

# VECTOR FIELDS

360

A vector field is an assignment of a vector at each point in space. Usually we take the vector field to be static, but you could imagine that the assignment itself had a time dependence.

Def<sup>n</sup>/ Let  $D$  be a subset of  $\mathbb{R}^2$ . A vector field on  $\mathbb{R}^2$  is a function  $\vec{F}$  that assigns to each point  $(x, y)$  in  $D$  a 2-dim'l vector  $\vec{F}(x, y) = \langle F_1(x, y), F_2(x, y) \rangle$ .

We call  $F_1$  and  $F_2$  the component functions of  $\vec{F}$ .

Def<sup>n</sup>/ Let  $E \subseteq \mathbb{R}^3$ . A vector field on  $\mathbb{R}^3$  is a function  $\vec{F}$  that assigns to each point  $(x, y, z)$  in  $E$  a 3-dim'l vector

$$\vec{F}(x, y, z) = \langle F_1(x, y, z), F_2(x, y, z), F_3(x, y, z) \rangle$$

The text likes to use the notation  $P, Q$  or  $P, Q, R$  for the component functions so  $\vec{F} = \langle P, Q \rangle$  in 2-dimensions or  $\vec{F} = \langle P, Q, R \rangle$  in  $\mathbb{R}^3$ .

- Each vector field is made of several scalar-valued functions. We could say a vector field is a vector-valued function of  $\mathbb{R}^n$ .

There are many interesting vector fields,

- $\vec{E}$  the Electric Field
- $\vec{B}$  the Magnetic Field
- $\vec{F}$  a force field
- $\vec{F} = -\frac{GmM}{r^2} \hat{r}$
- $\vec{v}$  the velocity field of some liquid
- $\vec{F} = -\nabla U$  a conservative vector field,  $U$  is a scalar function we call the potential energy.

Remark: We have already played with vector fields back when we consider the operation of gradient. The object  $\nabla f$  is a vector field. It turns out that not all vector fields

have the form  $\vec{F} = \nabla f$ . When  $\exists f$  such that  $\vec{F} = \nabla f$

we say  $\vec{F}$  is conservative. (this defines "conservative" for future reference)

E122 Consider  $x^2 + y^2 = 1$ . Diff. implicitly  $\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

in calc II. we plotted  $\frac{dy}{dx}$  at each  $(x, y)$  to describe the so-called direction-field. This is another important example of vector fields.

# WHAT'S THE DEAL WITH E12a?

360a

Let me elaborate on what I cut the discussion short on Tuesday, October 28, 2008.

Def<sup>n</sup>/  $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^n$  is an integral curve (or flow line) of a vector field  $\vec{X}$  on  $\mathbb{R}^n$  if we have that the vector field aligns itself with the tangent vectors to the curve, that is to say

$$\vec{X}(\vec{r}(t)) = \vec{r}'(t)$$

Now let's begin with the vector field  $\vec{F}(x,y) = \langle -y, x \rangle$  let's suppose  $\vec{r}(t) = \langle x(t), y(t) \rangle$  is an integral curve of  $\vec{F}$ . This means it must be a sol<sup>n</sup> to

$$\vec{F}(\vec{r}(t)) = \vec{r}'(t)$$

$$\langle -y, x \rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

This yields two coupled linear ordinary differential eq<sup>n</sup>s. These have a nice sol<sup>n</sup>, I'll show it here next,

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x \Rightarrow \frac{d^2y}{dt^2} = \frac{dx}{dt} = -y$$

Now the differential eq<sup>n</sup>  $\frac{d^2y}{dt^2} + y = 0$  has well-known sol<sup>n</sup>s of (\*)

$$y = A \sin(t + \varphi)$$

where  $A$  and  $\varphi$  are constants. You can substitute this into (\*) and check it works. Then

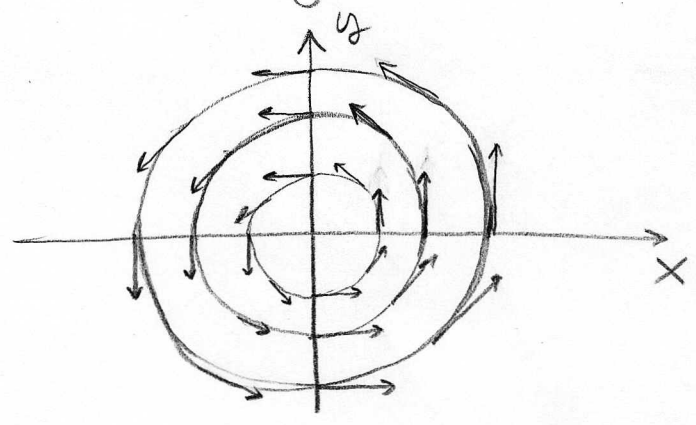
$$x = \frac{dy}{dt} = \frac{d}{dt}(A \sin(t + \varphi)) = A \cos(t + \varphi) = x$$

We find that  $\vec{F} = \langle -y, x \rangle$  has integral curve  $\vec{r}(t) = \langle A \cos(t + \varphi), A \sin(t + \varphi) \rangle$ .

We can rewrite this,

$$\vec{r}(t) = A \langle \cos(t + \varphi), \sin(t + \varphi) \rangle$$

It's clear  $|\langle \cos(t + \varphi), \sin(t + \varphi) \rangle| = 1$  so  $|\vec{r}(t)| = A$  for each  $t \in \mathbb{R}$ . This is a circle of radius A centered at the origin.



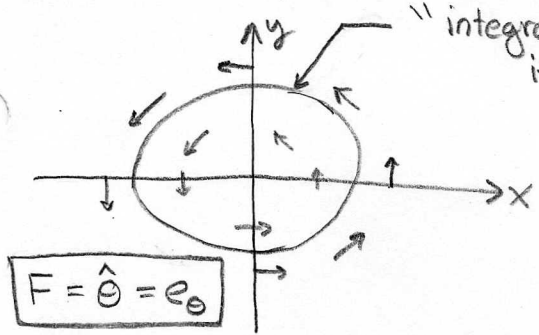
Sketch of  $\vec{F}$  and how it relates to a few token flow lines.

Now perhaps this reminds you of  $e_\theta$ .

$$\begin{aligned} \vec{F}(x,y) &= \langle -y, x \rangle = \langle -r \sin \theta, r \cos \theta \rangle \\ &= r \langle -\sin \theta, \cos \theta \rangle \\ &= r e_\theta \quad (\text{see pg. } \textcircled{361}) \end{aligned}$$

We see  $\vec{F}(r, \theta) = r e_\theta$ . It is quite natural that we found circles as the flow-lines for this vector field.

(Could find Clockwise sol<sup>n</sup>s to  $\frac{dy}{dx} = \frac{-x}{y}$  just the same way. There is no sense of orientation for the circle here)



"integral curve of the direction field"  
it is a sol<sup>n</sup> of  $\frac{dy}{dx} = \frac{-x}{y}$

the little arrows are not derived, (they're a choice)  
from the formula  $\frac{dy}{dx} = -x/y = \frac{\text{rise}}{\text{run}}$

its  $\approx \vec{F}(x,y) = \langle -y, x \rangle$  although  
perhaps  $\vec{F}(x,y) = -y\hat{i} + x\hat{j}$  is better here.

• flow lines of a vector field are paths whose tangents match the vector field

Standard Basis for Vectors in Polar Coordinates

We wish to describe unit vectors that point in the direction of increasing  $r$  and  $\theta$ . Notice first how we could find our familiar friends  $\hat{i}$  and  $\hat{j}$ :

$$\hat{i} = \langle 1, 0 \rangle = \nabla x \equiv \hat{x}$$

$$\hat{j} = \langle 0, 1 \rangle = \nabla y \equiv \hat{y}$$

observe that the gradient operation reveals just the vector we're looking for. Recall  $r = \sqrt{x^2 + y^2}$  and  $\tan \theta = y/x$

$$\nabla r = \frac{1}{2\sqrt{x^2 + y^2}} \langle 2x, 2y \rangle = \frac{x}{r} \hat{i} + \frac{y}{r} \hat{j}$$

$$\nabla \theta = \nabla (\tan^{-1}(y/x)) = \frac{1}{1+(y/x)^2} [\nabla(y/x)] = \frac{1}{1+y^2/x^2} \left[ \frac{-y}{x^2} \hat{i} + \frac{1}{x} \hat{j} \right]$$

$$\Rightarrow \nabla \theta = \frac{x^2}{x^2 + y^2} \left[ \frac{-y}{x^2} \hat{i} + \frac{x}{x^2} \hat{j} \right] = \frac{-y}{r^2} \hat{i} + \frac{x}{r^2} \hat{j}$$

We need unit vectors. It may help to use  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,

$$\nabla r = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\nabla \theta = \frac{1}{r} (-\sin \theta \hat{i} + \cos \theta \hat{j})$$

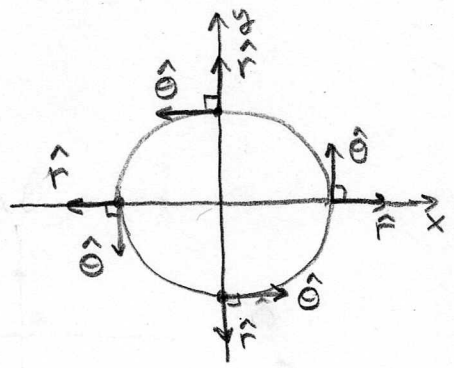
Clearly  $|\nabla r| = \cos^2 \theta + \sin^2 \theta = 1$ , on the other hand  $|\nabla \theta| = 1/r$ . We find then the Standard Polar basis is

$$\hat{r} \equiv e_r = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{\theta} \equiv e_{\theta} = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

