Introduction to Vectors

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1 Euclidean Space

We denote the real numbers as $\mathbb{R} = \mathbb{R}^1$. Naturally $\mathbb{R}$ is identified with a line as we are taught in our previous study of the number line. The *Cartesian products* of $\mathbb{R}$ with itself give us natural models for the plane, 3 dimensional space and more abstractly[1] n-dimensional space:

Definition 1.1. *Space is a collection of points:*

(1.) two-dimensional space: is the set of all **ordered pairs** of real numbers:

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

(2.) three-dimensional space: is the set of all **ordered triples** of real numbers:

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

(3.) n-dimensional space: is the set of all **ordered n-tuples** of real numbers:

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}} = \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{R} \text{ for each } j \in \mathbb{N}_n\}$$

We say (x, y) is a **point** in $\mathbb{R}^2$. Likewise, (x, y, z) is a **point** in $\mathbb{R}^3$ and $(x_1, \dots, x_n)$ is a **point** in $\mathbb{R}^n$.

The word *ordered* indicates points are equal if and only if each and every entry in the point match.

Example 1.2. *Note that* $(1, 1, 2) \neq (1, 1, 3)$ *since* $2 \neq 3$.

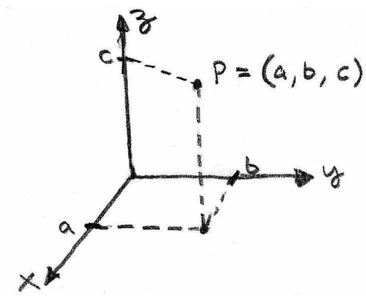
Example 1.3. *If* $(1, 2, 3, 4) = (a, b, c, d)$ *then* $a = 1, b = 2, c = 3$ *and* $d = 4$.

Conceptually, a *point* is something without any finite extent, it has no width, length or height. We characterize each point in terms of its components:

[1] picturing $\mathbb{R}^4$ is something we do with algebra rather than direct spatial intuition. Most people only have spatial intuition in two or three dimensions.

Definition 1.4. *point equality, components.*

In particular, $(v_1, v_2, \dots, v_n) = (w_1, w_2, \dots, w_n)$ iff $v_1 = w_1,\ v_2 = w_2,\ \dots, v_n = w_n$. In the context of $\mathbb{R}^2$ we say a is the **x-component** of (a, b) whereas b is the **y-component** of (a, b). In the context of $\mathbb{R}^3$ we say a is the **x-component** of (a, b, c) whereas b is the **y-component** of (a, b, c) and c is the **z-component** of (a, b, c). Generally, we say v_j the **j-th component** of $(v_1, v_2, \dots, v_n)$.



Sometimes the term **euclidean** is added to emphasize that we suppose distance between points is measured in the usual manner. Recall that in the one-dimensional case the distance between $x, y \in \mathbb{R}$ is given by the absolute value function; $d(x, y) = |y - x| = \sqrt{(y - x)^2}$. We define distance in n-dimensions by similar formulas:

Definition 1.5. *euclidean distance.*

(1.) distance in two-dimensional euclidean space: if $p_1 = (x_1, y_1), p_2 = (x_2, y_2) \in \mathbb{R}^2$ then the distance between points p_1 and p_2 is

$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

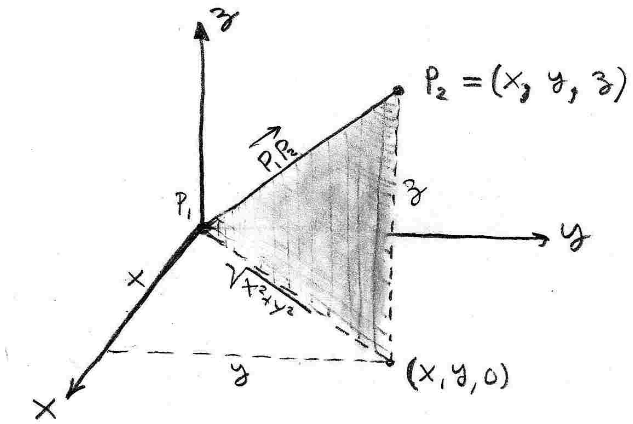
(2.) distance in three-dimensional euclidean space: if $p_1 = (x_1, y_1, z_1), p_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ then the distance between points p_1 and p_2 is

