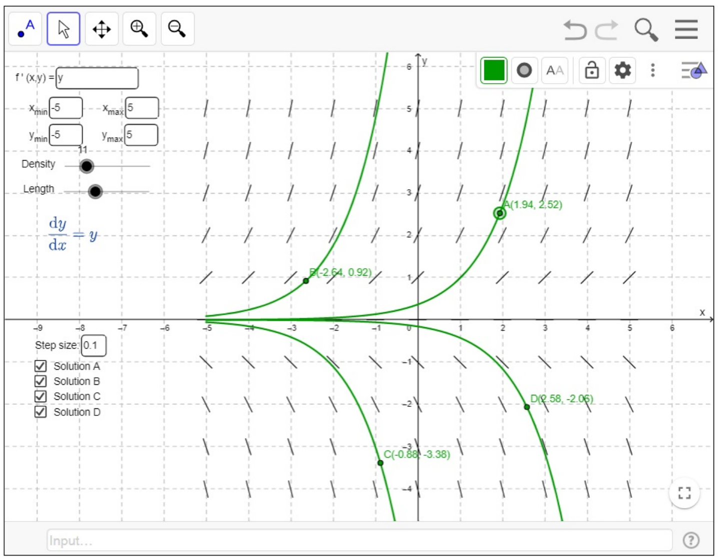
Beyond that we study how to solve a select subclass of really nice 2nd order problems which are very important to applications. The methods of this article are by no means complete or algorithmic. Solving arbitrary first order problems is an art. That said, it is not a hidden art, it is one we all must master. We begin with methods to visualize and approximate solutions to first order ODEs.

2 Visualizations and Euler's Method

The solution of $\frac{dy}{dx} = F(x,y)$ is a curve whose slope at (x,y) is $F(x,y)$. We can expect solutions will match the slopes prescribed by the given DEqn.

Example 2.1. *Consider* $\frac{dy}{dx} = y$ *this has solution* $y = ce^x$ *by math magic*[2]. *Notice* $\frac{d}{dx}(ce^x) = ce^x$ *thus* $y = ce^x$ *solves* $\frac{dy}{dx} = y$.

I found a nice Geogebra page(linked here) which visualizes the xy-plane and plots little dashes to indicate the slopes given by the DEqn. Look what happens for the example above:



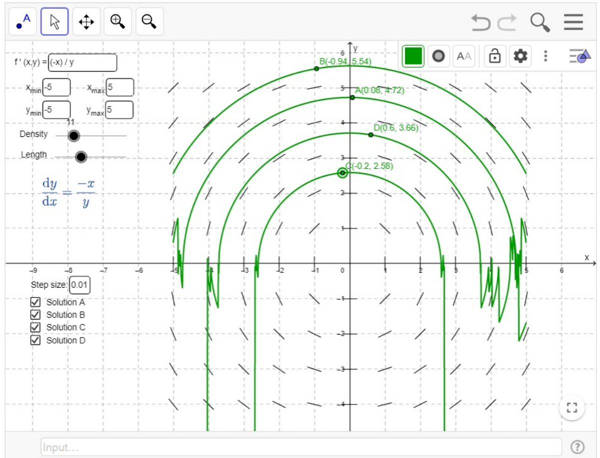
You can adjust the density of the dashes shown as well as their length and it plots four solutions through four initial values which you select by moving the little labeled dots. This website is very user friendly and I recommend it to study the geometry of solutions for $\frac{dy}{dx} = f(x,y)$. However, I should mention, there are pitfalls we can face from a lack of attention to domain.

For example, consider the differential equation $\frac{dy}{dx} = \frac{-x}{y}$. Technically this is not defined when $y = 0$. In contrast, if we study the corresponding Pfaffian form of the differential equation given by $xdx + ydy = 0$ then there is no apparent problem with the domain of the differential equation. In short, solving for $\frac{dy}{dx}$ limits our attention to solutions where y can be written as a function of x.

Example 2.2. *Observe* $\frac{dy}{dx} = \frac{-x}{y}$ *has implicit solution* $x^2 + y^2 = R^2$. *Differentiate implicity to see* $2x + 2y\frac{dy}{dx} = 0$ *thus* $\frac{dy}{dx} = \frac{-x}{y}$. *The differential of* $x^2 + y^2 = R^2$ *is* $2xdx + 2ydy = 0$ *hence* $xdx + ydy = 0$ *is a differential consequence of the equation* $x^2 + y^2 = R^2$. *In my opinion,* $xdx + ydy = 0$ *should be viewed as primary and the family of circles* $x^2 + y^2 = R^2$ *forms the general solution.*

When we limit our thinking to functions alone it causes problems. See what happens with the nice Geogebra page(linked here) when it tries to visualize the solutions for the example above. It gets stuck where $y = 0$ since the formula defining the differential equation blows up at that point and the numerical method which underlies the computer program produces garbage.

[2]we learn how to derive this in the next section, have patience my friend



There is a solution to the coordinate defect problem we encountered above. Suppose we wish to visualize the solution of $\frac{dy}{dx} = \frac{A}{B}$ where A, B are both functions of x, y in general. Suppose

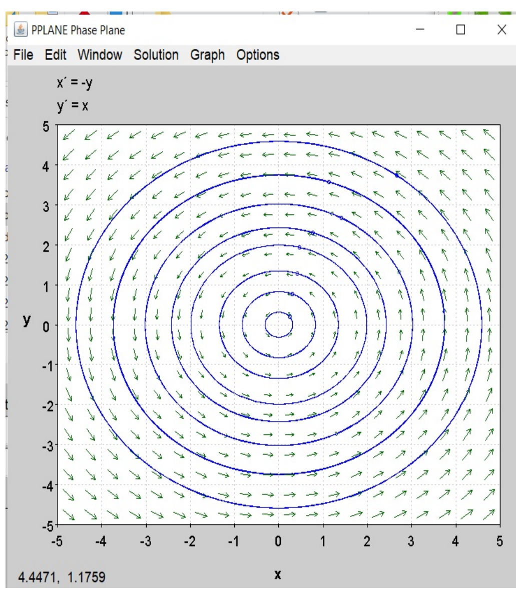
$$\frac{dy}{dt} = A \qquad \& \qquad \frac{dy}{dt} = B$$

then formally $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{A}{B}$.

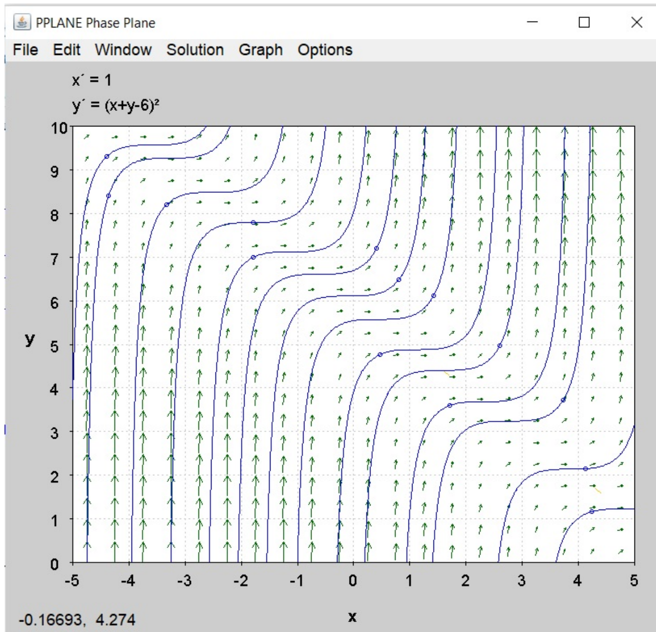
Remark 2.3.

To plot the direction field for $\frac{dy}{dx} = \frac{A}{B}$ we put $x' = B$ and $y' = A$ in a website or software which plots systems of differential equations in the plane. For example,this website hosted by the University of Arkansas based on the phase plane plotter in Matlab.

I use the pplane java applet available at (click here) to produce the direction field for $\frac{dy}{dx} = \frac{x}{-y}$ by setting $x' = -y$ and $y' = x$ in the pplane tool:

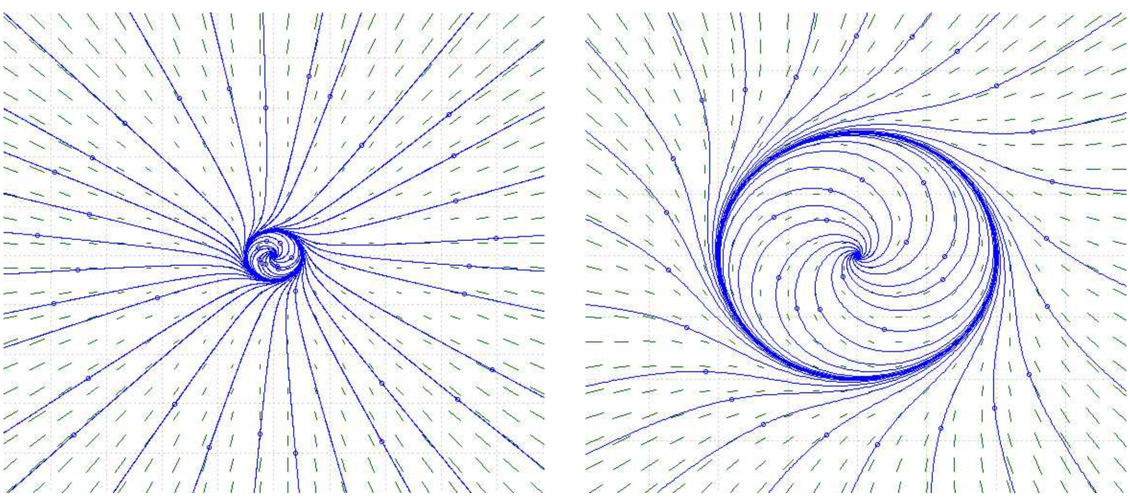


Example 2.4. Problem: *plot the direction field for* $\frac{dy}{dx} = (x+y-6)^2$ *and a few solutions.*
Solution: *use pplane with* $x' = 1$ *and* $y' = (x+y-6)^2$.