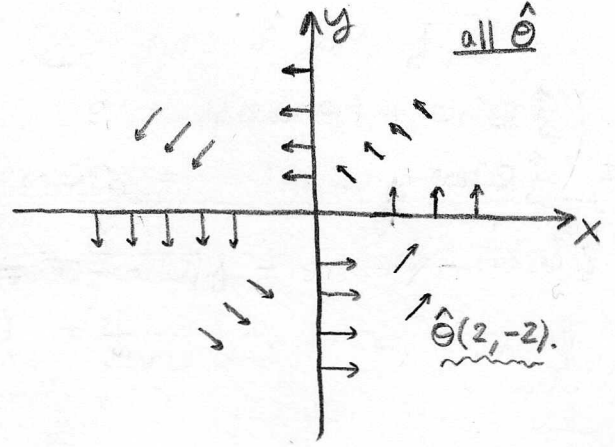
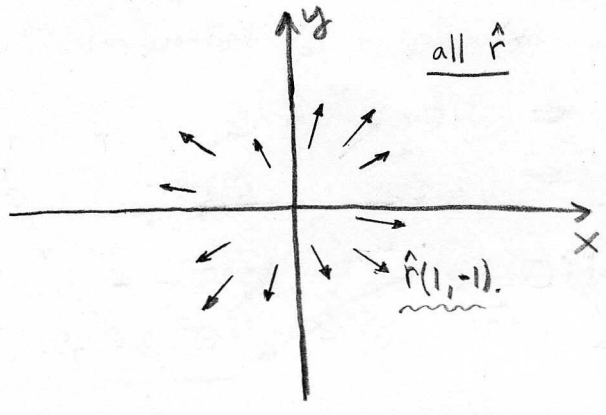
A basis is called orthogonal if the members are perpendicular. Note  $\hat{r} \cdot \hat{\theta} = 0$ .

$e_r$  and  $e_{\theta}$  are vector fields!



The Polar Coordinates Standard Basis :  $e_r, e_\theta$

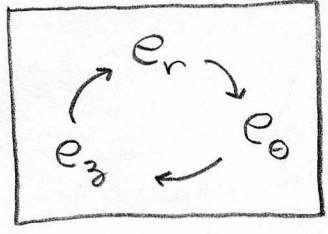
Before I have been quite reckless exchanging points and vectors. This bothered many of you (with good cause, they're different!) It was a special quirk of CARTESIAN COORDINATES that the Standard (CARTESIAN) Basis  $\hat{i}, \hat{j}$  is constant over the whole xy-plane. Now the Polar Basis or "frame" is coordinate dependent, we cannot simply move  $\hat{\theta}$  or  $\hat{r}$  around, their directions are given by  $\theta$ .



CYLINDRICAL STANDARD BASIS

Cylindrical coordinates are right-handed since  $e_z = e_r \times e_\theta$

$$e_z \stackrel{?}{=} e_r \times e_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$



Cross product works the same on these as  $\hat{i}, \hat{j}, \hat{k}$

$$= (\cos^2\theta + \sin^2\theta) \hat{k} = \hat{k} = e_z = \hat{z} = \langle 0, 0, 1 \rangle$$

We can work with vector fields of the form

$$F = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = F_r e_r + F_\theta e_\theta + F_z e_z$$

We would like to be able to use either coordinate system. Obvious question: how do the component functions  $F_x, F_y, F_z$  relate to  $F_r, F_\theta, F_z$ ? Its a simple matter of algebra, we tackle it next,

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = F_r e_r + F_\theta e_\theta + F_z e_z$$

Clearly the  $F_z$  is shared by both, not surprising. Now use (36) to write

$$\begin{aligned} F_x \hat{i} + F_y \hat{j} &= F_r (\cos\theta \hat{i} + \sin\theta \hat{j}) + F_\theta (-\sin\theta \hat{i} + \cos\theta \hat{j}) \\ &= (F_r \cos\theta - F_\theta \sin\theta) \hat{i} + (F_r \sin\theta + F_\theta \cos\theta) \hat{j} \end{aligned}$$

Therefore we find

$$\begin{cases} F_x = F_r \cos\theta - F_\theta \sin\theta \\ F_y = F_r \sin\theta + F_\theta \cos\theta \end{cases} \quad (*)$$

In matrix notation

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}$$

$$\begin{pmatrix} F_r \\ F_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} \cos\theta F_x + \sin\theta F_y \\ -\sin\theta F_x + \cos\theta F_y \end{pmatrix}$$

Thus the inverse relations are,

$$\begin{cases} F_r = \cos\theta F_x + \sin\theta F_y \\ F_\theta = -\sin\theta F_x + \cos\theta F_y \end{cases} \quad (**)$$

you could derive these relations geometrically if you prefer.

Or we could use  $F_r = \vec{F} \cdot e_r$  and  $F_\theta = \vec{F} \cdot e_\theta$  and that'd be quicker.

**E123** Suppose  $\vec{F} = \langle 1, 2, 3 \rangle$  find  $\vec{F}$  in cylindrical frame.

$$F_r = \cos\theta F_x + \sin\theta F_y = \cos\theta + 2\sin\theta$$

$$F_\theta = -\sin\theta F_x + \cos\theta F_y = -\sin\theta + 2\cos\theta$$

$$F_z = F_z = 3$$

$$\text{Thus } \vec{F} = (\cos\theta + 2\sin\theta)e_r + (2\cos\theta - \sin\theta)e_\theta + 3e_z$$

the components are coordinate dependent even though the original  $\vec{F}$  was constant, somehow these conspire to be constant.

Consider  $\theta = 0$  and  $\theta = \pi/2$  for example,

$$\vec{F}|_{\theta=0} = (e_r + 2e_\theta + 3e_z)|_{\theta=0} = \hat{i} + 2\hat{j} + 3\hat{k} = \langle 1, 2, 3 \rangle.$$

$$\vec{F}|_{\theta=\pi/2} = (2e_r - e_\theta + 3\hat{k})|_{\theta=\pi/2} = 2\hat{j} - (-\hat{i}) + 3\hat{k} = \langle 1, 2, 3 \rangle.$$

### SPHERICAL STANDARD BASIS

Spherical coordinates are  $\rho = \sqrt{x^2+y^2+z^2}$  and  $\theta, \varphi$  where  $\theta$  is the same polar angle as we discussed in the cylindrical case.

Thus  $e_\theta = -\sin\theta \hat{i} + \cos\theta \hat{j} = \frac{-y}{r} \hat{i} + \frac{x}{r} \hat{j}$  where  $r = \sqrt{x^2+y^2}$  and  $e_\rho$  is easily calculated,

$$e_\rho = \frac{\nabla \rho}{|\nabla \rho|} = \frac{x}{\rho} \hat{i} + \frac{y}{\rho} \hat{j} + \frac{z}{\rho} \hat{k} = \cos\theta \sin\varphi \hat{i} + \sin\theta \sin\varphi \hat{j} + \cos\varphi \hat{k}$$

finally construct  $e_\varphi$  as  $-e_\rho \times e_\theta$ :

oops should have done  $e_\varphi = e_\theta \times e_\rho$  but I already finished this

$$-e_\varphi = e_\rho \times e_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta \sin\varphi & \sin\theta \sin\varphi & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= (-\cos\varphi \cos\theta) \hat{i} - (\cos\varphi \sin\theta) \hat{j} + (\cos^2\theta \sin\varphi + \sin^2\theta \sin\varphi) \hat{k}$$

$$= -\cos\varphi \cos\theta \hat{i} - \cos\varphi \sin\theta \hat{j} + \sin\varphi \hat{k} = -e_\varphi$$

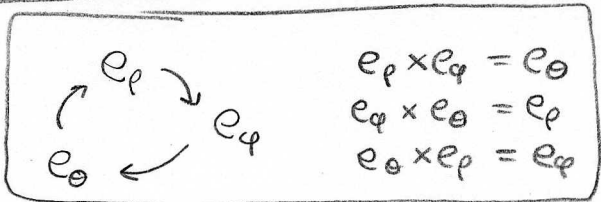
I would hope that  $e_\varphi = \frac{\nabla \varphi}{|\nabla \varphi|}$ . (You can check it).

To summarize:

$$\begin{aligned} e_\rho &= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \cos\theta \sin\varphi \hat{i} + \sin\theta \sin\varphi \hat{j} + \cos\varphi \hat{k} \\ e_\theta &= \frac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}} = -\sin\theta \hat{i} + \cos\theta \hat{j} \\ e_\varphi &= \frac{xz\hat{i} + yz\hat{j} - (x^2+y^2)\hat{k}}{\sqrt{x^2+y^2} \sqrt{x^2+y^2+z^2}} = \cos\varphi \cos\theta \hat{i} + \cos\varphi \sin\theta \hat{j} - \sin\varphi \hat{k} \end{aligned} \quad (*)$$

Then if  $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k} = F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta$  we have

$$\begin{aligned} F_\rho &= \vec{F} \cdot e_\rho = \cos\theta \sin\varphi F_x + \sin\theta \sin\varphi F_y + \cos\varphi F_z \\ F_\theta &= \vec{F} \cdot e_\theta = -\sin\theta F_x + \cos\theta F_y \\ F_\varphi &= \vec{F} \cdot e_\varphi = \cos\varphi \cos\theta F_x + \cos\varphi \sin\theta F_y - \sin\varphi F_z \end{aligned} \quad (**)$$



Sphericals are Right-Handed also.

