EXAMPLES OF TRIGONOMETRIC SUBSTITUTION:

<u>Remark</u>: I also include some hyperbolic subsitutions here. There are problems which are better served by hyperbolic and others by trig. Usually the goal is to remove some $\sqrt{}$. Remember, just as with u-substitution, we must change the integrand and measure (dx typically) and <u>bounds</u> if present.

1.) $$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta \, d\theta}{\sqrt{\cos^2\theta}} \quad \leftarrow \quad \boxed{\begin{array}{l} x = \sin\theta, \quad dx = \cos\theta \, d\theta \\ 1-x^2 = 1-\sin^2\theta = \cos^2\theta \end{array}}$$

$$= \int \frac{\cos\theta \, d\theta}{\cos\theta}$$

$$= \int d\theta$$

$$= \theta + C \qquad x = \sin\theta \Rightarrow \theta = \sin^{-1}(x)$$

$$= \underbrace{\boxed{\sin^{-1}(x) + C}}$$

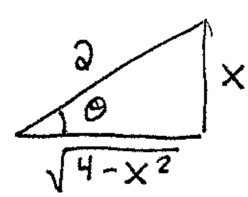
not surprising since $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ this is one of our <u>known</u> elementary integrals.

2.) $$\int \sqrt{4-x^2}\, dx = \int \sqrt{4\cos^2\theta}\, 2\cos\theta \, d\theta \qquad \boxed{\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta \, d\theta \\ 4-x^2 = 4-4\sin^2\theta \\ = 4(1-\sin^2\theta) \\ = 4\cos^2\theta \end{array}}$$

$$= \int 4\cos^2\theta \, d\theta$$

$$= \int 2(1+\cos(2\theta))\, d\theta$$

$$= 2\theta + \sin(2\theta) + C$$



$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) + \sin\left(2\sin^{-1}\left(\frac{x}{2}\right)\right) + C}$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) + \underbrace{2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)}_{2\sin\theta\cos\theta = \sin(2\theta)} + C = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C$$

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Remark: the * answer in 2. gives the result in terms of algebraic functions (as much as possible since $\sin^{-1}(\frac{x}{2})$ is not algebraic fnd.)

3.) $$\int \sqrt{x^2+4x+13}\,dx = \int \sqrt{(x+2)^2+9}\,dx$$

$$= \int \sqrt{9\sec^2\theta}\; 3\sec^2\theta\, d\theta$$

$$x+2 = 3\tan\theta$$
$$(x+2)^2+9 = 9(\underbrace{\tan^2\theta+1}_{\sec^2\theta})$$
$$dx = 3\sec^2\theta\, d\theta$$

$$= 9\underbrace{\int \sec^3\theta\, d\theta}_{\text{can be solved via IBP which loops,}} \;\star$$

$$I = \int \sec^3\theta\, d\theta = \int \underbrace{\sec\theta}_{u}\, \underbrace{\sec^2\theta\, d\theta}_{dv}$$

$$= \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta \quad : \quad \tan^2\theta + 1 = \sec^2\theta$$

$$= \sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= \sec\theta\tan\theta + \int \sec\theta\, d\theta - \int \sec^3\theta\, d\theta$$

$$= \sec\theta\tan\theta + \ln|\sec\theta+\tan\theta| - I$$

Hence, $\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$

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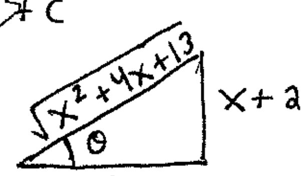
continuing from $\star$,

$$\int \sqrt{x^2+4x+13}\,dx = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta+\tan\theta| + C$$

$$= \frac{9}{2}\left(\frac{\sqrt{x^2+4x+13}}{3}\right)\left(\frac{x+2}{3}\right) + \frac{9}{2}\ln|\sec\theta+\tan\theta| + C$$

$$\tan\theta = \frac{x+2}{3}$$



$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+13} + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+4x+13}}{3} + \frac{x+2}{3}\right| + C$$

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3.) continued from ** we can simplify via properties of ln,

$$\ln\left|\frac{1}{3}\left(x+2+\sqrt{x^2+4x+13}\right)\right| = \underbrace{\ln\left|\tfrac{1}{3}\right|}_{\text{absorb this into } C} + \ln\left|x+2+\sqrt{x^2+4x+13}\right|$$

Hence,

$$\boxed{\int\sqrt{x^2+4x+13}\,dx = \underbrace{\frac{1}{2}(x+2)\sqrt{x^2+4x+13} + \frac{9}{2}\ln\left|x+2+\sqrt{x^2+4x+13}\right|}_{\text{answer in terms of algebraic functions as much as possible.}} + C}$$

4.) this is similar to 3.) but I'll use <u>hyperbolic</u> <u>subst.</u>

$$x = 4\sinh\theta\ , \quad dx = 4\cosh\theta\, d\theta$$

$$x^2+16 = \underline{16\sinh^2\theta + 16} = 16\cosh^2\theta$$

$$\cosh^2\theta - \sinh^2\theta = 1 \;\rightarrow\; \cosh^2\theta = 1+\sinh^2\theta$$

$$\int\sqrt{x^2+16}\,dx = \int\sqrt{16\cosh^2\theta}\; 4\cosh\theta\, d\theta$$

$$= 16\int\cosh^2\theta\, d\theta$$

$$= 8\int\left[1+\cosh(2\theta)\right]d\theta$$

$$= 8\left[\theta + \frac{1}{2}\sinh(2\theta)\right] + C$$

$$= \boxed{8\sinh^{-1}(x/4) + 4\sinh\left(2\sinh^{-1}(x/4)\right) + C}$$

<u>Remark</u>: 3.) could be accomplished via a substitution of $x+2 = 3\sinh\theta$. The identity

$$\sinh^{-1}(\theta) = \ln\left(\theta + \sqrt{\theta^2+1}\right)$$

helps us see how the answers in 3. & 4. are less different than they first seem.