We show solutions are given by $y = 6 + \tan(x+C) - x$ *in Example 5.1. This is the plot of that.*

Example 2.5. Problem: *plot the isocline field for* $\frac{dy}{dx} = \frac{y^3+x^2y-y-x}{xy^2+x^3+y-x}$ *and a few solutions.*
Solution: *we use pplane with* $x' = xy^2 + x^3 + y - x$ *and* $y' = y^3 + x^2y - y - x$.

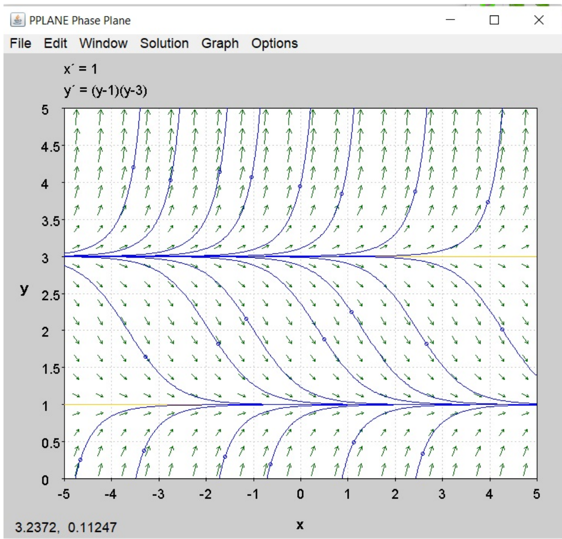


The solution of this requires a polar coordinate change which is helpful since this DEqn has a rotational symmetry. Also, notice that all solutions asymptotically are drawn to the unit circle. If the solution begins inside the circle it is drawn outwards to the circle whereas all solutions outside the circle spiral inward. The neat thing is we can appreciate how solutions behave for the given DEqn even though the solution is currently beyond our grasp.

Example 2.6. *Suppose we want to create functions which level out when $y = 1$ or $y = 3$ and are decreasing for $1 < y < 3$ and yet are increasing elsewhere. A moment's reflection reveals*

$$\frac{dy}{dx} = (y-1)(y-3)$$

has $\frac{dy}{dx} < 0$ for $1 < y < 3$ and $\frac{dy}{dx} > 0$ for $y > 3$ or $y < 1$. We can visually check our DEqn by plotting its direction field.

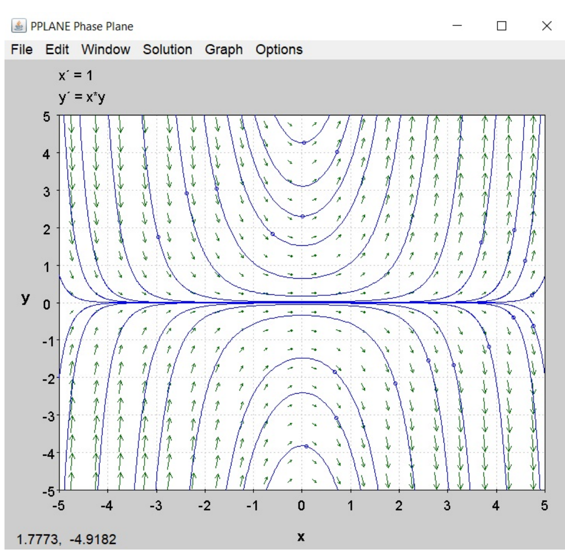


Notice I turned on the "show nullclines" feature which plots the solutions which have $\frac{dy}{dx} = 0$ along the whole solution. For this example, these equilbrium solutions are $y = 1$ and $y = 3$.

Example 2.7. *Suppose we want a functions whose graph is increasing in Quadrants I and III and yet is decreasing in Quadrants II and IV. An expression which comes to mind is xy since $xy > 0$ when both x and y are of the same sign and $xy < 0$ whenever x and y differ in sign. Solutions to*

$$\frac{dy}{dx} = xy$$

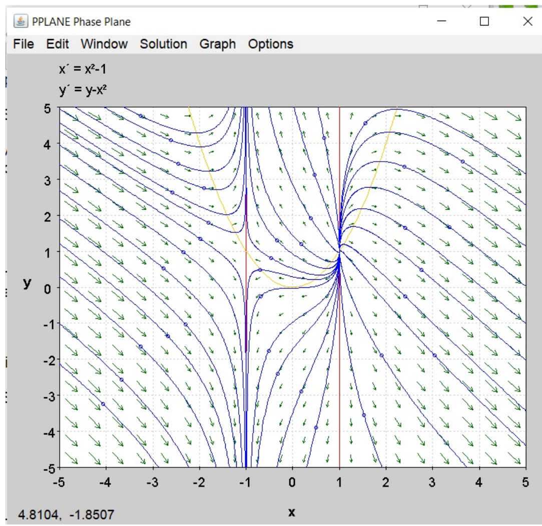
should have the desired traits. Set $x' = 1$ and $y' = xy$ in pplane and we plot:



Example 2.8. *Suppose we want a functions whose graph has horizontal tangents at any point of intersection with $y = x^2$ and whose graph have vertical tangents at $x = \pm 1$. We want a fraction whose numerator is zero for $y = x^2$ and whose denominator is zero for $x = \pm 1$. The natural candidate to try is:*

$$\frac{dy}{dx} = \frac{y - x^2}{x^2 - 1}.$$

Set $x' = x^2 - 1$ and $y' = y - x^2$ in pplane and we plot:

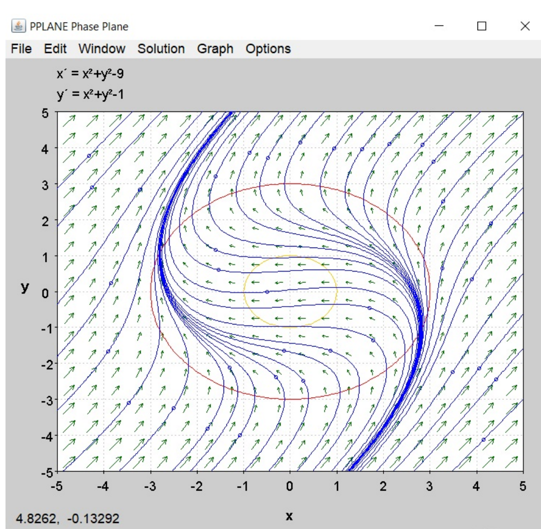


Pplane plotted nullclines of $x = \pm 1$ as well as $y = x^2$ in dark yellow/red.

Example 2.9. *Suppose we want a functions whose graph has horizontal tangents at any point of intersection with $x^2 + y^2 = 1$ and whose graph have vertical tangents at $x^2 + y^2 = 9$. Following the same idea as the previous example we study*

$$\frac{dy}{dx} = \frac{x^2 + y^2 - 1}{x^2 + y^2 - 9}.$$

Set $x' = x^2 + y^2 - 9$ and $y' = x^2 + y^2 - 1$ in pplane and we plot:



Pplane plotted nullclines of $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$ in dark yellow/red circles.

2.1 Euler's Method

Programs such as pplane are based on using a numerical method to piece together approximate solutions to the given DEqn. Euler's Method is possibly the simplest such method. Given a direction field we could implement Euler's Method simply by tracing curves which align with the arrows they intersect.

Basically the method is simply this: pick a point, find its tangent line as prescribed by the DEqn, follow the tangent line a bit, then repeat. Euler's Method is a **recursive method**. Each step is based on the previous step. Consider the differential equation $\frac{dy}{dx} = F(x, y)$ where $F(x, y)$ is a given expression involving x, y. Choose a step-size h and an initial point (x_0, y_0). Then the tangent line to the initial point is

$$y = y_0 + F(x_0, y_0)(x - x_0).$$

If we follow the tangent line to $x = x_0 + h$ we get $y = y_0 + F(x_0, y_0)h$. Therefore, the next point we find is (x_1, y_1) where $x_1 = x_0 + h$ and $y_1 = y_0 + F(x_0, y_0)h$. Continuing, build our second tangent line with base point (x_1, y_1) and slope $F(x_1, y_1)$

$$y = y_1 + F(x_1, y_1)(x - x_1).$$

If we follow the tangent line above to $x_2 = x_1 + h$ then we find $y_2 = y_1 + F(x_1, y_1)h$. Then we continue in this fashion as long as we wish in order to create an array of points which approximately follows the solution curves of the given DEqn.

Definition 2.10. *Euler's Method*

Consider the differential equation $\frac{dy}{dx} = F(x, y)$ where $F(x, y)$ is a given expression involving x, y. Choose a step-size h and an initial point (x_0, y_0). For $n = 0, 1, 2, \ldots$ define

$$x_{n+1} = x_n + h \qquad \& \qquad y_{n+1} = y_n + F(x_n, y_n)h$$

Example 2.11. *Consider $\frac{dy}{dx} = y$ with initial point $(0, 1)$ and step-size $h = 0.2$. We will estimate the value of the solution when $x = 2$. Identify $F(x, y) = y$ and $(x_0, y_0) = (0, 1)$ thus calculate $F(0, 1) = 1$*

(1.) $y_1 = 1 + 1(0.2) = 1.2$ *hence* $(x_1, y_1) = (0.2, 1.2)$ *and calculate* $F(0.2, 1.2) = 1.2$

(2.) $y_2 = 1.2 + 1.2(0.2) = 1.44$ *calculate* $F(0.4, 1.44) = 1.44$

(3.) $y_3 = 1.44 + 1.44(0.2) = 1.728$ *calculate* $F(0.6, 1.728) = 1.728$

(4.) $y_4 = 1.728 + 1.728(0.2) = 2.0736$ *calculate* $F(0.8, 2.0736) = 2.0736$

(5.) $y_5 = 2.0736 + 2.0736(0.2) = 2.48832$

Thus we approximate the value of the solution to $\frac{dy}{dx} = y$ by $y(1) \approx 2.48832$. Since the exact solution is given by $y = e^x$ we can compare our approximation to the real solution value of $y(1) = e^1 \approx 2.7183$.