I'd like to solve <sup>(364)</sup>(\*\*) for  $F_x, F_y, F_z$ . I'm going to use <sup>(364)</sup>(\*) and the fact  $F_x = \vec{F} \cdot \hat{i}$ ,  $F_y = \vec{F} \cdot \hat{j}$  and  $F_z = \vec{F} \cdot \hat{k}$  to do this

$$\begin{aligned} F_x &= (F_\rho e_\rho + F_\varphi e_\varphi + F_\theta e_\theta) \cdot \hat{i} \\ &= F_\rho (e_\rho \cdot \hat{i}) + F_\varphi (e_\varphi \cdot \hat{i}) + F_\theta (e_\theta \cdot \hat{i}) \\ &= \cos\theta \sin\varphi F_\rho + \cos\theta \cos\varphi F_\varphi - \sin\theta F_\theta \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{using (*) on (364)}$$

$$\begin{aligned} F_y &= F_\rho (e_\rho \cdot \hat{j}) + F_\varphi (e_\varphi \cdot \hat{j}) + F_\theta (e_\theta \cdot \hat{j}) \\ &= \sin\theta \sin\varphi F_\rho + \sin\theta \cos\varphi F_\varphi + \cos\theta F_\theta \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{using (*) on (364)}$$

$$\begin{aligned} F_z &= F_\rho (e_\rho \cdot \hat{k}) + F_\varphi (e_\varphi \cdot \hat{k}) + F_\theta (e_\theta \cdot \hat{k}) \\ &= \cos\varphi F_\rho - \sin\varphi F_\varphi \end{aligned}$$

Thus we find

$$\begin{aligned} F_x &= \cos\theta \sin\varphi F_\rho + \cos\theta \cos\varphi F_\varphi - \sin\theta F_\theta \\ F_y &= \sin\theta \sin\varphi F_\rho + \sin\theta \cos\varphi F_\varphi + \cos\theta F_\theta \\ F_z &= \cos\varphi F_\rho - \sin\varphi F_\varphi \end{aligned} \quad (*)$$

Remark: the reason I've gone to the trouble of calculating these relations is that it is a good warm-up to calculating the explicit formulas for gradient in other coordinates. Also it will help us find useful Non-CARTESIAN formulas for the "Curl" and "Divergence". Lets address the question of gradients for now.

Th<sup>y</sup> The gradient is found to be:

(1.)  $\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta$  : for  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

(2.)  $\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z$  : for  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$

(3.)  $\nabla f = \frac{\partial f}{\partial \rho} e_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} e_\varphi + \frac{1}{\rho \sin \phi} \frac{\partial f}{\partial \theta} e_\theta$  : for  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$

• Notice  $\nabla f$  is a vector field on  $\mathbb{R}^2$  or  $\mathbb{R}^3$  here.

Pf: We begin with case (1.). Recall that  $x = r \cos \theta$  and  $y = r \sin \theta$  and we found  $e_r = \cos \theta \hat{i} + \sin \theta \hat{j}$  &  $e_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}$ . We need to solve for  $\hat{i}$  and  $\hat{j}$  in terms of  $e_r$  &  $e_\theta$ , I'll use a standard trick,

$$\begin{aligned} \hat{i} &= a e_r + b e_\theta \quad : \text{find } a, b \text{ by using dot-product.} \\ &= (\hat{i} \cdot e_r) e_r + (\hat{i} \cdot e_\theta) e_\theta \quad : \text{generally } A = (A \cdot \hat{i}) \hat{i} + (A \cdot \hat{j}) \hat{j} \\ &= \boxed{\cos \theta e_r - \sin \theta e_\theta = \hat{i}} \quad \text{I'm just applying this idea to the polar basis.} \end{aligned}$$

$$\begin{aligned} \hat{j} &= (\hat{j} \cdot e_r) e_r + (\hat{j} \cdot e_\theta) e_\theta \quad : \text{same idea as in } \hat{i}. \\ &= \boxed{\sin \theta e_r + \cos \theta e_\theta = \hat{j}} \end{aligned}$$

Consider then,

$$\begin{aligned} \nabla f &= \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} \\ &= \frac{\partial f}{\partial x} (\cos \theta e_r - \sin \theta e_\theta) + \frac{\partial f}{\partial y} (\sin \theta e_r + \cos \theta e_\theta) \\ &= \left( \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right) e_r + \left( -\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} \right) e_\theta \\ &= e_r \left[ \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right] f + e_\theta \frac{1}{r} \left[ -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y} \right] f \\ &= e_r \frac{\partial}{\partial r} (f) + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} (f) \\ &= \boxed{e_r \frac{\partial f}{\partial r} + e_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} = \nabla f} \end{aligned}$$

see p. 309 we derived this during our discussion of the chain-rule.

In polar coordinates:  $\nabla = e_r \frac{\partial}{\partial r} + \frac{1}{r} e_\theta \frac{\partial}{\partial \theta}$

Proof Continued:

The proof of (2.) follows immediately from (1.) since in that case  $\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = e_r \frac{\partial f}{\partial r} + \frac{1}{r} e_\phi \frac{\partial f}{\partial \phi} + e_z \frac{\partial f}{\partial z}$

Since  $e_z = \hat{k}$ . The proof of (3.) is less obvious. To begin let's rewrite  $\hat{i}, \hat{j}, \hat{k}$  in the Spherical basis  $e_\rho, e_\phi, e_\theta$ ,

$$\begin{aligned} \hat{i} &= (\hat{i} \cdot e_\rho) e_\rho + (\hat{i} \cdot e_\phi) e_\phi + (\hat{i} \cdot e_\theta) e_\theta \\ &= \cos \theta \sin \phi e_\rho + \cos \theta \cos \phi e_\phi - \sin \theta e_\theta = \hat{i} \end{aligned} \quad \text{①}$$

$$\begin{aligned} \hat{j} &= (\hat{j} \cdot e_\rho) e_\rho + (\hat{j} \cdot e_\phi) e_\phi + (\hat{j} \cdot e_\theta) e_\theta \\ &= \sin \theta \sin \phi e_\rho + \sin \theta \cos \phi e_\phi + \cos \theta e_\theta = \hat{j} \end{aligned} \quad \text{②}$$

$$\begin{aligned} \hat{k} &= (\hat{k} \cdot e_\rho) e_\rho + (\hat{k} \cdot e_\phi) e_\phi + (\hat{k} \cdot e_\theta) e_\theta \\ &= \cos \phi e_\rho - \sin \phi e_\phi = \hat{k} \end{aligned} \quad \text{③}$$

I used (\*) of (364) to evaluate the dot products above. Consider,

$$\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

$$\begin{aligned} &= e_\rho \left( \cos \theta \sin \phi \frac{\partial f}{\partial x} + \sin \theta \sin \phi \frac{\partial f}{\partial y} + \cos \phi \frac{\partial f}{\partial z} \right) \\ &+ e_\phi \left( \cos \theta \cos \phi \frac{\partial f}{\partial x} + \sin \theta \cos \phi \frac{\partial f}{\partial y} - \sin \phi \frac{\partial f}{\partial z} \right) \\ &+ e_\theta \left( -\sin \theta \frac{\partial f}{\partial x} + \cos \theta \frac{\partial f}{\partial y} \right) \end{aligned} \quad \begin{array}{l} \text{: using} \\ \text{①, ②, ③} \\ \text{gathering like} \\ \text{terms.} \end{array}$$

$$= e_\rho \left( \frac{\partial x}{\partial \rho} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \rho} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \rho} \frac{\partial f}{\partial z} \right)$$

$$+ \frac{1}{\rho} e_\phi \left( \rho \cos \theta \cos \phi \frac{\partial f}{\partial x} + \rho \sin \theta \cos \phi \frac{\partial f}{\partial y} - \rho \sin \phi \frac{\partial f}{\partial z} \right)$$

$$+ \frac{1}{\rho \sin \theta} e_\theta \left( -\rho \sin \theta \sin \phi \frac{\partial f}{\partial x} + \rho \cos \theta \sin \phi \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial f}{\partial z} \right)$$

$$= e_\rho \frac{\partial f}{\partial \rho} + \frac{1}{\rho} e_\phi \left( \frac{\partial x}{\partial \phi} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial f}{\partial z} \right) + \frac{e_\theta}{\rho \sin \theta} \left( \frac{\partial x}{\partial \theta} \frac{\partial f}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial f}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial f}{\partial z} \right)$$

$$= \boxed{e_\rho \frac{\partial f}{\partial \rho} + \frac{1}{\rho} e_\phi \frac{\partial f}{\partial \phi} + \frac{1}{\rho \sin \theta} e_\theta \frac{\partial f}{\partial \theta} = \nabla f}$$

Observe in Sphericals  $\nabla = e_\rho \frac{\partial}{\partial \rho} + \frac{e_\phi}{\rho} \frac{\partial}{\partial \phi} + \frac{e_\theta}{\rho \sin \theta} \frac{\partial}{\partial \theta}$ . (\*)

**E124** Consider the function  $U = -\frac{GmM}{\rho}$ . Lets find the gradient of  $U$  where  $G$  is the gravitational constant,  $m$  is the mass of a planet and  $M$  is the mass of the sun which we place at  $\rho = 0$  so the distance between  $M$  &  $m$  is  $\rho = \sqrt{x^2 + y^2 + z^2}$ . Lets use our Th<sup>m</sup>,

$$\nabla U = -GmM \left[ e_\rho \frac{\partial}{\partial \rho} \left( \frac{1}{\rho} \right) + \frac{1}{\rho} e_\phi \frac{\partial}{\partial \phi} \left( \frac{1}{\rho} \right) + \frac{1}{\rho \sin \phi} e_\theta \frac{\partial}{\partial \theta} \left( \frac{1}{\rho} \right) \right]$$

$$\nabla U = \frac{GmM}{\rho^2} e_\rho \Rightarrow \vec{F} = -\nabla U = \boxed{-\frac{GmM}{\rho^2} e_\rho = \vec{F}}$$

this is Newton's Universal Law of Gravitation.

