We solved $L[y] = f$ by finding the homogenous solution $y_h = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ where $L[y_j] = 0$ for $j = 1, 2, \ldots, n$ and $y_p$ (the particular solution) with $L[y_p] = f$. Formally, you might like to find $L^{-1}$ such that

$$L^{-1} \left[ L[y] \right] = y \quad \text{hence} \quad y = L^{-1} [f].$$

Unfortunately $L^{-1}$ cannot be single-valued due to the infinity of possible homogeneous solutions. But, if we insist on finding a solution for $L[y] = f$ such that

$$y(0) = 0, \quad y'(0) = 0, \quad \ldots, \quad y^{(n-1)}(0) = 0$$

then a unique solution of $L[y] = f$ exists for each given forcing function $f$ (we assume $f$ is sufficiently nice to solve $L[y] = f$, certainly piecewise continuous will suffice, but I assume exponential order $\alpha$ to allow Laplace transform arguments to follow).

**First Order Case:**

Suppose $L = \frac{d}{dt} + p$. Consider $L[y] = Q$. We have

$$y' + py = Q \quad \Rightarrow \quad e^{\int p \, dt} \frac{dy}{dt} + pe^{\int p \, dt} y = Q e^{\int p \, dt}$$

$$\Rightarrow \quad \frac{d}{dt} \left( e^{\int p \, dt} y \right) = Q e^{\int p \, dt}$$

$$\Rightarrow \quad e^{\int p \, dt} y(t) = \int_0^t Q(s) e^{\int p \, ds} \, ds$$

$$\Rightarrow \quad y(t) = e^{-\int p \, dt} \int_0^t Q(u) e^{\int p \, du} \, du$$

$$\Rightarrow \quad y(t) = \int_0^t Q(u) K(u, t) \, du \quad \text{where}$$

We've defined $K(u, t) = \exp \left( \int p \, du - \int p \, dt \right)$

We define $L^{-1}(f)y = \int_0^t K(u, t)f(u) \, du$. You can verify that $L[y] = Q$ has solution $y(t) = L^{-1}(Q)(t)$. The function $K(u, t)$ is called the Green's function for $L$. 

Second Order Case

Consider \( ay'' + by' + cy = f \) and define \( L = aD^2 + bD + c \) where \( D = d/dt. \) Once more we seek \( L^{-1} \) such that
\[ L[y] = f \] has soln. \( L^{-1}[L[y]] = L^{-1}[f] \Rightarrow y = L^{-1}[f]. \)

I'll use variation of parameters. Suppose \( y_1, y_2 \) are solns of \( L[y] = 0 \) such that \( \{y_1, y_2\} \) is LI on the interval of interest \((0, \infty)\) (This argument can be made for other intervals, again I have Laplace transforms in mind...)

We derived that
\[ y = y_1 \int_0^t -y_2 f \, dt \frac{y_1 y_2}{w[y_1, y_2]} + y_2 \int_0^t y_1 f \, dt \frac{y_2 y_1}{w[y_1, y_2]} \]

Keep in mind \( y(0) = 0 \) and \( y'(0) = 0, \)
\[ y(t) = y_1(t) \int_0^t \frac{y_2(u) f(u) \, du}{y_1(u)y_2'(u) - y_2(u)y_1'(u)} + y_2(t) \int_0^t \frac{y_1(u) f(u) \, du}{y_1(u)y_2'(u) - y_2(u)y_1'(u)} \]
\[ y(t) = \int_0^t \left[ \frac{y_1(u)y_2(t) - y_1(t)y_2(u)}{y_1(u)y_2'(u) - y_2(u)y_1'(u)} \right] f(u) \, du \]

\[ K(y, t) \] is Green's Function for \( L \)

\[ L^{-1}(f)(t) = \int_0^t K(u, t) f(u) \, du \]

You can verify \( L[y] = f \) has soln. \( y(t) = L^{-1}(f)(t). \)

Clearly \( y(0) = \int_0^0 K(y, t) f(u) \, du = 0. \)

Comment: it appears \( K(y, t) \) may depend on our choice of fundamental solns \( y_1, y_2. \) However, this is not the case. Read Finney & Ostberg 163-177 of Differential Eq 2nd with Linear Algebra.