Fun fact, if you run Euler's Method with $h = 0.01$ on the above example you can calculate $y_{100} \approx 2.70$ which is much closer to the real answer. The next example demonstrates that there are some problems where Euler's Method fails.

Example 2.12. *Consider* $\frac{dy}{dx} = \sqrt{x^2+4y^3}$ *with initial point* $(2,1)$ *and step-size* $h = 0.5$. *We will estimate the value of the solution when* $x = 4$. *Identify* $F(x,y) = \sqrt{x^2+4y^3}$ *and* $(x_0, y_0) = (2,1)$ *thus calculate* $F(2,1) = \sqrt{2^2+4(1)^3} \approx 2.83$

(1.) $y_1 = 1 + 2.83(0.5) \approx 2.41$ *hence* $(x_1, y_1) = (2.5, 2.41)$ *and calculate* $F(2.5, 2.41) = \sqrt{2.5^2+4(2.41)^3} \approx 7.91$

(2.) $y_2 = 2.41 + 7.91(0.5) \approx 6.37$ *calculate* $F(3, 6.37) \approx 32.3$

(3.) $y_3 = 6.37 + 32.3(0.5) \approx 22.5$ *calculate* $F(3.5, 22.5) \approx 213.6$

(4.) $y_4 = 32.3 + 213.6(0.5) \approx 129.3$

Thus we approximate the value of the solution to $\frac{dy}{dx} = \sqrt{x^2+4y^3}$ *by* $y(4) \approx 129.3$.

Fun fact, if you run Euler's Method on the example above with a step-size of 0.1 then $y(4) = y_{20} = 2.44 \times 10^{30}$. I believe this example is numerically troublesome. If you look at the solutions generated by pplane for this problem you'll see the solutions grow very rapidly near the initial data point. This means Euler's Method has large error.

3 Separation of Variables

Suppose you are faced with the problem $\frac{dy}{dx} = f(x,y)$. If it happens that f can be factored into a product of functions $f(x,y) = g(x)h(y)$ then the problem is said to be **separable**. Proceed formally for now, suppose $h(y) \neq 0$,

$$\frac{dy}{dx} = g(x)h(y) \quad \Rightarrow \quad \frac{dy}{h(y)} = g(x)\,dx \quad \Rightarrow \quad \int \frac{dy}{h(y)} = \int g(x)\,dx$$

Ideally, we can perform the integrations above and solve for y to find an explicit solution. However, it may even be preferrable to not solve for y and capture the solution in an **implicit** form. Let me provide a couple examples before I prove the method at the end of this section.

Example 3.1. Problem: *Solve* $\frac{dy}{dx} = 2xy$.

Solution: *Separate variables to find* $\int \frac{dy}{y} = \int 2x\,dx$ *hence* $\ln|y| = x^2 + c$. *Exponentiate to obtain* $|y| = e^{x^2+c} = e^c e^{x^2}$. *The constant* $e^c \neq 0$ *however, the absolute value allows for either* $\pm$. *Moreover, we can also observe directly that* $y = 0$ *solves the problem. We find* $\boxed{y = ke^{x^2}}$ *is the general solution to the problem.*

An explicit solution of the differential equation is like an antiderivative of a given integrand. The general solution is like the indefinite integral of a given integrand. The general solution and the indefinite integral are not functions, instead, they are a family of functions of which each is an explicit solution or an antiderivative. Notice that for the problem of indefinite integration the constant can always just be thoughtlessly tacked on at the end and that will nicely index over all the possible antiderivatives. On the other hand, for a differential equation the constant could appear in many other ways.

Example 3.2. Problem: *Solve* $\frac{dy}{dx} = \frac{-2x}{2y}$.

Solution: *separate variables and find* $\int 2y\,dy = -\int 2x\,dx$ *hence* $y^2 = -x^2 + c$. *We find* $x^2 + y^2 = c$. *It is clear that* $c < 0$ *give no interesting solutions. Therefore, without loss of generality, we assume* $c \geq 0$ *and denote* $c = R^2$ *where* $R \geq 0$. *Altogether we find* $\boxed{x^2 + y^2 = R^2}$ *is the general* **implicit** *solution to the problem. To find an explicit solution we need to focus our efforts, there are two cases:*

1. *if* (a, b) *is a point on the solution and* $b > 0$ *then* $y = \sqrt{a^2 + b^2 - x^2}$.
2. *if* (a, b) *is a point on the solution and* $b < 0$ *then* $y = -\sqrt{a^2 + b^2 - x^2}$.

Notice here the constant appeared inside the square-root. I find the implicit formulation of the solution the most natural for the example above, it is obvious we have circles of radius R. To capture a single circle we need two function graphs. Generally, given an implicit solution we can solve for an explicit solution locally. The implicit function theorems of advanced calculus give explicit conditions on when this is possible.

Example 3.3. Problem: *Solve* $\frac{dy}{dx} = e^{x - 2\ln|y|}$.

Solution: *recall* $e^{x - \ln|y|^2} = e^x e^{\ln|y|^2} = e^x |y|^2 = e^x y^2$. *Separate variables in view of this algebra:*

$$\frac{dy}{y^2} = e^x\,dx \quad \Rightarrow \quad \frac{-1}{y} = e^x + C \quad \Rightarrow \quad \boxed{y = \frac{-1}{e^x + C}.}$$

When I began this section I mentioned the justification was *formal*. I meant that to indicate the calculation seems plausible, but it is not justified. We now show that the method is in fact justified. In short, I show that the notation works.

Proposition 3.4. *separation of variables:*

The differential equation $\frac{dy}{dx} = g(x)h(y)$ has an implicit solution given by

$$\int \frac{dy}{h(y)} = \int g(x)\,dx$$

for (x, y) such that $h(y) \neq 0$.

Proof: to say the integrals above are an implicit solution to $\frac{dy}{dx} = g(x)h(y)$ means that the differential equation is a differential consequence of the integral equation. In other words, if we differentiate the integral equation we should hope to recover the given DEqn. Let's see how this happens, differentiate implicitly,

$$\frac{d}{dx}\int \frac{dy}{h(y)} = \frac{d}{dx}\int g(x)\,dx \quad \Rightarrow \quad \frac{1}{h(y)}\frac{dy}{dx} = g(x) \quad \Rightarrow \quad \frac{dy}{dx} = h(y)g(x).$$

□

Remark 3.5.

Technically, there is a gap in the proof above. How did I know implicit differentiation was possible? Is it clear that the integral equation defines y as a function of x at least locally? We could use the implicit function theorem on the level curve $F(x, y) = \int \frac{dy}{h(y)} - \int g(x)\,dx = 0$. Observe that $\frac{\partial F}{\partial y} = \frac{1}{h(y)} \neq 0$ hence the implicit function theorem provides the existence of a function ϕ which has $F(x, \phi(x)) = 0$ at points near the given point with $h(y) \neq 0$. This comment comes to you from the advanced calculus course.

4 Integrating Factor Method

Let p and q be continuous functions. The following differential equation is called a **linear differential equation** in standard form:

$$\boxed{\frac{dy}{dx} + py = q \qquad (\star)}$$

Our goal in this section is to solve equations of this type. Fortunately, linear differential equations are very nice and the solution exists and is not too hard to find in general, well, at least up-to a few integrations.

Notice, we cannot directly separate variables because of the py term. A natural thing to notice is that it sort of looks like a product, maybe if we multiplied by some new function I then we could separate and integrate: multiply $\star$ by I,

$$I\frac{dy}{dx} + pIy = qI$$

Now, if we choose I such that $\frac{dI}{dx} = pI$ then the equation above separates by the product rule:

$$\frac{dI}{dx} = pI \quad \Rightarrow \quad I\frac{dy}{dx} + \frac{dI}{dx}y = qI \quad \Rightarrow \quad \frac{d}{dx}\big[Iy\big] = qI \quad \Rightarrow \quad Iy = \int qI\,dx \quad \Rightarrow \quad \boxed{y = \frac{1}{I}\int qI\,dx.}$$

Very well, but, is it possible to find such a function I? Can we solve $\frac{dI}{dx} = pI$? Yes. Separate variables,

$$\frac{dI}{dx} = pI \quad \Rightarrow \quad \frac{dI}{I} = p\,dx \quad \Rightarrow \quad \ln(I) = \int p\,dx \quad \Rightarrow \quad \boxed{I = e^{\int p\,dx}.}$$

Proposition 4.1. *integrating factor method:*

Suppose p, q are continuous functions which define the linear differential equation $\frac{dy}{dx} + py = q$ (label this $\star$). We can solve $\star$ by the following algorithm:

(1.) define $I = exp(\int p\,dx)$,

(2.) multiply $\star$ by I,

(3.) apply the product rule to write $I\star$ as $\frac{d}{dx}\big[Iy\big] = Iq$.

(4.) integrate both sides,

(5.) find general solution $y = \frac{1}{I}\int Iq\,dx$.

Proof: Define $I = e^{\int p\,dx}$, note that p is continuous thus the antiderivative of p exists by the FTC. Calculate,

$$\frac{dI}{dx} = \frac{d}{dx}e^{\int p\,dx} = e^{\int p\,dx}\frac{d}{dx}\int p\,dx = pe^{\int p\,dx} = pI.$$

Multiply $\star$ by I, use calculation above, and apply the product rule:

$$I\frac{dy}{dx} + Ipy = Iq \quad \Rightarrow \quad I\frac{dy}{dx} + \frac{dI}{dx}y = Iq \quad \Rightarrow \quad \frac{d}{dx}\big[Iy\big] = Iq.$$

Integrate both sides,

$$\int \frac{d}{dx}\big[Iy\big]\,dx = \int Iq\,dx \quad \Rightarrow \quad Iy = \int Iq\,dx \quad \Rightarrow \quad y = \frac{1}{I}\int Iq\,dx. \qquad \square$$

The integration in $y = \frac{1}{I}\int Iq\,dx$ is indefinite. It follows that we could write $y = \frac{C}{I} + \frac{1}{I}\int Iq\,dx$. Note once more that the constant is not simply added to the solution.