**E125** Consider the electric potential  $V = -\frac{kQ}{\rho}$  then

$$\vec{E} = -\nabla V = \boxed{\frac{kQ}{\rho^2} e_\rho = \vec{E}}$$

the Electric field of a static point charge  $Q$ .

to get the force on  $q$  due to  $Q$ 's electric field we would multiply by  $q$  so  $\vec{F} = \frac{kqQ}{\rho^2} e_\rho$  and the force is attractive if  $qQ < 0$  (opposite polarity) or repulsive if  $qQ > 0$  (like charges) in contrast to  $\vec{F}_{gravity}$  which has same  $1/\rho^2$  dependence but is always attractive.

**E126** Suppose  $x^2 + y^2 + z^2 = R^2$  find the normal to this level surface. Notice this is  $\rho^2 = R^2$  or  $\rho = R$  (assuming  $R > 0$ ) then  $F(\rho, \phi, \theta) = \rho$  gives sphere as  $F = R$ . The normal is simply  $\nabla F = e_\rho \frac{\partial \rho}{\partial \rho} + 0 + 0 = e_\rho$ .

Remark: these examples are probably to simple to grasp the power of using sphericals or cylindricals. Trust me they're very useful, and much easier to understand geometrically for a problem that has spherical or cylindrical symmetry.

## THE CURL OF A VECTOR FIELD:

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We now find the next use of our friend " $\nabla$ " (del or nabla). We are skipping ahead to §17.5 since it is our custom to differentiate before we integrate, and because it allows me to better explain certain topics in §17.2, 17.3, 17.4 and so on...

Def<sup>n</sup>/ Let  $\vec{F} = \langle P, Q, R \rangle$  where each component function is differentiable, and  $\vec{F} = \langle F_1, F_2, F_3 \rangle$  as well,

$$\text{curl}(\vec{F}) = \nabla \times \vec{F}$$

$$= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

$$= \langle \partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1 \rangle$$

$$= \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \times (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

**E127** Let  $\vec{F} = \langle x, y, z \rangle$  find  $\nabla \times \vec{F}$ .

$$\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y), \frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}(z), \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) \right\rangle = \mathbf{0}.$$

Hmm... notice  $\vec{F} = \nabla \left[ \frac{1}{2}(x^2 + y^2 + z^2) \right]$ . So  $\vec{F}$  is conservative.

**E128** Let  $\vec{F} = \langle -y, x, 0 \rangle$  we saw this in **E122** on **361**

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = \hat{i}(\partial_y(0) - \partial_z(x)) - \hat{j}(\partial_x(0) - \partial_z(y)) + \hat{k}(\partial_x x + \partial_y y)$$
$$\Rightarrow \boxed{\nabla \times \vec{F} = 2\hat{k}}$$

Using the  $\vec{F} = \langle P, Q, R \rangle$  notation we have  $P = -y$  &  $Q = x$  you might note  $\frac{\partial P}{\partial y} = -1$  whereas  $\frac{\partial Q}{\partial x} = 1$  thus

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ . We'll learn a little later this means  $\nexists U$  such that  $\vec{F} = \nabla U$ . That is  $\vec{F}$  is not "conservative"

Remark: we can deduce a few facts about the "curl" operation. It measures if the vector field twists or curls about some point. In **E127** we found  $\nabla \times \vec{F} = 0$  for vector field which could be written  $\vec{F} = \rho \hat{e}_\rho$  so it's purely "radial" it diverges straight away from the origin. On the other hand in **E128** we found  $\nabla \times \vec{F} = 2\hat{k}$  for the vector field  $\vec{F} = \langle -y, x, 0 \rangle$  which circles the origin in the xy-plane. We could say the curl measures the circulation of a vector field. Finally, we noticed that if  $\vec{F} = \nabla U$  then  $\nabla \times \vec{F} = \nabla \times (\nabla U) = 0$ . Intuitively this is appealing since  $\vec{A} \times \vec{A} = 0$ , however we know that result for an ordinary vector, not for a vector of operators which is what  $\nabla = \hat{i} \partial_x + \hat{j} \partial_y + \hat{k} \partial_z$  is. It is true though, we'll prove it.

**Th<sup>n</sup>** / If  $g$  has continuous 2<sup>nd</sup> order partials then  $\nabla \times \nabla g = 0$

Proof: follows from direct calculation + Clairaut's Th<sup>n</sup>

$$\begin{aligned} \nabla \times \nabla g &= \nabla \times \langle \partial_x g, \partial_y g, \partial_z g \rangle \\ &= \langle \partial_y(\partial_z g) - \partial_z(\partial_y g), \partial_z(\partial_x g) - \partial_x(\partial_z g), \partial_x(\partial_y g) - \partial_y(\partial_x g) \rangle \\ &= \langle \frac{\partial^2 g}{\partial y \partial z} - \frac{\partial^2 g}{\partial z \partial y}, \frac{\partial^2 g}{\partial z \partial x} - \frac{\partial^2 g}{\partial x \partial z}, \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial y \partial x} \rangle \\ &= \langle 0, 0, 0 \rangle \quad \left( \text{applying } \partial_i \partial_j g = \partial_j \partial_i g \text{ for continuous fnct. } g. \right) \end{aligned}$$

**Corollary**: If  $\vec{F}$  is conservative then  $\nabla \times \vec{F} = 0$

this means we can verify that  $\vec{F}$  fails to be conservative if  $\nabla \times \vec{F} \neq 0$ . It is not however guaranteed that if  $\nabla \times \vec{F} = 0$  then  $\vec{F}$  is conservative.

Remark:  $\vec{F}$  is said to be irrotational iff  $\nabla \times \vec{F} = 0$

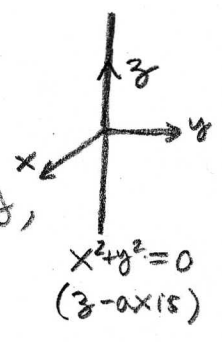
$$\begin{aligned} \vec{F} \text{ conservative} &\implies \nabla \times \vec{F} = 0 \text{ (always)} \\ \nabla \times \vec{F} = 0 &\implies \vec{F} = \nabla U \text{ (sometimes)} \end{aligned}$$

it turns out the topology matters.

**Th<sup>m</sup>** If  $\vec{F}$  is a vector field defined on all of  $\mathbb{R}^3$  whose components functions have continuous partial derivatives and  $\text{curl}(\vec{F}) = 0$  then  $\vec{F}$  is a conservative vector field

a stronger version says this Th<sup>m</sup> holds for simply-connected domains of  $\vec{F}$ . Simply-connected is a topological concept that means the space has no holes.

**E129** Let  $\vec{F} = \frac{-y}{x^2+y^2} \hat{i} + \frac{x}{x^2+y^2} \hat{j}$ . Lets check if  $\vec{F}$  is conservative. Later we'll show  $\nabla \times \vec{F} = 0$ . Thus we should be able to find  $f$  such that  $\vec{F} = \nabla f$  meaning,



$$\frac{\partial f}{\partial x} = \frac{-y}{x^2+y^2} = \frac{1}{1+(y/x)^2} \frac{-y}{x^2} = \frac{\partial}{\partial x} [\tan^{-1}(y/x)]$$

$$\frac{\partial f}{\partial y} = \frac{x}{x^2+y^2} = \frac{1}{1+(y/x)^2} \frac{1}{x} = \frac{\partial}{\partial y} [\tan^{-1}(y/x)]$$

Thus  $f = \tan^{-1}(y/x)$ . Here's the catch,  $\text{dom}(\vec{F}) = \{(x,y,z) \mid x^2+y^2 \neq 0\}$  whereas  $\text{dom}(f) = \{(x,y,z) \mid x \neq 0\}$  thus  $\vec{F}$  is not conservative despite the fact  $\nabla \times \vec{F} = 0$ . This example is a subtle one and ultimately it leads to many deep advances in modern quantum field theory (its the magnetic monopole)

**E130** Let  $\vec{F} = \langle 2x+y, z \cos(yz) + x, y \cos(yz) \rangle$ . Is  $\vec{F}$  conservative, if so find its potential function  $f$  such that  $\vec{F} = \nabla f$ . We can test if  $\vec{F}$  is conservative by examining  $\nabla \times \vec{F}$ , the  $\text{dom}(\vec{F}) = \mathbb{R}^3$  so we'll not have to worry about the exception to the Th<sup>m</sup>, in this case:  $\nabla \times \vec{F} = 0 \iff \vec{F}$  conservative

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+y & z \cos(yz) + x & y \cos(yz) \end{vmatrix} \\ &= \hat{i} (\cos(yz) - zy \sin(yz)) - \cos(yz) + zy \sin(yz) + \hat{j} (0 - 0) + \hat{k} (1 - 1) \\ &= \underline{0} = \nabla \times \vec{F} \implies \text{since } \text{dom}(\vec{F}) = \mathbb{R}^3 \text{ we find } \vec{F} \text{ conservative.} \end{aligned}$$