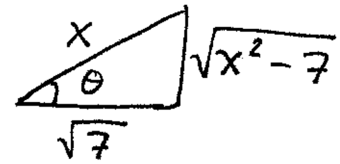
5.) $$\int \frac{x\,dx}{\sqrt{x^2-7}} = \int \frac{\sqrt{7}\sec\theta\,\cancel{\sqrt{7}}\sec\theta\,\cancel{\tan\theta}\,d\theta}{\sqrt{\cancel{7}\tan^2\theta}}$$

$$x = \sqrt{7}\sec\theta$$
$$x^2 - 7 = 7(\sec^2\theta - 1) = 7\tan^2\theta$$
$$dx = \sqrt{7}\sec\theta\tan\theta\,d\theta$$

$$= \int \sqrt{7}\sec^2\theta\,d\theta$$

$$= \sqrt{7}\tan\theta + C$$

$$\sec\theta = \frac{x}{\sqrt{7}} = \frac{\text{hyp}}{\text{adj.}}$$



$$= \sqrt{7}\left(\frac{\sqrt{x^2-7}}{\sqrt{7}}\right) + C$$

$$= \underbrace{\boxed{\sqrt{x^2-7} + C}}_{\text{answer is } \underline{\text{algebraic function}}}$$

6.) $$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$

$$x = \tan\theta, \quad dx = \sec^2\theta\,d\theta$$
$$x = 0 \Rightarrow \theta = 0$$
$$x = 1 \Rightarrow \theta = \pi/4$$
$$x^2 + 1 = \tan^2\theta + 1 = \sec^2\theta$$

$$= \int_0^{\pi/4} \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2}(1 + \cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - 0 - \frac{1}{2}\sin(0)\right)$$

$$= \boxed{\frac{\pi}{8} + \frac{1}{4}.}$$

7.) $$\int \frac{dx}{x^2\sqrt{x^2+4}} = \int \frac{2\sec^2\theta \, d\theta}{4\tan^2\theta \sqrt{4\sec^2\theta}}$$

$$\boxed{\begin{aligned} x &= 2\tan\theta \\ x^2+4 &= 4\sec^2\theta \\ dx &= 2\sec^2\theta \, d\theta \end{aligned}}$$

$$= \int \frac{2\sec^2\theta \, d\theta}{8\tan^2\theta \sec\theta}$$

$$= \int \frac{\sec\theta \, d\theta}{4\tan^2\theta}$$

$$= \int \frac{1}{4\cos\theta}\left[\frac{\cos^2\theta}{\sin^2\theta}\right] d\theta$$

$$= \int \frac{\cos\theta \, d\theta}{4\sin^2\theta}$$

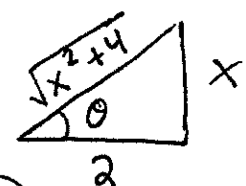
$$\left[\begin{aligned} u &= \sin\theta \\ du &= \cos\theta \, d\theta \end{aligned}\right]$$

$$= \int \frac{du}{4u^2}$$

$$= \frac{-1}{4u} + C$$

$$= \frac{-1}{4\sin\theta} + C$$

$$\tan\theta = \frac{x}{2}$$



$$= \frac{-1}{4\left(\frac{x}{\sqrt{x^2+4}}\right)} + C$$

$$= \boxed{\frac{-\sqrt{x^2+4}}{4x} + C}$$

8.) summary: key identities $\cos^2\theta + \sin^2\theta = 1 \longrightarrow 1 + \tan^2\theta = \sec^2\theta$

$\cosh^2\theta - \sinh^2\theta = 1 \longrightarrow 1 - \tanh^2\theta = \operatorname{sech}^2\theta$

$\sqrt{x^2 - a^2}$	$\sqrt{a^2 - x^2}$	$\sqrt{x^2 + a^2}$	
$x = a\sec\theta$ $x^2 - a^2 = a^2\tan^2\theta$ $dx = a\sec\theta\tan\theta\, d\theta$	$x = a\sin\theta$ $a^2 - x^2 = a^2\cos^2\theta$ $dx = a\cos\theta\, d\theta$	$x = a\tan\theta$ $x^2 + a^2 = a^2\sec^2\theta$ $dx = a\sec^2\theta\, d\theta$	TRIG SUBST.
$x = a\cosh\beta$ $x^2 - a^2 = a^2\sinh^2\beta$ $dx = a\sinh\beta\, d\beta$	$x = a\tanh\beta$ $a^2 - x^2 = a^2\operatorname{sech}^2\beta$ $dx = a\operatorname{sech}^2\beta\, d\beta$	$x = a\sinh\beta$ $x^2 + a^2 = a^2\cosh^2\beta$ $dx = a\cosh\beta\, d\beta$	HYPERBOLIC SUBST.

REMARK: the distinction between the above 6 substitutions and "u-substitution" is in the structure of the substition. Contrast $u = \text{stuff}$ to

$x = a\sec\theta$, $x = a\sin\theta$, $x = a\tan\theta$ (θ is new)

or $x = a\cosh\beta$, $x = a\tanh\beta$, $x = a\sinh\beta$ (β is new)

I think a better term for trigonometric substitution would be implicit trig subst. Anyway, words aside you should recognize this technique successfully removes $\sqrt{}$ by introducing squares in place of sum or diff. of given squared quantities. As usual, key is practice