Example 4.2. Problem: *find the general solution of* $\frac{dy}{dx} + \frac{2}{x}y = 3$

Solution: *Identify that* $p = 2/x$ *for this linear DE. Calculate, for* $x \neq 0$,

$$I = exp\bigg(\int \frac{2dx}{x}\bigg) = exp(2\ln|x|) = exp(\ln|x|^2) = |x|^2 = x^2$$

Multiply the DEqn by $I = x^2$ *and then apply the reverse product rule;*

$$x^2\frac{dy}{dx} + 2xy = 3x^2 \quad \Rightarrow \quad \frac{d}{dx}\big[\,x^2y\,\big] = 3x^2$$

Integrate both sides to obtain $x^2y = x^3 + c$ *therefore* $\boxed{y = x + c/x^2}$.

We could also write $y(x) = x + c/x^2$ to emphasize that we have determined y as a function of x.

Example 4.3. Problem: *let* r *be a real constant and suppose* g *is a continuous function, find the general solution of* $\frac{dy}{dt} - ry = g$

Solution: *Identify that* $p = r$ *for this linear DE with independent variable* t. *Calculate,*

$$I = exp\bigg(\int r\,dt\bigg) = e^{rt}$$

Multiply the DEqn by $I = e^{rt}$ *and then apply the reverse product rule;*

$$e^{rt}\frac{dy}{dt} + re^{rt}y = ge^{rt} \quad \Rightarrow \quad \frac{d}{dt}\big[\,e^{rt}y\,\big] = ge^{rt}$$

Integrate both sides to obtain $e^{rt}y = \int g(t)e^{rt}\,dt + c$ *therefore* $\boxed{y(t) = ce^{-rt} + e^{-rt}\int g(t)e^{rt}\,dt}$. *Now that we worked this in general it's fun to look at a few special cases:*

1. *if* $g = 0$ *then* $\boxed{y(t) = ce^{-rt}}$.
2. *if* $g(t) = e^{-rt}$ *then* $y(t) = ce^{-rt} + e^{-rt}\int e^{-rt}e^{rt}\,dt$ *hence* $\boxed{y(t) = ce^{-rt} + te^{-rt}}$.
3. *if* $r \neq s$ *and* $g(t) = e^{-st}$ *then* $y(t) = ce^{-rt} + e^{-rt}\int e^{-st}e^{rt}\,dt = ce^{-rt} + e^{-rt}\int e^{(r-s)t}\,dt$ *consqeuently we find that* $y(t) = ce^{-rt} + \frac{1}{r-s}e^{-rt}e^{(r-s)t}$ *and thus* $\boxed{y(t) = ce^{-rt} + \frac{1}{r-s}e^{-st}}$.

5 Substitutions

In this section we discuss a few common substitutions. The idea of substitution is simply to transform a given problem to one we already know how to solve. I'll illustrate via example.

Example 5.1. Problem: *solve* $\frac{dy}{dx} = (x+y-6)^2$. *(call this* $\star$*)*

Solution: *the substitution* $v = x+y-6$ *looks promising. We obtain* $y = v-x+6$ *hence* $\frac{dy}{dx} = \frac{dv}{dx} - 1$ *thus the DEQn* $\star$ *transforms to*

$$\frac{dv}{dx} - 1 = v^2 \quad \Rightarrow \quad \frac{dv}{dx} = v^2 + 1 \quad \Rightarrow \quad \frac{dv}{1+v^2} = dx \quad \Rightarrow \quad \tan^{-1}(v) = x + C$$

Hence, $\tan^{-1}(x+y-6) = x + C$ *is the general, implicit, solution to* $\star$*. In this case we can solve for* y *to find the explicit solution* $y = 6 + \tan(x+C) - x$.

Remark 5.2.

Generally the example above gives us hope that a DEqn of the form $\frac{dy}{dx} = F(ax+by+c)$ is solved through the substitution $v = ax+by+c$.

Example 5.3. Problem: *solve* $\frac{dy}{dx} = \frac{y/x+1}{y/x-1}$. *(call this* $\star$*)*

Solution: *the substitution* $v = y/x$ *looks promising. Note that* $y = xv$ *hence* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *by the product rule. We find* $\star$ *transforms to:*

$$v + x\frac{dv}{dx} = \frac{v+1}{v-1} \quad \Rightarrow \quad x\frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{v+1-v(v-1)}{v-1} = \frac{-v^2+2v+1}{v-1}$$

Hence, separating variables,

$$\frac{(v-1)\,dv}{-v^2+2v+1} = \frac{dx}{x} \quad \Rightarrow \quad -\frac{1}{2}\ln|v^2-2v-1| = \ln|x| + \tilde{C}$$

Thus, $\ln|v^2-2v-1| = \ln(1/x^2) + C$ *and after exponentiation and multiplication by* x^2 *we find the implicit solution* $\boxed{y^2 - 2xy - x^2 = K.}$

Remark 5.4.

Generally the example above gives us hope that a DEqn of the form $\frac{dy}{dx} = F(y/x)$ is solved through the substitution $v = y/x$.

Example 5.5. Problem: *Solve* $y' + xy = xy^3$. *(call this* $\star$*)*

Solution: *multiply by* y^{-3} *to obtain* $y^{-3}y' + xy^{-2} = x$*. Let* $z = y^{-2}$ *and observe* $z' = -2y^{-3}y'$ *thus* $y^{-3}y' = \frac{-1}{2}z'$*. It follows that:*

$$\frac{-1}{2}\frac{dz}{dx} + xz = x \quad \Rightarrow \quad \frac{dz}{dx} - 2xz = -2x$$

Identify this is a linear ODE and calculate the integrating factor is e^{-x^2} *hence*

$$e^{-x^2}\frac{dz}{dx} - 2xe^{-x^2}z = -2xe^{-x^2} \quad \Rightarrow \quad d(e^{-x^2}z) = -2xe^{-x^2}dx$$

Conquently, $e^{-x^2}z = e^{-x^2} + C$ *which gives* $z = y^{-2} = 1 + Ce^{x^2}$*. Finally, solve for* y

$$\boxed{y = \frac{\pm 1}{\sqrt{1 + Ce^{x^2}}}.}$$

Given an initial condition we would need to select either + or − as appropriate.

Remark 5.6.

This type of differential equation actually has a name; a differential equation of the type $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is called a **Bernoulli DEqn**. The procedure to solve such problems is as follows:

1. multiply $\frac{dy}{dx} + P(x)y = Q(x)y^n$ by y^{-n} to obtain $y^{-n}\frac{dy}{dx} + P(x)y^{-n+1} = Q(x)$,
2. make the substitution $z = y^{-n+1}$ and observe $z' = (1-n)y^{-n}y'$ hence $y^{-n}y' = \frac{1}{1-n}z'$,
3. solve the linear ODE in z; $\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$,
4. replace z with y^{-n+1} and solve if worthile for y.

Sometimes a second-order differential equation is easily reduced to a first-order problem. The examples below illustrate a technique called **reduction of order**.

Example 5.7. Problem: *solve* $y'' + y' = x^2$*. (call this* $\star$*)*

Solution: *Let* $y' = v$ *and observe* $y'' = v'$ *hence* $\star$ *transforms to*

$$\frac{dv}{dx} - v = e^{-x}$$

multiply the DEqn above by the integrating factor e^x*:*

$$e^x\frac{dv}{dx} - ve^x = 1 \quad \Rightarrow \quad \frac{d}{dx}\Big[\, e^x v \,\Big] = 1$$

thus $e^x v = x + c_1$ *and we find* $v = xe^{-x} + c_1e^{-x}$*. Then as* $v = \frac{dy}{dx}$ *we can integrate once more to find the solution:*

$$y = \int \big[xe^{-x} + c_1e^{-x}\big]dx = -xe^{-x} - e^{-x} - c_1e^{-x} + c_2$$

cleaning it up a bit,

$$\boxed{y = -e^{-x}(x - 1 + c_1) + c_2.}$$

Remark 5.8.

Generally, given a differential equation of the form $y'' = F(y', x)$ we can solve it by a two-step process:

1. substitute $v = y'$ to obtain the first-order problem $v' = F(v, x)$. Solve for v.
2. recall $v = y'$, integrate to find y.

There will be two constants of integration. This is a typical feature of second-order ODE.

Example 5.9. Problem: *solve* $\frac{d^2y}{dt^2} + y = 0$. *(call this* $\star$*)*

Solution: *once more let* $v = \frac{dy}{dt}$. *Notice that*

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dy}{dt}\frac{dv}{dy} = v\frac{dv}{dy}$$

thus $\star$ *transforms to the first-order problem:*

$$v\frac{dv}{dy} + y = 0 \quad \Rightarrow \quad vdv + ydy = 0 \quad \Rightarrow \quad \frac{1}{2}v^2 + \frac{1}{2}y^2 = \frac{1}{2}C^2.$$

assume the constant $C > 0$, *note nothing is lost in doing this except the point solution* $y = 0, v = 0$. *Solving for* v *we obtain* $v = \pm\sqrt{C - y^2}$. *However,* $v = \frac{dy}{dt}$ *so we find:*

$$\frac{dy}{\sqrt{C^2 - y^2}} = \pm dt \quad \Rightarrow \quad \sin^{-1}(y/C) = \pm t + \phi$$

Thus, $y = C\sin(\pm t + \phi)$. *We can just as well write* $y = A\sin(t + \phi)$. *Moreover, by trigonometry, this is the same as* $y = B\cos(t + \gamma)$, *it's just a matter of relabeling the constants in the general solution.*

Remark 5.10.

Generally, given a differential equation of the form $y'' = F(y)$ we can solve it by a two-step process:

1. substitute $v = y'$ and use the identity $\frac{dv}{dt} = v\frac{dv}{dy}$ to obtain the first-order problem $v\frac{dv}{dy} = F(y)$. Solve for v.
2. recall $v = y'$, integrate to find y.

There may be several cases possible as we solve for v, but in the end there will be two constants of integration.