E130 continued: We found that  $\vec{F}$  is conservative with the help of the  $\nabla \times \vec{F}$  test. In principle this is not necessary, we alternatively could simply assume that  $\exists f$  such that  $\vec{F} = \nabla f$ . If  $\vec{F}$  is not conservative we'll not be successful in finding the potential function.

$$\nabla f = \langle 2x + y, z \cos(yz) + x, y \cos(yz) \rangle$$

yields three partial differential eq<sup>n</sup>'s, (PDEs)

$$\frac{\partial f}{\partial x} = 2x + y, \quad \frac{\partial f}{\partial y} = z \cos(yz) + x, \quad \frac{\partial f}{\partial z} = y \cos(yz)$$

generally PDEs aren't easy to solve, but these are easy. I'll illustrate a procedure you should know (this comes up in other math courses like Differential Eq<sup>n</sup>'s and applications)

$$f = \int \frac{\partial f}{\partial x} dx = \int (2x + y) dx = x^2 + xy + C_1(y, z)$$

I've used the notation " $\partial x$ " to emphasize that we hold  $y$  &  $z$  constant in the integration, the constant with respect to  $x$  can be a function of  $y$  &  $z$  as I emphasize with the notation  $C_1(y, z)$ . Now we'll use the remaining two PDEs to pin down the explicit form of  $C_1(y, z)$ .

$$\frac{\partial f}{\partial y} = z \cos(yz) + x = \frac{\partial}{\partial y} [x^2 + xy + C_1(y, z)] = x + \frac{\partial C_1}{\partial y}$$

$$\Rightarrow z \cos(yz) = \frac{\partial C_1}{\partial y}$$

$$\Rightarrow C_1 = \int \frac{\partial C_1}{\partial y} dy = \int z \cos(yz) dy = \frac{z \sin(yz)}{z} + C_2(z)$$

Notice that  $C_1 = C_1(y, z)$  so I knew that  $C_2 = C_2(z)$ , it can only depend on  $z$  otherwise  $C_1$  might acquire an  $x$ -dependence.

$$\frac{\partial f}{\partial z} = y \cos(yz) = \frac{\partial}{\partial z} [x^2 + xy + \sin(yz) + C_2(z)]$$

$$\Rightarrow y \cos(yz) = y \cos(yz) + \frac{\partial C_2}{\partial z} \Rightarrow \frac{dC_2}{dz} = 0 \therefore \underline{C_2 = \text{constant}}$$

Here  $\partial C_2 / \partial z = dC_2 / dz$  since  $C_2$  only depends on  $z$ . In total,

$$f = x^2 + xy + \sin(yz) + C_2 \text{ is the potential function for } F$$

Remark: this seems longer than it is, I've put extra comments here to try to clarify the method. Also see §13.5 #13 & 16 for more examples.



**Th<sup>m</sup>** / If  $\vec{F} = \langle P, Q, R \rangle$  is a vector field such that  $P, Q, R$  have continuous 2<sup>nd</sup> order partial derivatives then

$$\text{div}(\text{curl}(\vec{F})) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

Proof: follows from Clairaut's Th<sup>m</sup> again, use  $\vec{F} = \langle F_1, F_2, F_3 \rangle$

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{F}) &= \nabla \cdot \langle \partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1 \rangle \\ &= \cancel{\partial_1 \partial_2 F_3} - \cancel{\partial_2 \partial_3 F_2} + \cancel{\partial_3 \partial_3 F_1} - \cancel{\partial_2 \partial_1 F_3} + \cancel{\partial_3 \partial_1 F_2} - \cancel{\partial_3 \partial_2 F_1} \\ &= 0 // \end{aligned}$$

**E133** Is it possible that  $\vec{F} = \langle xz, xyz, -y^2 \rangle = \nabla \times \vec{G}$  for some  $\vec{G}$ ? Notice if this were true then  $\nabla \cdot \vec{F} = \nabla \cdot (\nabla \times \vec{G}) = 0$  however  $\nabla \cdot \vec{F} \neq 0$  as we may easily calculate,

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz \neq 0$$

∴ No such  $\vec{G}$  exists

**E134** One of Maxwell's Eq<sup>s</sup> can be written as  $\nabla \cdot \vec{B} = 0$ , find another vector field which automatically solves this eq<sup>n</sup>. The so-called vector potential  $\vec{A}$  does the job, it is req<sup>d</sup> that

$$\vec{B} = \nabla \times \vec{A} \quad \text{thus} \quad \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0$$

Notice that  $\vec{A}$  is not unique since  $\nabla \times \nabla g = 0$  we'll find the same magnetic field  $\vec{B}$  from  $\vec{A}$  or  $\vec{A} + \nabla g$ . This is gauge symmetry.

Laplace Operator

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

this can be applied to real-valued functions or vectors.

**E135** Gauss Law states  $\nabla \cdot \vec{E} = \rho/\epsilon_0$  and in electrostatics the electric field is given by the potential  $V$  according to  $\vec{E} = -\nabla V$  thus Gauss Law becomes  $\nabla \cdot (-\nabla V) = \rho/\epsilon_0$ . this gives  $\nabla^2 V = -\rho/\epsilon_0$ .

**E136** In or 17.5 # 38 in Calculus Ed. 6 §13.5 # 36 we encounter the Laplace Operator acting on vector fields

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \quad \& \quad \nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2}$$

the sol<sup>n</sup>s to these eq<sup>s</sup> are electromagnetic waves, that is light.

# CURL AND DIVERGENCE IN CURVED COORDINATES

We found the gradients already, this will hurt a little. We begin with the divergence in cylindricals,  $(r, \theta, z)$ , we need a few preliminary facts

$$\begin{aligned} e_r &= \cos\theta \hat{i} + \sin\theta \hat{j} & x &= r \cos\theta \\ e_\theta &= -\sin\theta \hat{i} + \cos\theta \hat{j} & y &= r \sin\theta \\ e_z &= \hat{k} & z &= z \end{aligned}$$

Then from the chain-rule we find

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z} = -r \sin\theta \frac{\partial}{\partial x} + r \cos\theta \frac{\partial}{\partial y} \end{aligned}$$

We need to solve these for  $\partial/\partial x$  and  $\partial/\partial y$ .

$$\begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r \sin\theta & r \cos\theta \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \quad \text{note} \quad \begin{pmatrix} \cos\theta & \sin\theta \\ -r \sin\theta & r \cos\theta \end{pmatrix}^{-1} = \frac{1}{r \cos^2\theta + r \sin^2\theta} \begin{pmatrix} r \cos\theta & -\sin\theta \\ r \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r \cos\theta & -\sin\theta \\ r \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \partial_r \\ \partial_\theta \end{pmatrix} \Rightarrow \begin{cases} \partial_x = \cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \\ \partial_y = \sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \end{cases}$$

Calculate,

$$\begin{aligned} \nabla \cdot \vec{F} &= (\hat{i} \partial_x + \hat{j} \partial_y + \hat{k} \partial_z) \cdot (F_r e_r + F_\theta e_\theta + F_z e_z) \\ &= \left[ \hat{i} \left( \cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \right) + \hat{j} \left( \sin\theta \partial_r + \frac{\cos\theta}{r} \partial_\theta \right) + \hat{k} \partial_z \right] \cdot \\ &\quad \cdot \left[ \hat{i} (F_r \cos\theta - F_\theta \sin\theta) + \hat{j} (F_r \sin\theta + F_\theta \cos\theta) + \hat{k} F_z \right] \\ &= \left( \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \right) (F_r \cos\theta - F_\theta \sin\theta) + \\ &\quad + \left( \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta} \right) (F_r \sin\theta + F_\theta \cos\theta) + \frac{\partial F_z}{\partial z} \\ &= \underbrace{\cos\theta \frac{\partial}{\partial r} [F_r \cos\theta - F_\theta \sin\theta]}_{\text{I}} - \underbrace{\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} [F_r \cos\theta - F_\theta \sin\theta]}_{\text{II}} \\ &\quad + \underbrace{\sin\theta \frac{\partial}{\partial r} [F_r \sin\theta + F_\theta \cos\theta]}_{\text{III}} + \underbrace{\frac{\cos\theta}{r} \frac{\partial}{\partial \theta} [F_r \sin\theta + F_\theta \cos\theta]}_{\text{IV}} + \frac{\partial F_z}{\partial z} \end{aligned}$$

Continuing to find  $\nabla \cdot \vec{F}$  in cylindrical coordinates. We focus on the pieces  $\textcircled{\text{I}}, \textcircled{\text{II}}, \textcircled{\text{III}}, \textcircled{\text{IV}}$  one at a time. To begin we pull out constant pieces. Remember generally  $F_r$  and  $F_\theta$  functions of both  $r, \theta,$  and  $z$ .