Remark: I've constructed the Green's Function in ways which do not immediately connect \( L^{-1} \) with \( L. \) In contrast, the calculation 2 shows how the coefficient functions of \( L = a_n D^n + \ldots + a_2 D^2 + a_1 D + a_0 \) give rise to the formula for \( L^{-1}. \)
Consider \( a_n y^{(n)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = f(t) \) or \( L[y] = f \).

Take the Laplace transform, use \( y(0) = y'(0) = \ldots = y^{(n-1)}(0) = 0 \) to derive that

\[
\left( a_n s^n + \ldots + a_1 s + a_0 \right) \mathcal{L}\{y(t)\}(s) = \mathcal{L}\{f(t)\}(s) = F(s)
\]

Hence,

\[
\mathcal{L}\{y(t)\}(s) = \frac{1}{p(s)} F(s)
\]

Thus,

\[
\mathcal{L}^{-1}\left\{ \frac{1}{p(s)} F(s) \right\} \quad \text{where} \quad \mathcal{L}^{-1}\{F(s)\}(t) = f(t).
\]

Define \( H(s) = \frac{\mathcal{L}\{y(t)\}(s)}{F(s)} = \frac{1}{p(s)} \) the "transfer function."

We let \( \mathcal{L}^{-1}\{H\} = h \) as usual. To find the green's function \( K(t, u) \) we need to somehow write \( \mathcal{L}\{y(t)\}(s) = \mathcal{L}^{-1}\left\{ \frac{1}{p(s)} F(s) \right\} = \mathcal{L}^{-1}\{H\} \cdot \mathcal{L}\{f\}(s) = \int_0^t K(t, u) f(u) du.\)

**Def:** If \( \mathcal{L}\{f\} = F \) and \( \mathcal{L}\{g\} = G \) then \( f \ast g \) is the function such that \( \mathcal{L}\{f \ast g\} = FG. \) In other words,

\[
\mathcal{L}^{-1}\{FG\} = \mathcal{L}^{-1}\{F\} \ast \mathcal{L}^{-1}\{G\} \quad ( \ast \text{ is called the convolution product} )
\]

(we'll attempt a derivation of the formula for \( f \ast g \) shortly but for now, let's appreciate the connection to Green's functions...)

\[
\mathcal{L}^{-1}\{H\} \cdot \mathcal{L}\{f\}(s) = \mathcal{L}^{-1}\{H\} \ast \mathcal{L}^{-1}\{f\} = (h \ast f)(t)
\]

Thus, finding the convolution product will allow us to find \( \mathcal{L}^{-1} \) and the Green's function. Note that

\[
\mathcal{L}^{-1}\left\{ \frac{1}{p(s)} \right\}
\]

is a well-understood problem which we can handle by partial fractions for a given \( L \).
Derivation of Convolution Product

We desire a formula for \( f \ast g \) such that

\[
\mathcal{L} \{ f \ast g \} = \mathcal{L} \{ f \} \cdot \mathcal{L} \{ g \}
\]

\[
\int_0^\infty e^{-st} (f \ast g)(t) \, dt = \int_0^\infty e^{-st} f(t) \, dt \int_0^\infty e^{-su} g(u) \, du
\]

\[
= \int_0^\infty \int_0^\infty e^{-s(t+u)} f(t) g(u) \, dt \, du
\]

\[
= \int_0^\infty \int_0^\infty e^{-sw} f(w-u) g(u) \, dw \, du
\]

\[
= \int_0^\infty e^{-sw} \left( \int_0^\infty f(w-u) g(u) \, du \right) \, dw
\]

\[
= \int_0^\infty e^{-st} \left( \int_0^t f(t-u) g(u) \, du \right) \, dt
\]

\[\text{note: } f(x) = 0 \text{ for } x < 0\]

We define

\[
(f \ast g)(t) = \int_0^t f(t-u) g(u) \, du
\]

Observe that it is clear that

\(0 \ast g = 0\) and \((c f) \ast g = c (f \ast g) = f \ast (c g)\)

and

\((f_1 + f_2) \ast g = f_1 \ast g + f_2 \ast g\)

Consider

\[
(g \ast f)(t) = \int_0^t g(t-u) f(u) \, du
\]

\[
= \int_0^t g(w) f(t-w) \, dw
\]

\[
= \int_0^t f(t-w) g(w) \, dw
\]

\[
= (f \ast g)(t).
\]

You can also show \( f \ast (g \ast h) = (f \ast g) \ast h \).
Continuing the \( n \)th order constant coefficient \( L = P(D) \) discussion.