6 Physics and Applications

I've broken this section into two parts. The initial subsection examines how we can use differential-equations techniques to better understand Newton's Laws and energy in classical mechanics. This sort of discussion is found in many of the older classic texts on differential equations. The second portion of this section is a collection of isolated application examples which are focused on a particular problems from a variety of fields.

6.1 Physics

In physics we learn that $\vec{F}_{net} = m\vec{a}$ or, in terms of momentum $\vec{F}_{net} = \frac{d\vec{p}}{dt}$. We consider the one-dimensional problem hence we have no need of the vector notation and we generally are faced with the problem:

$$F_{net} = m\frac{dv}{dt} \qquad \text{or} \qquad F_{net} = \frac{dp}{dt}$$

where the momentum p for a body with mass m is given by $p = mv$ where v is the velocity as defined by $v = \frac{dx}{dt}$. The acceleration a is defined by $a = \frac{dv}{dt}$. It is also customary to use the dot and double dot notation for problems of classical mechanics. In particular: $v = \dot{x}$, $a = \dot{v} = \ddot{x}$. Generally the net-force can be a function of position, velocity and time; $F_{net} = F(x, v, t)$. For example,

1. the spring force is given by $F = -kx$
2. the force of gravity near the surface of the earth is given by $F = \pm mg$ ($\pm$ depends on interpretation of x)
3. force of gravity distance x from center of mass M given by $F = -\frac{GmM}{x^2}$
4. thrust force on a rocket depends on speed and rate at which mass is ejected
5. friction forces which depend on velocity $F = \pm bv^n$ ($\pm$ needed to insure friction force is opposite the direction of motion)
6. an external force, could be sinusoidal $F = A\cos(\omega t)$, ...

Suppose that the force only depends on x; $F = F(x)$ consider Newton's Second Law:

$$m\frac{dv}{dt} = F(x)$$

Notice that we can use the identity $\frac{dv}{dt} = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$ hence

$$mv\frac{dv}{dx} = F(x) \quad \Rightarrow \quad \int_{v_o}^{v_f} mv\,dv = \int_{x_o}^{x_f} F(x)\,dx \quad \Rightarrow \quad \boxed{\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_o^2 = \int_{x_o}^{x_f} F(x)\,dx.}$$

The equation boxed above is the **work-energy theorem**, it says the change in the kinetic energy $K = \frac{1}{2}mv^2$ is given by $\int_{x_o}^{x_f} F(x)\,dx$. which is the **work** done by the force F. This result holds for any net-force, however, in the case of a conservative force we have $F = -\frac{dU}{dx}$ for the **potential energy** function U hence the work done by F simplifies nicely

$$\int_{x_o}^{x_f} F(x)\,dx = -\int_{x_o}^{x_f} \frac{dU}{dx}\,dx = -U(x_f) + U(x_o)$$

and we obtain the **conservation of total mechanical energy** $\frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2 = -U(x_f) + U(x_o)$ which is better written in terms of energy $E(x,v) = \frac{1}{2}mv^2 + U(x)$ as $E(x_o, v_o) = E(x_f, v_f)$. The total energy of a conservative system is constant. We can also see this by a direct-argument on the differential equation below:

$$m\frac{dv}{dt} = -\frac{dU}{dx} \quad \Rightarrow \quad m\frac{dv}{dt} + \frac{dU}{dx} = 0$$

multiply by $\frac{dx}{dt}$ and use the identity $\frac{d}{dt}\left[\frac{1}{2}v^2\right] = v\frac{dv}{dt}$:

$$m\frac{dx}{dt}\frac{dv}{dt} + \frac{dx}{dt}\frac{dU}{dx} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2\right] + \frac{dU}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2 + U\right] = 0 \quad \Rightarrow \quad \boxed{\frac{dE}{dt} = 0.}$$

Once more we have derived that the energy is constant for a system with a net-force which is conservative. Note that as time evolves the expression $E(x,v) = \frac{1}{2}mv^2 + U(x)$ is invariant. It follows that the motion of the system is in described by an **energy-level** curve in the xv-plane. This plane is commonly called the **phase plane** in physics literature. Much information can be gleaned about the possible motions of a system by studying the energy level curves in the phase plane. I'll return to that topic later in the course.

We now turn to a mass m for which the net-force is of the form $F(x,v) = -\frac{dU}{dx} \mp b|v|^n$. Here we insist that $-$ is given for $v > 0$ whereas the $+$ is given for the case $v < 0$ since we assume $b > 0$ and this friction force ought to point opposite the direction of motion. Once more consider Newton's Second Law:

$$m\frac{dv}{dt} = -\frac{dU}{dx} \mp bv^n \quad \Rightarrow \quad m\frac{dv}{dt} - \frac{dU}{dx} = \mp b|v|^n$$

multiply by the velocity and use the identity as we did in the conservative case:

$$m\frac{dx}{dt}\frac{dv}{dt} - \frac{dx}{dt}\frac{dU}{dx} = \mp bv|v|^n \quad \Rightarrow \quad \frac{d}{dt}\left[\frac{1}{2}mv^2 + U\right] = \mp bv|v|^n \quad \Rightarrow \quad \boxed{\frac{dE}{dt} = \mp bv|v|^n.}$$

The friction force reduces the energy. For example, if $n = 1$ then we have $\frac{dE}{dt} = -bv^2$.

Remark 6.1.

> The concept of energy is implicit within Example 5.9. I should also mention that the trick of multiplying by the velocity to reveal a conservation law is used again and again in the junior-level classical mechanics course.

6.2 Applications

Example 6.2. Problem: *Suppose x is the position of a mass undergoing one-dimensional, constant acceleration motion. You are given that initially we have velocity v_o at position x_o and later we have velocity v_f at position x_f. Find how the initial and final velocities and positions are related.*

Solution: *recall that $a = \frac{dv}{dt}$ but, by the chain-rule we can write $a = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx}$. We are given that a is a constant. Separate variables, and integrate with respect to the given data*

$$a = \frac{dx}{dt}\frac{dv}{dx} = v\frac{dv}{dx} \quad \Rightarrow \quad a\,dx = v\,dv \quad \Rightarrow \quad \int_{x_o}^{x_f} a\,dx = \int_{v_o}^{v_f} v\,dv \quad \Rightarrow \quad a(x_f - x_o) = \frac{1}{2}\left(v_f^2 - v_o^2\right).$$

Therefore, $\boxed{v_f^2 = v_o^2 + 2a(x_f - x_o)}$. *I hope you recognize this equation from physics.*

Example 6.3. Problem: *suppose the population P grows at a rate which is directly proportional to the population. Let k be the proportionality constant. Find the population at time t in terms of the initial population P_o.*

Solution: *the given problem translates into the differential equation $\frac{dP}{dt} = kP$ with $P(0) = P_o$. Seperate variables and integrate, note $P > 0$ so I drop the absolute value bars in the integral,*

$$\frac{dP}{dt} = kP \quad \Rightarrow \quad \int \frac{dP}{P} = \int k\,dt \quad \Rightarrow \quad \ln(P(t)) = kt + C$$

Apply the initial condition; $\ln(P(0)) = k(0) + C$ hence $C = \ln(P_o)$. Consequently $\ln(P(t)) = \ln(P_o) + kt$. Exponentiate to derive $\boxed{P(t) = P_o e^{kt}}$.

In the example above I have in mind $k > 0$, but if we allow $k < 0$ that models exponential population decline. Or, if we think of P as the number of radioactive particles then the same mathematics for $k < 0$ models radioactive decay.

Example 6.4. Problem: *the voltage dropped across a resistor R is given by the product of R and the current I through R. The voltage dropped across a capacitor C depends on the charge Q according to $C = Q/V$ (this is actually the definition of capacitance). If we connect R and C end-to-end making a loop then they are in parallel hence share the same voltage: $IR = \frac{Q}{C}$. As time goes on the charge on C flows off the capacitor and through the resistor. It follows that $I = -\frac{dQ}{dt}$. If the capacitor initially has charge Q_o then find $Q(t)$ and $I(t)$ for the* **discharging capacitor**

Solution: *We must solve*

$$-R\frac{dQ}{dt} = \frac{Q}{C}$$

Separate variables, integrate, apply $Q(0) = Q_o$:

$$\frac{dQ}{Q} = -\frac{dt}{RC} \quad \Rightarrow \quad \ln|Q| = -\frac{t}{RC} + c_1 \quad \Rightarrow \quad Q(t) = \pm e_1^c e^{-t/RC} \quad \Rightarrow \quad \boxed{Q(t) = Q_o e^{-t/RC}}$$

Another application of first order differential equations is simply to search for curves with particular properties. The next example illustrates that concept.

Example 6.5. Problem: *find a family of curves which are increasing whenever $y < -2$ or $y > 2$ and are decreasing whenever $-2 < y < 2$.*

Solution: *while many examples exist, the simplest example is one for which the derivative is quadratic in y. Think about the quadratic $(y+2)(y-2)$. This expression is positive for $|y| > 2$ and negative for $|y| < 2$. It follows that solutions to the differential equation $\frac{dy}{dx} = (y+2)(y-2)$ will have the desired properties. Note that $y = \pm 2$ are exceptional solutions for the give DEqn. Proceed* *by separation of variables, recall the technique of partial fractions,*

$$\frac{dy}{(y+2)(y-2)} = dx \quad \Rightarrow \quad \int \left[\frac{1}{4(y-2)} - \frac{1}{4(y+2)}\right] dy = \int dx \quad \star$$

$$\Rightarrow \quad \ln|y-2| - \ln|y+2| = 4x + C$$

$$\Rightarrow \quad \ln\left|\frac{y-2}{y+2}\right| = 4x + C$$

$$\Rightarrow \quad \ln\left|\frac{y+2}{y+2} - \frac{4}{y+2}\right| = 4x + C$$

$$\Rightarrow \quad \ln\left|1 - \frac{4}{y+2}\right| = 4x + C$$

$$\Rightarrow \quad \left|1 - \frac{4}{y+2}\right| = e^{4x+C} = e^C e^{4x}$$

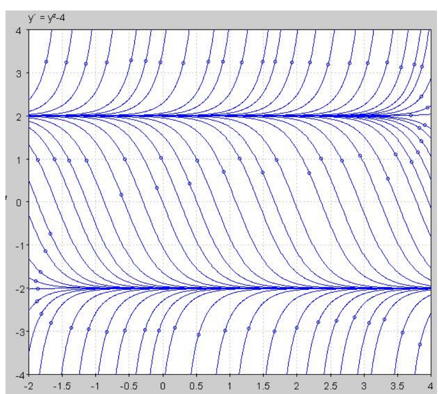
$$\Rightarrow \quad 1 - \frac{4}{y+2} = \pm e^C e^{4x} = Ke^{4x}$$

$$\Rightarrow \quad \frac{1}{y+2} = \frac{1-Ke^{4x}}{4}$$

$$\Rightarrow \quad \boxed{y = -2 + \frac{4}{1-Ke^{4x}}, \quad \textit{for } K \neq 0.}$$

It is neat that $K = 0$ returns the exceptional solution $y = 2$ whereas the other exceptional solution is lost since we have division by $y + 2$ in the calculation above. If we had multiplied $\star$ by -1 then the tables would turn and we would recover $y = -2$ in the general formula.