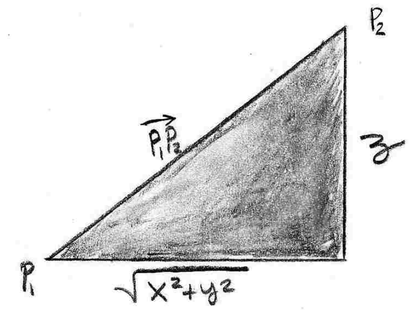
$$d(p_1, p_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

(3.) distance in n-dimensional euclidean space: if $a, b \in \mathbb{R}^n$ where $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ then the distance between points a and point b is

$$d(a, b) = \sqrt{\sum_{j=1}^{n} (b_j - a_j)^2} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + \cdots + (b_n - a_n)^2}.$$

It is simple to verify that the definition above squares with our traditional ideas about distance from previous math courses. In particular, notice these follow from the Pythagorean theorem applied to appropriate triangles. The picture below shows the three dimensional distance formula is consistent with the two dimensional formula.





2 Vectors in Two or Three Dimensions

The directed line-segment from P_1 to P_2 is denoted $\overrightarrow{P_1P_2}$ in the above diagram. Directed line-segments are called **vectors**. In contrast to points, a nonzero directed line-segment has an extent in one-direction.

Definition 2.1. *Two Dimensional Vectors:*

If $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ then $\overrightarrow{PQ}$ is the vector from P to Q given by:

$$\overrightarrow{PQ} = Q - P = \langle Q_1 - P_1, Q_2 - P_2 \rangle$$

If $P = (P_1, P_2)$ then $\vec{P} = \langle P_1, P_2 \rangle$; we write $\vec{P}$ for the vector from the origin to the point P. The arrow notation is used to emphasize the object is a directed-line segment. If $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$ then we define **addition** and **scalar multiplication** by $c \in \mathbb{R}$ as follows:

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle, \qquad \& \qquad c\vec{v} = \langle cv_1, cv_2 \rangle.$$

Furthermore, the **length** or **magnitude** of the vector $\vec{v} = \langle v_1, v_2 \rangle$ is defined by:

$$\|\vec{v}\| = v = \sqrt{v_1^2 + v_2^2}.$$

If $\vec{v} \neq 0$ then $\hat{v} = \frac{1}{v}\vec{v}$ and we call $\hat{v}$ the **direction-vector** or **unit-vector** of $\vec{v}$.

Notice $\vec{v} \neq 0$ can be written as the product of its magnitude and direction; $\vec{v} = v\hat{v}$. Moreover, our definition of vector length makes the length of $\overrightarrow{PQ}$ simply the distance from P to Q.

Example 2.2. *If* $P = (-2, 4)$ *and* $Q = (8, 7)$ *then* $\overrightarrow{PQ} = \langle 8 - (-2), 7 - 4 \rangle = \langle 10, 3 \rangle$*. The magnitude* $\|\overrightarrow{PQ}\| = \sqrt{10^2 + 3^2} = \sqrt{109}$ *is the distance from* P *to* Q*.*

Example 2.3. *Let* $\vec{A} = \langle 1, 3 \rangle$ *and* $\vec{B} = \langle -1, 0 \rangle$ *then*

$$\vec{A} + \vec{B} = \langle 1, 3 \rangle + \langle -1, 0 \rangle = \langle 1 - 1, 3 + 0 \rangle = \langle 0, 3 \rangle.$$

We find magnitudes $A = \sqrt{1^2 + 3^2} = \sqrt{10}$ *and* $B = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$*. Thus unit-vectors in the* $\vec{A}$ *and* $\vec{B}$ *directions are given by:*