Recall, \( P(D) [y] = f \) 

\[ L \{ P(D) [y] \} = L \{ f \} \]

\[ P(s) \ Y(s) = F(s) \]

\[ Y(s) = \left( \frac{1}{P(s)} \right) F(s) \]

\[ Y(s) = H(s) F(s) \]

Use convolution to take inverse Laplace transform,

\[ y(t) = L^{-1} \{ H(s) F(s) \} \]

\[ = L^{-1} \{ H(s) \} * L^{-1} \{ F(s) \} \]

\[ = (h * f)(t) \]

\[ = \int_{0}^{t} h(t-u) f(u) \, du \]

Identify that \( K(u,t) = h(t-u) \), the Green's function is the inverse transform of the transform function evaluated at \( t-u \).}

\[ L^{-1} (f)(t) = \int_{0}^{t} h(t-u) f(u) \, du \]

The formula above shows how the force \( f \) applied to \( L[y] = 0 \) system gives a response dictated by the transfer function (or Green's function if you prefer) of the system. The general solution has the form

\[ y(t) = c_{1} y_{1} + c_{2} y_{2} + \ldots + c_{n} y_{n} + \int_{0}^{t} h(t-u) f(u) \, du \]

We constructed \( y_{p}(t) = \int_{0}^{t} h(t-u) f(u) \, du \) to have initial conditions \( y_{p}(0) = 0 \), \( y'_{p}(0) = 0 \), \ldots \( y^{(n-1)}_{p}(0) = 0 \). You might read Rutgan & Rose for further physical comments on the transfer function, or Finney and Ostberg for more mathematical details on Green's function. There are many interesting generalities to comment upon.
Example: find \( h \) for the general distinct root \( n=2 \) problem. we have \( p(s) = (s-r_1)(s-r_2) \) for \( r_1 \neq r_2 \) if the coefficient of \( y'' \) is 1. Then the transfer function

\[
H(s) = \frac{1}{p(s)} = \frac{1}{(s-r_1)(s-r_2)}
\]

We can simplify by partial fractions,

\[
\frac{1}{(s-r_1)(s-r_2)} = \frac{A}{s-r_1} + \frac{B}{s-r_2}
\]

\[
\begin{align*}
1 &= A(s-r_2) + B(s-r_1) \\
\begin{cases} 
1 = A(r_2-r_1) & \Rightarrow A = \frac{1}{r_1-r_2} \\
1 = B(r_1-r_2) & \Rightarrow B = \frac{-1}{r_1-r_2}
\end{cases}
\end{align*}
\]

\[
H(s) = \frac{1}{r_1-r_2} \left( \frac{1}{s-r_1} - \frac{1}{s-r_2} \right)
\]

Therefore,

\[
h(t) = \mathcal{L}^{-1}\{H(s)\} \{t\} = \frac{1}{r_1-r_2} \left( e^{r_1 t} - e^{r_2 t} \right)
\]

The general solution to \( (D-r_1)(D-r_2) \{y\} = f \) with \( y(0) = y'(0) = 0 \) is

\[
y(t) = \int_0^t \frac{1}{r_1-r_2} \left( e^{r_1 (t-v)} - e^{r_2 (t-v)} \right) f(v) \, dv
\]

This gives an integral solution for any distinct-root constant coeff, nonhomogeneous problem. Let us apply it to a particular problem,

\[
y'' + 3y' + 2y = e^{-t}
\]

\[
p(s) = s^2 + 3s + 2 = (s+1)(s+2) \Rightarrow r_1 = -1, \ r_2 = -2
\]