The plot of the solutions below was prepared with pplane which is a feature of Matlab as I discussed earlier in this article.



If you study the solutions in the previous example you'll find that all solutions tend to either $y = 2$ or $y = -2$ in some limit. You can also show that all the solutions which cross the x-axis have inflection points at their x-intercept. We can derive that from the differential equation directly:

$$\frac{dy}{dx} = (y+2)(y-2) = y^2 - 4 \quad \Rightarrow \quad \frac{d^2y}{dx^2} = 2y\frac{dy}{dx} = 2y(y+2)(y-2).$$

We can easily reason when solutions have $y > 2$ or $-2 < y < 0$ they are concave up whereas solutions with $0 < y < 2$ or $y < -2$ are concave down. It follows that a solution crossing $y = 0, -2$ or 2 is at a point of inflection. Careful study of the solutions show that solutions do not cross $y = -2$ or $y = 2$ thus only $y = 0$ has solutions with genuine points of inflection.

Example 6.6. Problem: *suppose you are given a family S of curves which satisfy $\frac{dy}{dx} = f(x,y)$. Find a differential equation for a family of curves which are orthogonal to the given set of curves. In other words, find a differential equation whose solution consists of curves $S^{\perp}$ whose tangent vectors are perpendicular to the tangent vectors of curves in S at points of intersection.*

Solution: *Consider a point (x_o, y_o), note that the solution to $\frac{dy}{dx} = f(x,y)$ has slope $f(x_o, y_o)$ at that point. The perpendicular to the tangent has slope $-1/f(x_o, y_o)$. Thus, we should use the differential equation $\frac{dy}{dx} = -\frac{1}{f(x,y)}$ to obtain orthogonal trajectories.*

Let me give a concrete example of orthogonal trajectories:

Example 6.7. Problem: *find orthogonal trajectories of $xdx + ydy = 0$.*

Solution: *we find $\frac{dy}{dx} = \frac{-x}{y}$ hence the orthogonal trajectories are found in the solution set of $\frac{dy}{dx} = \frac{-y}{x}$. Separate variables to obtain:*

$$\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \ln|y| = \ln|x| + C \quad \Rightarrow \quad y = \pm e^C x.$$

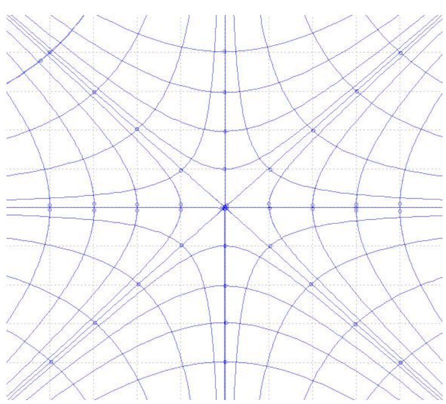
In other words, the orthogonal trajectories are lines through the origin $y = kx$. Technically, by our derivation, we ought not allow $k = 0$ but when you understand the solutions of $xdx + ydy = 0$ are simply circles $x^2 + y^2 = R^2$ it is clear that $y = 0$ is indeed an orthogonal trajectory.

Example 6.8. Problem: *find orthogonal trajectories of $x^2 - y^2 = 1$.*

Solution: *observe that the hyperbola above is a solution of the differential equation $2x - 2y\frac{dy}{dx} = 0$ hence $\frac{dy}{dx} = \frac{x}{y}$. Orthogonal trajectories are found from $\frac{dy}{dx} = \frac{-y}{x}$. Separate variables,*

$$\frac{dy}{y} = \frac{-dx}{x} \quad \Rightarrow \quad \ln|y| = -\ln|x| + C \quad \Rightarrow \quad y = k/x.$$

Once more, the case $k = 0$ is exceptional, but it is clear that $y = 0$ is an orthogonal trajectory of the given hyperbola.



Orthogonal trajectories are important to the theory of electrostatics. The field lines which are *integral curves* of the electric field form orthogonal trajectories to the *equpotential* curves. Or, in the study of heatflow, the isothermal curves are orthgonal to the curves which line-up with the flow of heat.

Example 6.9. Problem: *Suppose the force of friction on a speeding car is given by $F_f = -bv^2$. If the car has mass m and initial speed v_o and position x_o then find the velocity and position as a function of t as the car glides to a stop. Assume that the net-force is the friction force since the normal force and gravity cancel.*

Solution: *by Newton's second law we have $m\frac{dv}{dt} = -bv^2$. Separate variables, integrate. apply initital condition,*

$$\frac{dv}{v^2} = -\frac{bdt}{m} \quad \Rightarrow \quad \frac{-1}{v} = \frac{-bt}{m} + c_1 \quad \Rightarrow \quad \frac{-1}{v_o} = \frac{-b(0)}{m} + c_1 \quad \Rightarrow \quad c_1 = \frac{-1}{v_o}$$

Thus,

$$\frac{1}{v(t)} = \frac{bt}{m} + \frac{1}{v_o} \quad \Rightarrow \quad v(t) = \frac{1}{\frac{bt}{m} + \frac{1}{v_o}} \quad \Rightarrow \quad \boxed{v(t) = \frac{v_o}{\frac{btv_o}{m} + 1}}.$$

Since $v = \frac{dx}{dt}$ we can integrate the velocity to find the position

$$x(t) = c_1 + \frac{m}{b}\ln\left|1 + \frac{bv_ot}{m}\right| \quad \Rightarrow \quad x(0) = c_1 + \ln(1) = x_o \quad \Rightarrow \quad \boxed{x(t) = x_o + \frac{m}{b}\ln\left|1 + \frac{bv_ot}{m}\right|}.$$

Notice the slightly counter-intuitive nature of this solution, the position is unbounded even though the velocity tends to zero. Common sense might tell you that if the car slows to zero for large time then the total distance covered must be finite. Well, common sense fails, math wins. The point is that the velocity actually goes too zero too slowly to give bounded motion.

Example 6.10. Problem: *Newton's Law of Cooling states that the change in temperature T for an object is proportional to the difference between the ambient temperature R and T; in particular: $\frac{dT}{dt} = -k(T-R)$ for some constant k and R is the room-temperature. Suppose that $T(0) = 150$ and $T(1) = 120$ if $R = 70$, find $T(t)$*

Solution: *To begin let us examine the differential equation for arbitrary k and R,*

$$\frac{dT}{dt} = -k(T-R) \quad \Rightarrow \quad \frac{dT}{dt} + kT = kR$$

Identify that $p = k$ hence $I = e^{kt}$ and we find

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = ke^{kt}R \quad \Rightarrow \quad \frac{d}{dt}\big[\, e^{kt}T \,\big] = ke^{kt}R \quad \Rightarrow \quad e^{kt}T = Re^{kt} + C \quad \Rightarrow \quad \boxed{T(t) = R + Ce^{-kt}.}$$

Now we may apply the given data to find both C and k, we already know $R = 70$ from the problem statement;

$$T(0) = 70 + C = 150 \qquad \& \qquad T(1) = 70 + Ce^{-k} = 120$$

Hence $C = 80$ which implies $e^{-k} = 5/8$ thus $e^k = 8/5$ and $k = \ln(8/5)$. Therefore, $\boxed{T(t) = 70 + 80e^{t\ln(5/8)}}$. *To understand this solution note that $\ln(5/8) < 0$ hence the term $80e^{t\ln(5/8)} \to 0$ as $t \to \infty$ hence $T(t) \to 70$ as $t \to \infty$. After a long time, Newton's Law of Cooling predicts objects will assume room temperature.*

Example 6.11. *Suppose you decide to have coffee with a friend and you both get your coffee ten minutes before the end of a serious presentation by your petty boss who will be offended if you start drinking during his fascinating talk on maximal efficiencies for production of widgets. You both desire to drink your coffee with the same amount of cream and you both like the coffee as hot as possible. Your friend puts the creamer in immediately and waits quitely for the talk to end. You on the other hand think you wait to put the cream in at the end of talk. Who has hotter coffee and why?* **Discuss.**

Example 6.12. Problem: *the voltage dropped across a resistor R is given by the product of R and the current I through R. The voltage dropped across an inductor L depends on the change in the current according to $L\frac{dI}{dt}$. An inductor resists a change in current whereas a resistor just resists current. If we connect R and L in series with a voltage source $\mathcal{E}$ then the Kirchoff's voltage law yields the differential equation*

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0$$

Given that $I(0) = I_o$ find $I(t)$ for the circuit.