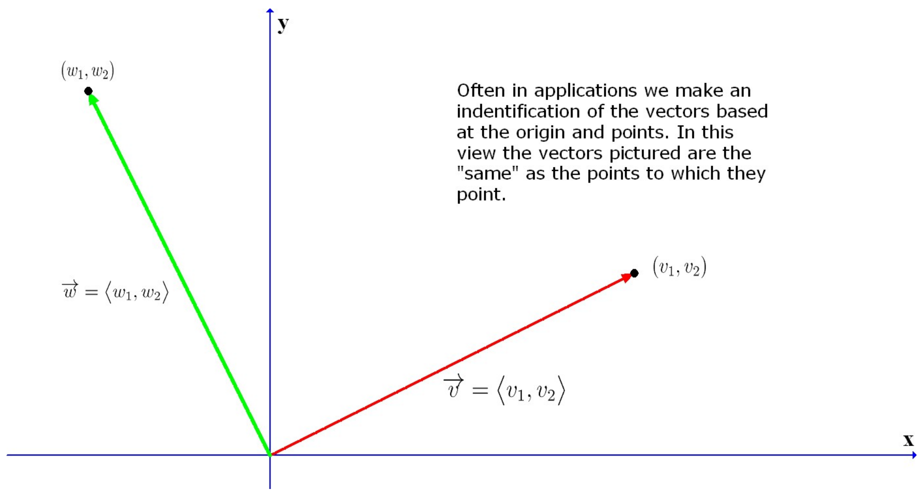
$$\widehat{A} = \frac{1}{A}\vec{A} = \frac{1}{\sqrt{10}}\langle 1, 3 \rangle = \langle 1/\sqrt{10}, 3/\sqrt{10} \rangle \qquad \& \qquad \widehat{B} = \frac{1}{B}\vec{B} = \langle -1, 0 \rangle.$$

Example 2.4. *Let* $\vec{A} = \langle 3, 4\rangle$ *then* $||A|| = \sqrt{3^2+4^2} = \sqrt{25} = 5$*. Therefore,* $\widehat{A} = \frac{1}{5}\langle 3, 4\rangle$*.*

Example 2.5. *Find a vector* $\vec{B}$ *with length 7 and the same direction as* $\vec{A} = \langle 1, 1\rangle$*. Observe* $\widehat{A} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle$ *hence* $\vec{B} = B\widehat{B} = \frac{7}{\sqrt{2}}\langle 1, 1\rangle$*.*

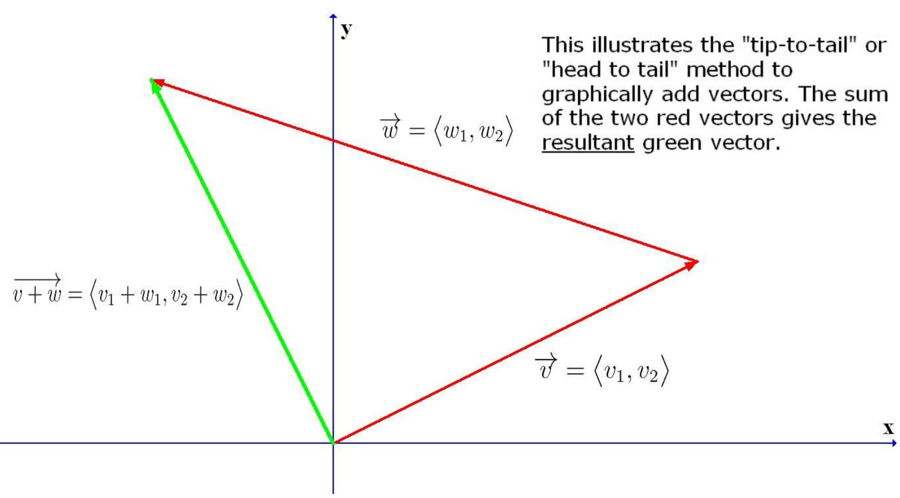
The solution given in the preceding example is *geometrically* motivated. An alternative *algebraic* approach would be to solve $\vec{B} = k\vec{A}$ and $B = 7$ for k. Both approaches have merit. I used the geometric approach to induce insight for the direction vector concept.

There is a natural correspondence between points and directed line-segments from the origin.

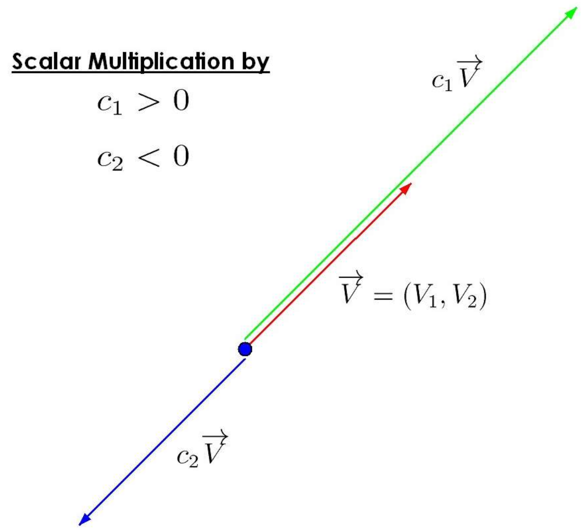


We will use the notation $\vec{p}$ for vectors throughout the remainder of these notes to emphasize the fact that $\vec{p}$ is a vector. Some texts use **bold** to denote vectors, but I prefer the over-arrow notation which is easily duplicated in hand-written work.