$$\textcircled{\text{I}} = \cos^2 \theta \frac{\partial F_r}{\partial r} - \cos \theta \sin \theta \frac{\partial F_\theta}{\partial r} \quad \textcircled{1}$$

$$\begin{aligned} \textcircled{\text{II}} &= -\frac{\sin \theta}{r} \left[ \frac{\partial F_r}{\partial \theta} \cos \theta - F_r \sin \theta - \frac{\partial F_\theta}{\partial \theta} - F_\theta \cos \theta \right] \\ &= -\frac{\sin \theta \cos \theta}{r} \frac{\partial F_r}{\partial \theta} + \frac{\sin^2 \theta}{r} F_r + \frac{\sin^2 \theta}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\sin \theta \cos \theta}{r} F_\theta \quad \textcircled{2} \end{aligned}$$

$$\textcircled{\text{III}} = \sin^2 \theta \frac{\partial F_r}{\partial r} + \sin \theta \cos \theta \frac{\partial F_\theta}{\partial r} \quad \textcircled{1}$$

$$\begin{aligned} \textcircled{\text{IV}} &= \frac{\cos \theta}{r} \left[ \frac{\partial F_r}{\partial \theta} \sin \theta + F_r \cos \theta + \frac{\partial F_\theta}{\partial \theta} - F_\theta \sin \theta \right] \\ &= \frac{\sin \theta \cos \theta}{r} \frac{\partial F_r}{\partial \theta} + \frac{\cos^2 \theta}{r} F_r + \frac{\cos^2 \theta}{r} \frac{\partial F_\theta}{\partial \theta} - \frac{\sin \theta \cos \theta}{r} F_\theta \quad \textcircled{2} \end{aligned}$$

Now lets assemble  $\nabla \cdot \vec{F}$  given the above results, notice certain terms cancel,

$$\begin{aligned} \nabla \cdot \vec{F} &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial F_r}{\partial r} + \frac{1}{r} (\sin^2 \theta + \cos^2 \theta) F_r + \frac{\sin^2 \theta + \cos^2 \theta}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \\ &= \frac{\partial F_r}{\partial r} + \frac{1}{r} F_r + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z} \quad (*) \\ &= \frac{1}{r} \left[ r \frac{\partial F_r}{\partial r} + F_r + \frac{\partial F_\theta}{\partial \theta} + r \frac{\partial F_z}{\partial z} \right] \\ &= \frac{1}{r} \left[ \frac{\partial}{\partial r} (r F_r) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial}{\partial z} (r F_z) \right] = \nabla \cdot \vec{F} \end{aligned}$$

The last formula and (\*) are probably both useful. Notice that on  $\mathbb{R}^2$  for  $\vec{F} = F_r \hat{e}_r + F_\theta \hat{e}_\theta = F_1 \hat{i} + F_2 \hat{j}$  we will find the same result without the  $z$ -terms,

$$\nabla \cdot \vec{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

$$\text{Th}^n / \nabla \cdot \vec{F} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} [\rho^2 F_\rho] + \frac{1}{\rho \sin \varphi} \frac{\partial}{\partial \varphi} [\sin \varphi F_\varphi] + \frac{1}{\rho \sin \varphi} \frac{\partial F_\theta}{\partial \theta}$$

where  $\vec{F} = F_\rho \mathbf{e}_\rho + F_\varphi \mathbf{e}_\varphi + F_\theta \mathbf{e}_\theta$  in spherical coordinates  $\rho, \varphi, \theta$   
 where  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi$

Proof: We'll convert the def<sup>n</sup>  $\nabla \cdot \vec{F} = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$  to sphericals.  
 Lets recall what we already know from earlier work, (\*) (365)

$$F_x = \cos \theta \sin \varphi F_\rho + \cos \theta \cos \varphi F_\varphi - \sin \theta F_\theta$$

$$F_y = \sin \theta \sin \varphi F_\rho + \sin \theta \cos \varphi F_\varphi + \cos \theta F_\theta$$

$$F_z = \cos \varphi F_\rho - \sin \varphi F_\varphi$$

We also calculated that

$$\nabla = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \frac{1}{\rho} \mathbf{e}_\varphi \frac{\partial}{\partial \varphi} + \frac{\mathbf{e}_\theta}{\rho \sin \varphi} \frac{\partial}{\partial \theta} \quad : (*) \text{ on } (367)$$

$$= \left( \hat{i} \cos \theta \sin \varphi + \sin \theta \sin \varphi \hat{j} + \cos \varphi \hat{k} \right) \frac{\partial}{\partial \rho} \quad : \text{Using } (*) \text{ on } (364)$$

$$+ \frac{1}{\rho} \left( \hat{i} \cos \theta \cos \varphi + \hat{j} \sin \theta \cos \varphi - \sin \varphi \hat{k} \right) \frac{\partial}{\partial \varphi}$$

$$+ \frac{1}{\rho \sin \varphi} \left( -\sin \theta \hat{i} + \cos \theta \hat{j} \right) \frac{\partial}{\partial \theta}$$

$$= \left. \begin{aligned} & \hat{i} \left[ \cos \theta \sin \varphi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \cos \theta \cos \varphi \frac{\partial}{\partial \varphi} - \frac{\sin \theta}{\rho \sin \varphi} \frac{\partial}{\partial \theta} \right] \text{ (I)} \\ & + \hat{j} \left[ \sin \theta \sin \varphi \frac{\partial}{\partial \rho} + \frac{1}{\rho} \sin \theta \cos \varphi \frac{\partial}{\partial \varphi} + \frac{\cos \theta}{\rho \sin \varphi} \frac{\partial}{\partial \theta} \right] \text{ (II)} \\ & + \hat{k} \left[ \cos \varphi \frac{\partial}{\partial \rho} - \frac{\sin \varphi}{\rho} \frac{\partial}{\partial \varphi} \right] \text{ (III)} \end{aligned} \right\} (\star)$$

$$= \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

We could derive  $(\star)$  from the chain rule, I took an alternative route.

Now  $\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$  and we may compute

$$\nabla \cdot \vec{F} = \text{(I)} F_x + \text{(II)} F_y + \text{(III)} F_z \quad \text{where } F_x, F_y, F_z \text{ are}$$

expressed in terms of  $\rho, \varphi, \theta$ . We'll work out the gory details

$$\textcircled{I} F_x = \cos \theta \sin \varphi \frac{\partial}{\partial \rho} [\cos \theta \sin \varphi F_\rho + \cos \theta \cos \varphi F_\varphi - \sin \theta F_\theta] \\ + \frac{1}{\rho} \cos \theta \cos \varphi \frac{\partial}{\partial \phi} [\cos \theta \sin \varphi F_\rho + \cos \theta \cos \varphi F_\varphi - \sin \theta F_\theta] \\ - \frac{\sin \theta}{\rho \sin \phi} \frac{\partial}{\partial \theta} [\cos \theta \sin \varphi F_\rho + \cos \theta \cos \varphi F_\varphi - \sin \theta F_\theta]$$

$$= \cos^2 \theta \sin^2 \varphi \frac{\partial F_\rho}{\partial \rho} + \cos^2 \theta \sin \varphi \cos \varphi \frac{\partial F_\varphi}{\partial \rho} - \cos \theta \sin \theta \sin \varphi \frac{\partial F_\theta}{\partial \rho} \\ + \frac{1}{\rho} \cos^2 \theta \cos \varphi [\cos \varphi F_\rho + \sin \varphi \frac{\partial F_\rho}{\partial \phi} - \sin \varphi F_\varphi + \cos \varphi \frac{\partial F_\varphi}{\partial \phi}] - \frac{\cos \theta \sin \theta \cos \varphi}{\rho} \frac{\partial F_\theta}{\partial \phi} \\ - \frac{\sin \theta}{\rho} [-\sin \theta F_\rho + \cos \theta \frac{\partial F_\rho}{\partial \theta}] - \frac{\sin \theta \cos \varphi}{\rho \sin \phi} [-\sin \theta F_\varphi + \cos \theta \frac{\partial F_\varphi}{\partial \theta}] \\ - \frac{\sin \theta}{\rho \sin \phi} [-\cos \theta F_\theta - \sin \theta \frac{\partial F_\theta}{\partial \theta}]$$

$$\textcircled{II} F_y = \sin \theta \sin \varphi \frac{\partial}{\partial \rho} [\sin \theta \sin \phi F_\rho + \sin \theta \cos \phi F_\phi + \cos \theta F_\theta] \\ + \frac{\sin \theta \cos \varphi}{\rho} \frac{\partial}{\partial \phi} [\sin \theta \sin \phi F_\rho + \sin \theta \cos \phi F_\phi + \cos \theta F_\theta] \\ + \frac{\cos \theta}{\rho \sin \phi} \frac{\partial}{\partial \theta} [\sin \theta \sin \phi F_\rho + \sin \theta \cos \phi F_\phi + \cos \theta F_\theta]$$

$$= \sin^2 \theta \sin^2 \phi \frac{\partial F_\rho}{\partial \rho} + \sin^2 \theta \sin \phi \cos \phi \frac{\partial F_\phi}{\partial \rho} + \sin \theta \cos \theta \sin \phi \frac{\partial F_\theta}{\partial \rho} \\ + \frac{\sin^2 \theta \cos \phi}{\rho} [\cos \phi F_\rho + \sin \phi \frac{\partial F_\rho}{\partial \phi} - \sin \phi F_\phi + \cos \phi \frac{\partial F_\phi}{\partial \phi}] + \frac{\cos \theta \sin \theta \cos \phi}{\rho} \frac{\partial F_\theta}{\partial \phi} \\ + \frac{\cos \theta}{\rho} [\cos \theta F_\rho + \sin \theta \frac{\partial F_\rho}{\partial \theta}] + \frac{\cos \theta \cos \phi}{\rho \sin \phi} [\cos \theta F_\phi + \sin \theta \frac{\partial F_\phi}{\partial \theta}] \\ + \frac{\cos \theta}{\rho \sin \phi} [-\sin \theta F_\theta + \cos \theta \frac{\partial F_\theta}{\partial \theta}]$$

$$\textcircled{III} F_z = \cos \varphi \frac{\partial}{\partial \rho} [\cos \varphi F_\rho - \sin \varphi F_\phi] - \frac{\sin \varphi}{\rho} \frac{\partial}{\partial \phi} [\cos \phi F_\rho - \sin \phi F_\phi] \\ = \cos^2 \varphi \frac{\partial F_\rho}{\partial \rho} - \sin \varphi \cos \varphi \frac{\partial F_\phi}{\partial \rho} + \frac{\sin^2 \varphi}{\rho} F_\rho - \frac{\sin \phi \cos \phi}{\rho} \frac{\partial F_\rho}{\partial \phi} + \frac{\partial}{\partial \phi} \\ + \frac{\sin \phi \cos \phi}{\rho} F_\phi + \frac{\sin^2 \varphi}{\rho} \frac{\partial F_\phi}{\partial \phi}$$