Solution: *Identify that this is a linear DE with independent variable t,*

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{\mathcal{E}}{L}$$

The integrating factor is simply $\mu = e^{\frac{Rt}{L}}$ (using I here would be a poor notation). Multiplying the DEqn above by μ to obtain,

$$e^{\frac{Rt}{L}}\frac{dI}{dt} + \frac{R}{L}e^{\frac{Rt}{L}}I = \frac{\mathcal{E}}{L}e^{\frac{Rt}{L}} \quad \Rightarrow \quad \frac{d}{dt}\big[\, e^{\frac{Rt}{L}}I\,\big] = \frac{\mathcal{E}}{L}e^{\frac{Rt}{L}}$$

Introduce a dummy variable of integration τ and integrate from $\tau = 0$ to $\tau = t$,

$$\int_0^t \frac{d}{d\tau}\big[\, e^{\frac{R\tau}{L}}I\,\big]d\tau = \int_0^t \frac{\mathcal{E}}{L}e^{\frac{R\tau}{L}}\,d\tau \quad \Rightarrow \quad e^{\frac{Rt}{L}}I(t) - I_o = \int_0^t \frac{\mathcal{E}}{L}e^{\frac{R\tau}{L}}\,d\tau.$$

Therefore, $I(t) = I_oe^{\frac{-Rt}{L}} + e^{\frac{-Rt}{L}}\int_0^t \frac{\mathcal{E}}{L}e^{\frac{R\tau}{L}}\,d\tau$. If the voltage source is constant then $\mathcal{E}(t) = \mathcal{E}_o$ for all t and the solution yields to $I(t) = I_oe^{\frac{-Rt}{L}} + e^{\frac{-Rt}{L}}\frac{\mathcal{E}_o}{L}\frac{L}{R}\big(e^{\frac{Rt}{L}} - 1\big)$ which simplifies to

$$\boxed{I(t) = \left[I_o - \frac{\mathcal{E}_o}{R}\right]e^{\frac{-Rt}{L}} + \frac{\mathcal{E}_o}{R}}.$$

The **steady-state** *current found from letting $t \to \infty$ where we find $I(t) \to \frac{\mathcal{E}_o}{R}$. After a long time it is approximately correct to say the inductor is just a short-circuit. What happens is that as the current changes in the inductor a magnetic field is built up. The magnetic field contains energy and the maximum energy that can be stored in the field is governed by the voltage source. So, after a while, the field is approximately maximal and all the voltage is dropped across the resistor. You could think of it like saving money in a piggy-bank which cannot fit more than $\mathcal{E}_o$ dollars. If every week you get an allowance then eventually you have no choice but to spend the money if the piggy-bank is full and there is no other way to save.*

Example 6.13. Problem: *Suppose a tank of salty water has* $15kg$ *of salt disolved in* $1000L$ *of water at time* $t = 0$*. Furthermore, assume pure water enters the tank at a rate of* $10L/min$ *and salty water drains out at a rate of* $10L/min$*. If* $y(t)$ *is the number of kg of salt at time t then find* $y(t)$ *for* $t > 0$*. Also, how much salt is left in the tank when* $t = 20$ *(minutes). We suppose that this tank is arranged such that the concentration of salt is constant throughout the liquid in this* **mixing tank.**

Solution: *Generally when we work such a problem we are interested in the rate of salt enetering the tank and the rate of salt exiting the tank ;* $\frac{dy}{dt} = R_{in} - R_{out}$*. However, this problem only has a nonzero out-rate:* $R_{out} = \frac{10\,L}{min}\frac{y}{1000L} = \frac{y}{100min}$*. We omit the "min" in the math below as we assume t is in minutes,*

$$\frac{dy}{dt} = -\frac{y}{100} \quad \Rightarrow \quad \frac{dy}{y} = -\frac{dt}{100} \quad \Rightarrow \quad \ln|y| = -\frac{t}{100} + C \quad \Rightarrow \quad y(t) = ke^{-\frac{t}{100}}.$$

However, we are given that $y(0) = 15$ *hence* $k = 15$ *and we find*[3]*:*

$$\boxed{y(t) = 15e^{-0.01t}.}$$

Evaluating at $t = 20min$ *yields* $y(20) = 12.28\,kg$.

Example 6.14. Problem: *Suppose a water tank has* $100L$ *of pure water at time* $t = 0$*. Suppose salty water with a concentration of* $1.5kg$ *of salt per L enters the tank at a rate of* $8L/min$ *and gets quickly mixed with the initially pure water. There is a drain in the tank where water drains out at a rate of* $6L/min$*. If* $y(t)$ *is the number of kg of salt at time t then find* $y(t)$ *for* $t > 0$*. If the water tank has a maximum capacity of* $1000L$ *then what are the physically reasonable values for the solution? For what t does your solution cease to be reasonable?*

Solution: *Generally when we work such a problem we are interested in the rate of salt enetering the tank and the rate of salt exiting the tank ;* $\frac{dy}{dt} = R_{in} - R_{out}$*. The input-rate is constant and is easily found from multiplying the given concentration by the flow-rate:*

$$R_{in} = \frac{1.5\,kg}{L}\frac{8\,L}{min} = \frac{12\,kg}{min}$$

notice how the units help us verify we are setting-up the model wisely. That said, I omit them in what follows to reduce clutter for the math. The output-rate is given by the product of the flow-rate $6L/min$ *and the salt-concentration* $y(t)/V(t)$ *where* $V(t)$ *is the volume of water in L at time t. Notice that the* $V(t)$ *is given by* $V(t) = 100 + 2t$ *for the given flow-rates, each minute the volume increases by* $2L$*. We find (in units of kg and min):*

$$R_{out} = \frac{6y}{100 + 2t}$$

Therefore, we must solve:

$$\frac{dy}{dt} = 12 - \frac{6y}{100 + 2t} \quad \Rightarrow \quad \frac{dy}{dt} + \frac{3dt}{50 + t}y = 12.$$

This is a linear ODE, we can solve it by the integrating factor method.

$$I(t) = exp\Big(\int \frac{3dt}{50 + t}\Big) = exp\Big(3\ln(50 + t)\Big) = (50 + t)^3.$$

[3] to be physically explicit, $y(t) = (15kg)exp(\frac{-0.01t}{min})$, but the units clutter the math here so we omit them

Multiplying by I yields:

$$(50+t)^3\frac{dy}{dt}+3(50+t)^2y=12(50+t)^3 \quad \Rightarrow \quad \frac{d}{dt}\Big[(50+t)^3y\Big]=12(50+t)^3$$

Integrating yields $(50+t)^3y(t)=3(50+t)^4+C$ *hence* $y(t)=3(50+t)+\frac{C}{(50+t)^3}$. *The water is initially pure thus* $y(0)=0$ *thus* $0=150+C/50^3$ *which gives* $C=-150(50)^3$. *The solution is*[4]

$$y(t)=3(50+t)-150\left(\frac{50}{50+t}\right)^3$$

Observe that $V(t)\leq 1000\,L$ *thus we need* $100+2t\leq 1000$ *which gives* $t\leq 450$. *The solution is only appropriate physically for* $0\leq t\leq 450$.

Example 6.15. Problem: *suppose the population P grows at a rate which is directly proportional to the population. Let k_1 be the proportionality constant for the growth rate. Suppose further that as the population grows the death-rate is proportional to the square of the population. Suppose k_2 is the proportionality constant for the death-rate. Find the population at time t in terms of the initial population P_o.*

Solution: *the given problem translates into the IVP of* $\frac{dP}{dt}=k_1P-k_2P^2$ *with* $P(0)=P_o$. *Observe that* $k_1P-k_2P^2=k_1P(1-k_2P/k_1)$. *Introduce* $C=k_1/k_2$). *Separate variables:*

$$\frac{dP}{P(1-P/C)}=k_1dt$$

Recall the technique of partial fractions:

$$\frac{1}{P(1-P/C)}=\frac{-C}{P(P-C)}=\frac{A}{P}+\frac{B}{P-C} \quad \Rightarrow \quad -C=A(P-C)+BP$$

Set $P=0$ *to obtain* $-C=-AC$ *hence* $A=1$ *and set* $P=C$ *to obtain* $-C=BC$ *hence* $B=-1$ *and we find:*

$$\int\left[\frac{1}{P}-\frac{1}{P-C}\right]dP=k_1dt \quad \Rightarrow \quad \ln|P|-\ln|P-C|=k_1t+c_1$$

It follows that letting $c_2=e^{c_1}$ *and* $c_3=\pm c_2$

$$\left|\frac{P}{P-C}\right|=c_2e^{k_1t} \quad \Rightarrow \quad P=(P-C)c_3e^{k_1t}$$

hence, $P[1-c_3e^{k_1t}]=-c_3Ce^{k_1t}$

$$P(t)=\frac{c_3Ce^{k_1t}}{c_3e^{k_1t}-1} \quad \Rightarrow \quad P(t)=\frac{C}{1-c_4e^{-k_1t}}$$

where I let $c_4=1/c_3$ *for convenience. Let us work on writing this general solution in-terms of the initial population* $P(0)=P_o$*:*

$$P_o=\frac{C}{1-c_4} \quad \Rightarrow \quad P_o(1-c_4)=C \quad \Rightarrow \quad P_o-C=P_oc_4 \quad \Rightarrow \quad c_4=\frac{P_o-C}{P_o}.$$

[4]following the formatting of Example 7 of § 2.7 of Rice & Strange's Ordinary Differential Equations with Applications

This yields,

$$P(t) = \frac{C}{1 - \frac{P_o - C}{P_o} e^{-k_1 t}} \quad \Rightarrow \quad \boxed{P(t) = C\left[\frac{P_o}{P_o - [P_o - C]e^{-k_1 t}}\right]}$$

The quantity C is called the **carrying capacity** *for the system. As we defined it here it is given by the quotient of the birth-rate and death-rate constants $C = k_1/k_2$. Notice that as $t \to \infty$ we find $P(t) \to C$. If $P_o > C$ then the population decreases towards C whereas if $P_o < C$ then the population increases towards C. If $P_o = C$ then we have a special solution where $\frac{dP}{dt} = 0$ for all t, the* **equilbrium solution***. A a bit of fun trivia, these models are notoriously incorrect for human populations. For example, in 1920 a paper by R. Pearl and L. J. Reed found $P(t) = \frac{210}{1+51.5e^{-0.03t}}$. The time t is the number of years past 1790 ($t = 60$ for 1850 for example). As discussed in Ritger and Rose page 85 this formula does quite well for 1950 where is well-approximates the population as 151 million. However, the carrying capacity of 210 million people is not even close to correct. Why? Because there are many factors which influence population which are simply not known. The same problem exists for economic models. You can't model game-changing events such as an interfering government. It doesn't flow from logic or optimal principles, political convenience whether it benefits or hurts a given market cannot be factored in over a long-term. Natural disasters also spoil our efforts to model populations and markets. That said, the exponential and logarthmic population models are important to a wide-swath of reasonably isolated populations which are free of chaotic events.*