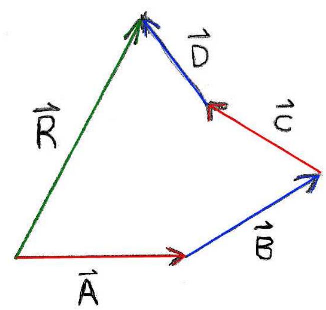
We add vectors geometrically by the tip-to-tail method as illustrated below.



Also, we rescale them by shrinking or stretching their length by a scalar multiple:

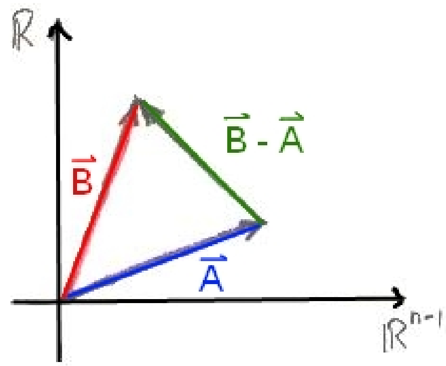


In the diagram below we illustrate the geometry behind the vector equation $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$.



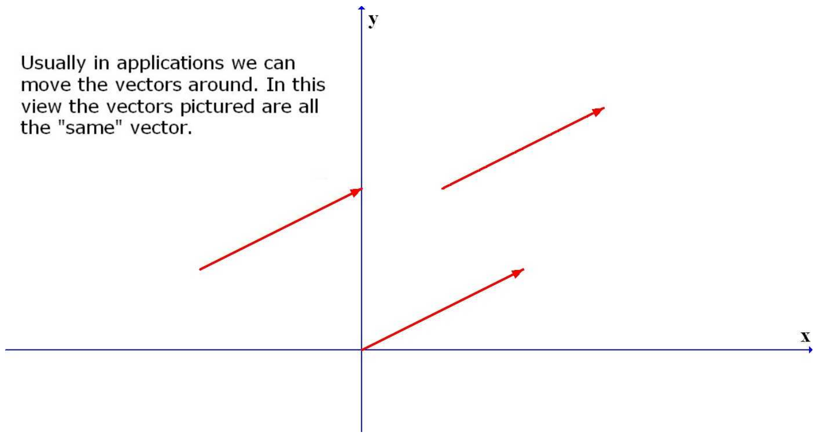
Continuing in this way we can add any finite number of vectors in the same *tip-2-tail* fashion. I used $\vec{R}$ in the diagram above because physicists often call the result of a vector addition the **resultant** vector.

It is sometimes useful to see how $\vec{A}$ and $\vec{B}$ are connected by the vector $\vec{B} - \vec{A}$:



Notice that $\vec{A} + (\vec{B} - \vec{A}) = \vec{B}$ by the tip-2-tail diagram above.

In most applications of vectors we are free to move a given vector around the plane in such a way that we maintain its direction and length:



If we wish to keep track of the base point of vectors then additional comment is required. I think of vectors as based at the origin unless there is reason from the context to think of them based elsewhere. For example, if I think about a force applied to a lever arm then I imagine the force as acting on its point of application.

I have mostly emphasized two-dimensional vectors up to this point, but we can easily extend the discussion to three-dimensional vectors.

Definition 2.6. *Three Dimensional Vectors:*

If $P = (P_1, P_2, P_3)$ and $Q = (Q_1, Q_2, Q_3)$ then $\overrightarrow{PQ}$ is the vector from P to Q given by:

$$\overrightarrow{PQ} = Q - P = \langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \rangle$$

If $P = (P_1, P_2, P_3)$ then $\vec{P} = \langle P_1, P_2, P_3 \rangle$; we write $\overrightarrow{P}$ for the vector from the origin to the point P. The arrow notation is used to emphasize the object is a directed-line segment. If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$ then we define **addition** and **scalar multiplication** by $c \in \mathbb{R}$ as follows:

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle, \qquad \& \qquad c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle.$$

Furthermore, the **length** or **magnitude** of the vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is defined by:

$$||\vec{v}|| = v = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

If $\vec{v} \neq 0$ then $\hat{v} = \frac{1}{v}\vec{v}$ and we call $\hat{v}$ the **direction-vector** or **unit-vector** of $\vec{v}$.