Proof Continued: sum ①  $F_x$  + ②  $F_y$  + ③  $F_z$  using appropriate trig-identity and cancelations,

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$$\begin{aligned}
 \nabla \cdot \vec{F} &= \frac{\partial F_\rho}{\partial \rho} [\cos^2 \theta \sin^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \phi] \\
 &+ \frac{\partial F_\phi}{\partial \rho} [\cos^2 \theta \sin \phi \cos \phi + \sin^2 \theta \sin \phi \cos \phi - \sin \phi \cos \phi] \\
 &+ \frac{\partial F_\theta}{\partial \rho} [-\cos \theta \sin \theta \sin \phi + \sin \theta \cos \theta \sin \phi] \\
 &+ \frac{\partial F_\rho}{\partial \phi} \left[ \frac{1}{\rho} \cos^2 \theta \cos \phi \sin \phi + \frac{1}{\rho} \sin^2 \theta \cos \phi \sin \phi - \frac{1}{\rho} \sin \phi \cos \phi \right] \\
 &+ \frac{\partial F_\phi}{\partial \phi} \left[ \frac{1}{\rho} \cos^2 \theta \cos^2 \phi + \frac{1}{\rho} \sin^2 \theta \cos^2 \phi + \frac{\sin^2 \phi}{\rho} \right] \\
 &+ \frac{\partial F_\theta}{\partial \theta} \left[ -\frac{\sin \theta \cos \theta}{\rho} + \frac{\sin \theta \cos \theta}{\rho} \right] \\
 &+ \frac{\partial F_\phi}{\partial \theta} \left[ -\frac{\sin \theta \cos \theta}{\rho} + \frac{\cos \theta \sin \theta}{\rho} \right] \\
 &+ \frac{\partial F_\theta}{\partial \theta} \left[ +\frac{\sin^2 \theta}{\rho \sin \phi} + \frac{\cos^2 \theta}{\rho \sin \phi} \right] \\
 &+ F_\rho \frac{1}{\rho} [\cos^2 \theta \cos^2 \phi + \sin^2 \theta + \sin^2 \theta \cos^2 \phi + \cos^2 \theta + \sin^2 \phi] \\
 &+ F_\phi \frac{1}{\rho} [-\cancel{\cos^2 \theta \cos \phi \sin \phi} + \frac{\sin^2 \theta \cos \phi}{\sin \phi} - \cancel{\sin^2 \theta \cos \phi \sin \phi} + \frac{\cos^2 \theta \cos \phi}{\sin \phi} + \cancel{\sin \phi \cos \phi}] \\
 &+ F_\theta \frac{1}{\rho} \left[ \frac{\cancel{\sin \theta \cos \theta}}{\sin \phi} - \frac{\cancel{\sin \theta \cos \theta}}{\sin \phi} \right] \\
 \hline
 &= \left( \frac{\partial F_\rho}{\partial \rho} + \frac{2}{\rho} F_\rho \right) + \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\cos \phi}{\rho \sin \phi} F_\phi + \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta}
 \end{aligned}$$

Then notice that

$$\begin{aligned}
 \frac{1}{\rho^2} \frac{\partial}{\partial \rho} [\rho^2 F_\rho] &= \frac{2\rho}{\rho^2} F_\rho + \frac{\rho^2}{\rho^2} \frac{\partial F_\rho}{\partial \rho} = \frac{2}{\rho} F_\rho + \frac{\partial F_\rho}{\partial \rho} \\
 \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} [\sin \phi F_\phi] &= \frac{\cos \phi}{\rho \sin \phi} F_\phi + \frac{1}{\rho} \frac{\sin \phi}{\sin \phi} \frac{\partial F_\phi}{\partial \phi} = \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} + \frac{\cos \phi}{\rho \sin \phi} F_\phi
 \end{aligned}$$

$$\therefore \nabla \cdot \vec{F} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} [\rho^2 F_\rho] + \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} [\sin \phi F_\phi] + \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta}$$

$$\text{Th}^2 / \nabla \times \vec{F} = \left( \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \mathbf{e}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} [r F_\theta] - \frac{\partial F_r}{\partial \theta} \right) \mathbf{e}_z$$

where  $\vec{F} = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_z \mathbf{e}_z$  in cylindrical coordinates.

Proof: the definition was given in CARTESIAN'S, focus on  $\mathbf{e}_r \cdot (\nabla \times \vec{F})$

$$\begin{aligned} (\nabla \times \vec{F}) \cdot \mathbf{e}_r &= (\nabla \times \vec{F}) \cdot (\cos \theta \hat{i} + \sin \theta \hat{j}) \\ &= \cos \theta (\nabla \times \vec{F}) \cdot \hat{i} + \sin \theta (\nabla \times \vec{F}) \cdot \hat{j} \\ &= \cos \theta [\partial_y F_z - \partial_z F_y] + \sin \theta [\partial_z F_x - \partial_x F_z] \\ &= \underbrace{\cos \theta \partial_y F_z}_{\text{I}} - \underbrace{\cos \theta \partial_z F_y}_{\text{II}} + \underbrace{\sin \theta \partial_z F_x}_{\text{III}} - \underbrace{\sin \theta \partial_x F_z}_{\text{IV}} \end{aligned}$$

We'll proceed to convert  $\text{I}, \text{II}, \text{III}, \text{IV}$  to cylindricals, need to convert the partial derivatives, we already have formulas for  $F_x, F_y, F_z$  in terms of  $F_r, F_\theta, F_z$  (see (\*) on (363)) and likewise for  $\partial_x, \partial_y, \partial_z$  in terms of  $\partial_r, \partial_\theta, \partial_z$  on (375). Calculate them,

$$\text{I} = \cos \theta \left[ \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \right] [F_z] = \cancel{\cos \theta \sin \theta \partial_r F_z} + \frac{\cos^2 \theta}{r} \partial_\theta F_z$$

$$\text{II} = -\cos \theta [\partial_z] [F_r \sin \theta + F_\theta \cos \theta] = -\cancel{\cos \theta \sin \theta \partial_z F_r} - \cos^2 \theta \partial_z F_\theta$$

$$\text{III} = \sin \theta [\partial_z] [F_r \cos \theta - F_\theta \sin \theta] = \cancel{\cos \theta \sin \theta \partial_z F_r} - \sin^2 \theta \partial_z F_\theta$$

$$\text{IV} = -\sin \theta \left[ \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \right] [F_z] = -\cancel{\sin \theta \cos \theta \partial_r F_z} + \frac{\sin^2 \theta}{r} \partial_\theta F_z$$

We see some encouraging cancellations, we find,

$$\boxed{(\nabla \times \vec{F}) \cdot \mathbf{e}_r = \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}}$$

One component down, two to go,

- Our strategy here is to work on the curl one component at a time, this makes the calculation more manageable.

$$\begin{aligned}
 (\nabla \times \vec{F}) \cdot \mathbf{e}_\theta &= (\nabla \times \vec{F}) \cdot [-\sin \theta \hat{i} + \cos \theta \hat{j}] \\
 &= -\sin \theta (\nabla \times \vec{F}) \cdot \hat{i} + \cos \theta (\nabla \times \vec{F}) \cdot \hat{j} \\
 &= \underbrace{-\sin \theta \partial_y F_z}_{\text{I}} + \underbrace{\sin \theta \partial_z F_y}_{\text{II}} + \underbrace{\cos \theta \partial_z F_x}_{\text{III}} - \underbrace{\cos \theta \partial_x F_z}_{\text{IV}}
 \end{aligned}$$

Proceed as before, convert to cylindrical coordinates,

$$\begin{aligned}
 \text{I} &= -\sin \theta [\sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta] [F_z] = -\sin^2 \theta \partial_r F_z - \frac{\sin \theta \cos \theta}{r} \partial_\theta F_z \\
 \text{II} &= \sin \theta [\partial_z] [F_r \sin \theta + F_\theta \cos \theta] = \sin^2 \theta \partial_z F_r + \sin \theta \cos \theta \partial_z F_\theta \\
 \text{III} &= \cos \theta [\partial_z] [F_r \cos \theta - F_\theta \sin \theta] = \cos^2 \theta \partial_z F_r - \sin \theta \cos \theta \partial_z F_\theta \\
 \text{IV} &= -\cos \theta [\cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta] [F_z] = -\cos^2 \theta \partial_r F_z + \frac{\sin \theta \cos \theta}{r} \partial_\theta F_z
 \end{aligned}$$

After a few cancellations in  $\text{I} + \text{II} + \text{III} + \text{IV}$ ,

$$\boxed{(\nabla \times \vec{F}) \cdot \mathbf{e}_\theta = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}}$$