Example 6.16. Problem: *Suppose a raindrop falls through a cloud and gathers water from the cloud as it drops towards the ground. Suppose the mass of the raindrop is m and the suppose the rate at which the mass increases is proportional to the mass; $\frac{dm}{dt} = km$ for some constant $k > 0$. Find the equation of the velocity for the drop.*

Solution: *Newton's equation is $-mg = \frac{dp}{dt}$. This follows from the assumption that, on average, there is no net-momentum of the water vapor which adheres to the raindrop thus the momentum change is all from the gravitational force. Since $p = mv$ the product rule gives:*

$$-mg = \frac{dm}{dt}v + m\frac{dv}{dt} \quad \Rightarrow \quad -mg = kmv + m\frac{dv}{dt}$$

Consequently, dividing by m and applying the integrating factor method gives:

$$\frac{dv}{dt} + kv = -g \quad \Rightarrow \quad e^{kt}\frac{dv}{dt} + ke^{kt}v = -ge^{kt} \quad \Rightarrow \quad \frac{d}{dt}\left[e^{kt}v\right] = -ge^{kt}$$

Integrate to obtain $e^{kt}v = \frac{-g}{k}e^{kt} + C$ from which it follows $\boxed{v(t) = -\frac{g}{k} + Ce^{-kt}.}$ Consider the limit $t \to \infty$, we find $v_\infty(t) = -\frac{g}{k}$. This is called the **terminal velocity***. Physically this is a very natural result; the velocity is constant when the forces balance. There are two forces at work here (1.) gravity $-mg$ and (2.) water friction $-kmv$ and we look at*

$$m\frac{dv}{dt} = -mg - kmv$$

If $v = -\frac{g}{k}$ then you obtain $ma = 0$. You might question if we should call the term $-kmv$ a "force". Is it really a force? In any event, you might note we can find the terminal velocity without solving the DEqn, we just have to look for an equilbrium of the forces.

Not all falling objects have a terminal velocity... well, at least if you believe the following example. To be honest, I'm not so sure it is very physical. I would be interested in your thoughts on the analysis if your thoughts happen to differ from my own.

Example 6.17. Problem: *Suppose a raindrop falls through a cloud and gathers water from the cloud as it drops towards the ground. Suppose the mass of the raindrop is* m *and the suppose the drop is spherical and the rate at which the mass adheres to the drop is proportional to the cross-sectional area relative the vertical drop (*$\frac{dm}{dt} = k\pi R^2$*). Find the equation of the velocity for the drop.*

Solution: *we should assume the water in the could is motionless hence the water collected from cloud does not impart momentum directly to the raindrop. It follows that Newton's Law is* $-mg = \frac{dp}{dt}$ *where the momentum is given by* $p = mv$ *and* $v = \dot{y}$ *and* y *is the distance from the ground. The mass* m *is a function of time. However, the density of water is constant at* $\rho = 1000 kg/m^3$ *hence we can relate the mass* m *to the volume* $V = \frac{4}{3}\pi R^3$ *we have*

$$\rho = \frac{4\pi R^3}{3m}$$

Solve for R^2*,*

$$R^2 = \left[\frac{3\rho m}{4\pi}\right]^{2/3}$$

As the drop falls the rate of water collected should be proportional to the cross-sectional area πR^2 *the drop presents to cloud. It follows that:*

$$\frac{dm}{dt} = km^{2/3}$$

Newton's Second Law for varying mass,

$$-mg = \frac{d}{dt}\big[mv\big] = \frac{dm}{dt}v + m\frac{dv}{dt} = km^{2/3}v + m\frac{dv}{dt}$$

This is a linear ODE in velocity,

$$\frac{dv}{dt} + \left(\frac{k}{m^{1/3}}\right)v = -g$$

We should find the mass as a function of time,

$$\frac{dm}{dt} = km^{2/3} \quad \Rightarrow \quad \frac{dm}{m^{2/3}} = kdt \quad \Rightarrow \quad 3m^{1/3} = kt + C_1 \quad \Rightarrow \quad m = \frac{1}{27}\big[kt + C_1\big]^3$$

where m_o *is the initial mass of the droplet.*

$$\frac{dv}{dt} + \frac{3kv}{kt + C_1} = -g$$

The integrating factor is found from integrating the coefficient of v*,*

$$I = exp\left[\int \frac{3kdt}{kt + C_1}\right] = exp\Big[3\ln(kt + C_1)\Big] = (kt + C_1)^3$$

Hence,

$$(kt + C_1)^3\frac{dv}{dt} + 3(kt + C_1)^2 v = -g(kt + C_1)^3 \quad \Rightarrow \quad \frac{d}{dt}\Big[(kt + C_1)^3 v\Big] = -g(kt + C_1)^3$$

Hence $\boxed{v(t) = -\frac{gt}{4} - C_3 + C_2/(kt+C_1)^3.}$ *The constants* C_1, C_2, C_3 *have to do with the geometry of the drop, its initial mass and its initial velocity. Suppose* $t = 0$ *marks the intiial formation of the raindrop, it is interesting to consider the case* $t \to \infty$*, we find*

$$v_\infty(t) = -\frac{gt}{4} - C_3$$

which says that the drop accelerates at approximately constant acceleration $-g/4$ *as it falls through the cloud. There is no terminal velocity in contrast to the previous example. You can integrate* $v(t) = \frac{dy}{dt}$ *to find the equation of motion for* y*.*

Example 6.18. Problem: *Rocket flight. Rockets fly by ejecting mass with momentum to form thrust. We analyze the upward motion of a vertically launched rocket in this example. In this case Netwon's Second Law takes the form:*

$$\frac{d}{dt}\bigg[\ mv\ \bigg] = F_{external} + F_{thrust}$$

the external force could include gravity as well as friction and the thrust arises from conservation of momentum. Suppose the rocket expells gas downward at speed u *relative the rocket. Suppose that the rocket burns mass at a uniform rate* $m(t) = m_o - \alpha t$ *and find the resulting equation of motion. Assume air friction is neglible.*

Solution: *If the rocket has velocity* v *then the expelled gas has velocity* $v - u$ *relative the ground's frame of reference. It follows that:*

$$F_{thrust} = (v-u)\frac{dm}{dt}$$

Since $F_{external} = -mg$ *and* $\frac{dm}{dt} = -\alpha$ *we must solve*

$$\frac{d}{dt}\bigg[\ mv\ \bigg] = -mg + (v-u)\frac{dm}{dt} \quad \Rightarrow \quad \frac{dm}{dt}v + m\frac{dv}{dt} = -mg + v\frac{dm}{dt} - u\frac{dm}{dt}$$

Thus,

$$m\frac{dv}{dt} = -u\frac{dm}{dt} - mg$$

Suppose, as was given, that $m(t) = m_o - \alpha t$ *hence* $\frac{dm}{dt} = -\alpha$

$$(m_o - \alpha t)\frac{dv}{dt} = \alpha u - (m_o - \alpha t)g \quad \Rightarrow \quad \frac{dv}{dt} = \frac{\alpha u}{m_o - \alpha t} - g$$

We can solve by integration: assume $v(0) = 0$ *as is physically reasonable,*

$$v(t) = -u\ln(m_o - \alpha t) + u\ln(m_o) - gt = -u\ln\left(1 - \frac{\alpha t}{m_o}\right) - gt.$$

The initial mass m_o *consists of fuel and the rocket itself:* $m_o = m_f + m_r$*. This model is only physical for time* t *such that* $m_r \leq m_f + m_r - \alpha t$ *hence* $0 \leq t \leq m_f/\alpha$*. Once the fuel is finished* *the empty rocket completes the flight by projectile motion. You can integrate $v = dy/dt$ to find the equation of motion. In particular:*

$$\begin{aligned} y(t) &= \int_0^t \big[-u\ln(m_o - \alpha\tau) + u\ln(m_o) - g\tau\big]d\tau \qquad (1) \\ &= \left(-\frac{u}{\alpha}\left[(\alpha\tau - m_o)\ln(m_o - \alpha\tau) - \alpha\tau\right] + u\tau\ln(m_o) - \frac{1}{2}g\tau^2\right)\Bigg|_0^t \\ &= -\frac{u}{\alpha}\left[(\alpha t - m_o)\ln(m_o - \alpha t) - \alpha t\right] + ut\ln(m_o) - \frac{1}{2}gt^2 - \frac{m_o u}{\alpha}\ln(m_o) \\ &= ut - \frac{1}{2}gt^2 - u\frac{m_o}{\alpha}\left(1 - \frac{\alpha t}{m_o}\right)\ln\left(1 - \frac{\alpha t}{m_o}\right) \end{aligned}$$

Suppose $-\int_0^{\frac{m_f}{\alpha}} u\ln\left(1-\frac{\alpha t}{m_o}\right)dt = A$ *then* $y(t) = A - \frac{1}{2}g\left(t - \frac{m_f}{\alpha}\right)^2$ *for* $t > \frac{m_f}{\alpha}$ *as the rocket freefalls back to earth having exhausted its fuel.*

Technically, if the rocket flies more than a few miles vertically then we ought to use the variable force of gravity which correctly accounts for the weaking of the gravitational force with increasing altitude. Mostly this example is included to show how variable mass with momentum transfer is handled.

Other interesting applications include chemical reactions, radioactive decay, blood-flow, other population models, dozens if not hundreds of modifications of the physics examples we've considered, rumor propogation, etc... the math here is likely found in any discipline which uses math to quantiatively describe variables.