Thus, \( h(t) = \frac{1}{-1-2} \left( e^{-t} - e^{-2t} \right) = e^{-t} - e^{-2t} \). Therefore, the soln with \( y(0) = y'(0) = 0 \) is

\[
y(t) = \left( h \ast f \right) (t)
\]

\[
= \int_0^t h(t-v) f(v) \, dv
\]

\[
= \int_0^t \left[ e^{-(t-v)} - e^{-2(t-v)} \right] e^{-v} \, dv
\]

\[
= \int_0^t \left[ e^{-v} - e^{-v-2t} \right] \, dv
\]

\[
= \left[ ve^{-v} - e^{-v-2t} \right]_0^t
\]

\[
= te^{-t} - e^{-t} + e^{-2t}
\]

To create a general soln we simply add \( c_1 e^{-t} + c_2 e^{-2t} \), these terms are called \( \text{transient} \) as they vanish as \( t \to \infty \). Actually the same is true for \( y(t) \) here, but if \( f = \cos(\omega t) \) etc... often \( y(t) \to 0 \).
Example: find transfer function and h for the arbitrary complex root case with $a = 1$; $P(s) = (s - \alpha + i\beta)(s - \alpha - i\beta)$.

We should work out the partial fractal decomp for

\[
\frac{1}{P(s)} = \frac{1}{(s - \alpha)^2 + \beta^2} = \frac{A\beta}{(s - \alpha)^2 + \beta^2} + \frac{B(s - \alpha)}{(s - \alpha)^2 + \beta^2}
\]

\[
1 = A\beta + B(s - \alpha)
\]

\[
1 = A\beta - \alpha B + BS \Rightarrow B = 0 \quad \& \quad A = \frac{1}{\beta}
\]

(well duh.)

\[
H(s) = \frac{1}{\beta} \left[ \frac{B}{(s - \alpha)^2 + \beta^2} \right]
\]

\[
\Rightarrow h(t) = \frac{1}{\beta} e^{\alpha t} \sin \beta t.
\]

Therefore, to solve $[1 - (s - \alpha)^2 + \beta^2](y) = f$ we may calculate

\[
y(t) = \int_0^t \frac{1}{\beta} e^{\alpha(t - \nu)} \sin[\beta(t - \nu)] f(\nu) d\nu
\]

For example, to solve $y'' + y = f = \sin t$ solve, for $h(t) = \sin t$:

\[
y(t) = \int_0^t \sin(t - \nu) \sin(\nu) d\nu
\]

\[
= \int_0^t \left( \sin t \cos \nu - \sin \nu \cos t \right) \sin \nu d\nu
\]

\[
= \sin t \int_0^t \cos \nu \sin \nu d\nu - \cos t \int_0^t \sin^2 \nu d\nu
\]

\[
= \sin t \left( \frac{1}{2} \sin^2(\nu) \right)_0^t - \cos t \left( \frac{\nu}{2} - \frac{1}{4} \sin(2\nu) \right)_0^t
\]

\[
= \frac{1}{2} \sin^2 t - \frac{1}{2} t \cos t + \frac{1}{4} \cos(t) \sin(2t)
\]

\[
= \frac{1}{2} \sin^2 t - \frac{1}{2} t \cos t
\]

\[
(\text{you can check, } y(0) = 0, y'(0) = 0.)
\]
Example III: Examine the repeated root case.

\[ y'' - 2ry' + r^2y = f \]

\[ P(s) = s^2 - 2sr + r^2 = (s-r)^2 \Rightarrow H(s) = \frac{1}{(s-r)^2} \]

I'll use convolution to calculate \( L^{-1}\{H\} \).

\[ h(t) = L^{-1}\{\frac{1}{(s-r)^2}\} \]
\[ = L^{-1}\{\frac{1}{s-r}\} \cdot L^{-1}\{\frac{1}{s-r}\} \]
\[ = e^{rt} \ast e^{rt} \]
\[ = \int_0^t e^{r(t-v)} e^{rv} \, dv \]
\[ = \int_0^t e^{r(t-v)} \, dv \]
\[ = e^{rt} \int_0^t \, dv \]
\[ = \frac{1}{r} e^{rt} \Rightarrow y(t) = \int_0^t (t-v) e^{r(t-v)} f(v) \, dv \]