The example below illustrates a nice trick for constructing vectors.

Example 2.7. *If* $\vec{A} = \langle 1, 2, -2 \rangle$ *then* $A = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$ *thus* $\widehat{A} = \langle 1/3, 2/3, -2/3 \rangle$. *If you want to construct a vector* $\vec{B}$ *of length* 18 *in the direction of* $\vec{A}$ *then simply use* $\vec{B} = 18\widehat{A} = 18\langle 1/3, 2/3, -2/3 \rangle = \langle 6, 12, -12 \rangle$.

3 Decomposing Vectors into Components

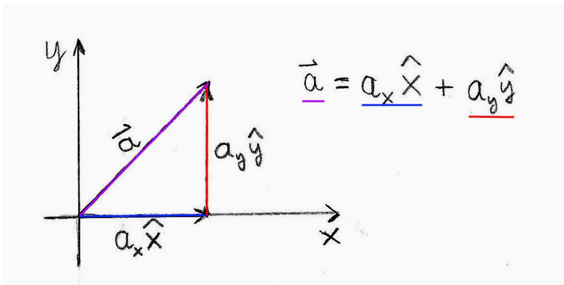
For $\mathbb{R}^2$, **define**[2] $\widehat{x} = \langle 1, 0\rangle$ **and** $\widehat{y} = \langle 0, 1\rangle$ hence:

$$\begin{aligned} \langle a, b\rangle &= \langle a, 0\rangle + \langle 0, b\rangle \\ &= a\langle 1, 0\rangle + b\langle 0, 1\rangle \\ &= a\,\widehat{x} + b\,\widehat{y} \end{aligned}$$

Definition 3.1. *vector and scalar components of two-vectors.*

The **vector component** of $\langle a, b\rangle$ in the x-direction is simply $a\,\widehat{x}$ whereas the **vector component** of $\langle a, b\rangle$ in the y-direction is simply $b\,\widehat{y}$. In contrast, a, b are the **scalar components** of $\langle a, b\rangle$ in the x, y-directions respective.

Scalar components are scalars whereas vector components are vectors. These are entirely different objects if $n \neq 1$, please keep clear this distinction in your mind. Notice that the vector components are what we use to reproduce a given vector by the tip-to-tail sum:



Example 3.2. *Let $\vec{v} = \langle 2, -3\rangle$ then $2\,\widehat{x}$ is the x-vector component of $\vec{v}$ and 2 is the scalar component of $\vec{v}$ in the x-direction. Likewise, $-3\,\widehat{y}$ is the y-vector component of $\vec{v}$.*

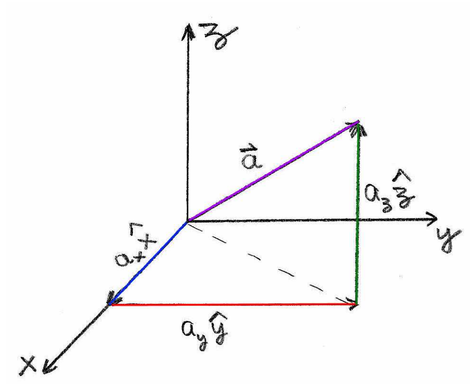
Example 3.3. Problem: *find a vector $\vec{A}$ of length 10 which has $6\,\widehat{x}$ as its x-vector component.*
Solution: *we seek to find y such that $\vec{A} = \langle 6, y\rangle$ has length 10. Notice $A^2 = 6^2 + y^2 = 10^2$ hence $y^2 = 64$ which gives $y = \pm 8$. We find two vectors which solve this problem, $\vec{A} = \langle 6, \pm 8\rangle$.*