And now the last component,

$$\begin{aligned}
 (\nabla \times \vec{F}) \cdot \mathbf{e}_z &= \partial_x F_y - \partial_y F_x \\
 &= [\cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta] [F_r \sin \theta + F_\theta \cos \theta] - \\
 &\quad - [\sin \theta \partial_r + \frac{1}{r} \cos \theta \partial_\theta] [F_r \cos \theta - F_\theta \sin \theta] \\
 &= \left[ \cancel{\sin \theta \cos \theta \partial_r F_r} + \cos^2 \theta \partial_r F_\theta - \frac{1}{r} \sin^2 \theta \partial_\theta F_r - \frac{1}{r} \sin \theta \cos \theta F_r} + \frac{1}{r} \cancel{\sin \theta \cos \theta \partial_\theta F_\theta} + \frac{1}{r} \sin^2 \theta F_\theta \right] \\
 &\quad - \left[ \cancel{\sin \theta \cos \theta \partial_r F_r} - \sin^2 \theta \partial_r F_\theta + \frac{1}{r} \cos^2 \theta \partial_\theta F_r - \frac{1}{r} \cancel{\cos \theta \sin \theta F_r} \right. \\
 &\quad \left. + \frac{1}{r} \cancel{\cos \theta \sin \theta \partial_\theta F_\theta} - \frac{\cos^2 \theta}{r} F_\theta \right] \\
 &= \partial_r F_\theta - \frac{1}{r} \partial_\theta F_r + \frac{1}{r} F_\theta = \boxed{\frac{1}{r} \frac{\partial}{\partial r} [r F_\theta] - \frac{1}{r} \frac{\partial F_r}{\partial \theta} = (\nabla \times \vec{F}) \cdot \mathbf{e}_z}
 \end{aligned}$$

So we find that

$$\nabla \times \vec{F} = [(\nabla \times \vec{F}) \cdot \mathbf{e}_r] \mathbf{e}_r + [(\nabla \times \vec{F}) \cdot \mathbf{e}_\theta] \mathbf{e}_\theta + [(\nabla \times \vec{F}) \cdot \mathbf{e}_z] \mathbf{e}_z$$

$$\boxed{\nabla \times \vec{F} = \left( \frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \mathbf{e}_\theta + \frac{1}{r} \left( \frac{\partial}{\partial r} [r F_\theta] - \frac{\partial F_r}{\partial \theta} \right) \mathbf{e}_z}$$

Bonus Point: Derive the formula for  $\nabla \times \vec{F}$  in Spherical coordinates, its on next page

# SUMMARY OF DIFFERENTIAL VECTOR CALCULUS

This table is borrowed from Thomas' Calculus 10<sup>th</sup> ed. Notice that  $e_r, e_\theta, e_\phi$  are denoted  $u_r, u_\theta, u_\phi$  and  $e_\rho$  is  $u_\rho$ . We have derived much of what is here. You should be equipt to derive the formulas for the laplacian  $\nabla^2 f$  given the calculations I've shown you in these notes.

## Vector Operator Formulas in Cartesian, Cylindrical, and Spherical Coordinates; Vector Identities

### Formulas for Grad, Div, Curl, and the Laplacian

	<b>Cartesian (x, y, z)</b> $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. $F_x, F_y,$ and $F_z$ are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.	<b>Cylindrical (r, <math>\theta,</math> z)</b> $\mathbf{u}_r, \mathbf{u}_\theta,$ and $\mathbf{k}$ are unit vectors in the directions of increasing r, $\theta,$ and z. $F_r, F_\theta,$ and $F_z$ are the scalar components of $\mathbf{F}(r, \theta, z)$ in these directions.	<b>Spherical (<math>\rho, \phi, \theta</math>)</b> $\mathbf{u}_\rho, \mathbf{u}_\phi,$ and $\mathbf{u}_\theta$ are unit vectors in the directions of increasing $\rho, \phi,$ and $\theta.$ $F_\rho, F_\phi,$ and $F_\theta$ are the scalar components of $\mathbf{F}(\rho, \phi, \theta)$ in these directions.
<b>Gradient</b>	$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{u}_\theta + \frac{\partial f}{\partial z} \mathbf{k}$	$\nabla f = \frac{\partial f}{\partial r} \mathbf{u}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{u}_\phi + \frac{1}{\rho \sin \phi} \frac{\partial f}{\partial \theta} \mathbf{u}_\theta$
<b>Divergence</b>	$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (rF_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$	$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} (\rho^2 F_\rho) + \frac{1}{\rho \sin \phi} \frac{\partial}{\partial \phi} (F_\phi \sin \phi) + \frac{1}{\rho \sin \phi} \frac{\partial F_\theta}{\partial \theta}$
<b>Curl</b>	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{1}{r} \mathbf{u}_r & \mathbf{u}_\theta & \frac{1}{r} \mathbf{k} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & F_\theta & F_z \end{vmatrix}$	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{u}_\rho & \mathbf{u}_\phi & \mathbf{u}_\theta \\ \rho^2 \sin \phi \frac{\partial}{\partial \rho} & \rho \sin \phi \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_\rho & \rho F_\phi & \rho \sin \phi F_\theta \end{vmatrix}$
<b>Laplacian</b>	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$	$\nabla^2 f = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial f}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 f}{\partial \theta^2}$

### Vector Triple Products

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

### Vector Identities for the Cartesian Form of the Operator $\nabla$

In the identities listed here,  $f(x, y, z)$  and  $g(x, y, z)$  are differentiable scalar functions and  $\mathbf{u}(x, y, z)$  and  $\mathbf{v}(x, y, z)$  are differentiable vector functions.

$$\nabla \cdot f\mathbf{v} = f\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla f = f\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla) f$$

$$\nabla \times f\mathbf{v} - f\nabla \times \mathbf{v} + \nabla f \times \mathbf{v}$$

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\nabla \times (\nabla f) = \mathbf{0}$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u})$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u})$$

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - (\nabla \cdot \nabla)\mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}$$

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = (\mathbf{v} \cdot \nabla)\mathbf{v} - \frac{1}{2} \nabla(\mathbf{v} \cdot \mathbf{v})$$

# PHYSICS' CONVENTIONS FOR VECTOR CALCULUS

Probably some of you will encounter the other type of spherical & cylindrical coordinates. In my experience physicists will always use these conventions where we have the following dictionary

Math	Physics
$\rho$	$r = \sqrt{x^2 + y^2 + z^2}$
$\Theta$	$\phi, 0 \leq \phi \leq 2\pi$
$\phi$	$\theta, 0 \leq \theta \leq \pi$
$r$	$s = \sqrt{x^2 + y^2}$

For your convenience I include a few results borrowed from David Griffith's EXCELLENT (!) text on Electromagnetics.

## SPHERICAL AND CYLINDRICAL COORDINATES

### Spherical

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad \begin{cases} \hat{x} = \sin \theta \cos \phi \hat{r} + \cos \theta \cos \phi \hat{\theta} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \theta \sin \phi \hat{r} + \cos \theta \sin \phi \hat{\theta} + \cos \phi \hat{\phi} \\ \hat{z} = \cos \theta \hat{r} - \sin \theta \hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \quad \begin{cases} \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \\ \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \end{cases}$$

### Cylindrical

$$\begin{cases} x = s \cos \phi \\ y = s \sin \phi \\ z = z \end{cases} \quad \begin{cases} \hat{x} = \cos \phi \hat{s} - \sin \phi \hat{\phi} \\ \hat{y} = \sin \phi \hat{s} + \cos \phi \hat{\phi} \\ \hat{z} = \hat{z} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \quad \begin{cases} \hat{s} = \cos \phi \hat{x} + \sin \phi \hat{y} \\ \hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y} \\ \hat{z} = \hat{z} \end{cases}$$

# PHYSICS' CONVENTIONS CONTINUED

There is nothing really new here, we just take what we did and change notation according to the dictionary. These formulas are taken from the cover of Griffith's E&M text.

## VECTOR DERIVATIVES

**Cartesian.**  $d\mathbf{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}; \quad d\tau = dx dy dz$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial x} \hat{x} + \frac{\partial t}{\partial y} \hat{y} + \frac{\partial t}{\partial z} \hat{z}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

*Curl:*  $\nabla \times \mathbf{v} = \left( \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{x} + \left( \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{y} + \left( \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{z}$

*Laplacian:*  $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

**Spherical.**  $d\mathbf{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}; \quad d\tau = r^2 \sin \theta dr d\theta d\phi$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial t}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial t}{\partial \phi} \hat{\phi}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta v_\theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi}$

*Curl:*  $\nabla \times \mathbf{v} = \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta v_\phi) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi}$

*Laplacian:*  $\nabla^2 t = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial t}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial t}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 t}{\partial \phi^2}$

**Cylindrical.**  $d\mathbf{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}; \quad d\tau = s ds d\phi dz$

*Gradient:*  $\nabla t = \frac{\partial t}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial t}{\partial \phi} \hat{\phi} + \frac{\partial t}{\partial z} \hat{z}$

*Divergence:*  $\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (s v_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$

*Curl:*  $\nabla \times \mathbf{v} = \left[ \frac{1}{s} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right] \hat{s} + \left[ \frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} \right] \hat{\phi} + \frac{1}{s} \left[ \frac{\partial}{\partial s} (s v_\phi) - \frac{\partial v_s}{\partial \phi} \right] \hat{z}$

*Laplacian:*  $\nabla^2 t = \frac{1}{s} \frac{\partial}{\partial s} \left( s \frac{\partial t}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$