For $\mathbb{R}^3$ we define the following notation[3]: $\widehat{x} = \langle 1, 0, 0\rangle$, $\widehat{y} = \langle 0, 1, 0\rangle$, **and** $\widehat{z} = \langle 0, 0, 1\rangle$ hence:

$$\begin{aligned} \langle a, b, c\rangle &= \langle a, 0, 0\rangle + \langle 0, b, 0\rangle + \langle 0, 0, c\rangle \\ &= a\langle 1, 0, 0\rangle + b\langle 0, 1, 0\rangle + c\langle 0, 0, 1\rangle \\ &= a\,\widehat{x} + b\,\widehat{y} + c\,\widehat{z} \end{aligned}$$

[2] I should mention that often $\widehat{i}$ is used for $\widehat{x}$ and $\widehat{j}$ is used for $\widehat{y}$, I choose this less popular notation because it is far more descriptive than the traditional notation, I trust the reader can adapt in future studies if need be. Incidentially, another popular notation in linear algebra is that $e_1 = (1, 0)$ and $e_2 = (0, 1)$ in the context of $\mathbb{R}^2$.

[3] yes, in the context of $\mathbb{R}^3$ we have $\widehat{x} = \widehat{i} = e_1 = (1, 0, 0)$ whereas $\widehat{y} = \widehat{j} = e_2 = (0, 1, 0)$ and $\widehat{z} = \widehat{k} = e_3 = (0, 0, 1)$, notice the number of zeros depends on the context.



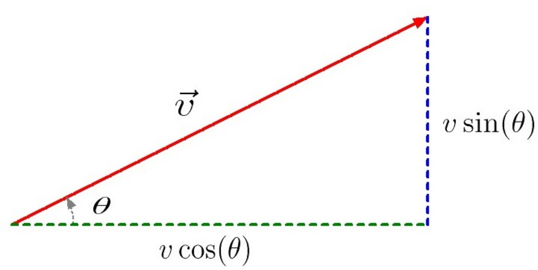
Definition 3.4. *vector and scalar components of three-vectors.*

The **vector components** of $\langle a, b, c\rangle$ are: $a\,\widehat{x}$ in the x-direction, $b\,\widehat{y}$ in the y-direction and $c\,\widehat{z}$ in the z-direction. In contrast, a, b, c are the **scalar components** of $\langle a, b, c\rangle$ in the x, y, z-directions respective.

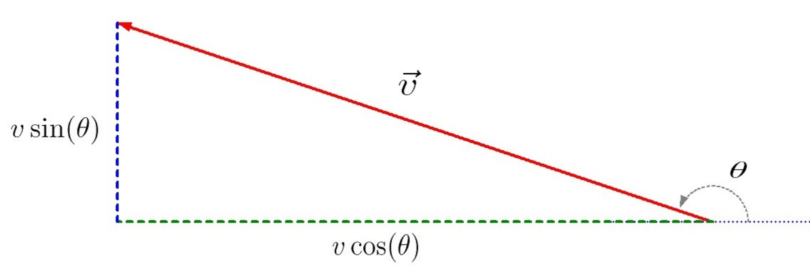
Example 3.5. *Observe,* $\langle 1, 2, 3\rangle = \langle 1, 0, 0\rangle + \langle 0, 2, 0\rangle + \langle 0, 0, 3\rangle = \widehat{x} + 2\,\widehat{y} + 3\,\widehat{z}$.

Example 3.6. Problem: *find a vector $\vec{A}$ of length 5 which has $2\,\widehat{y}$ as its y-vector component and $-3\,\widehat{z}$ as its z-vector component.*
Solution: *we seek to find x such that $\vec{A} = \langle x, 2, -3\rangle$ has length 5. Notice $A^2 = x^2 + 2^2 + (-3)^2 = 5^2$ hence $x^2 = 12$ which gives $x = \pm\sqrt{12}$. We find two solutions $\vec{A} = \langle \pm\sqrt{12}, 2 - 3\rangle$.*

We conclude this section by discussing how trigonometry is often applied to the study of vectors in the plane. It is not uncommon to be faced with vectors which are described by a length and a direction in the plane. In such a case we need to rely on trigonometry to *break-down* the vector into it's Cartesian components.



Example 3.7. *Suppose a vector $\vec{v}$ has a length $v = 5$ at $\theta = \pi/3$ then $v\cos\theta = 5\cos(\pi/3) = \frac{5}{2}$ and $v\sin\theta = 5\sin(\pi/3) = \frac{5\sqrt{3}}{2}$. Therefore, in view of the diagram above this example,* $\vec{v} = \left\langle \dfrac{5}{2}, \dfrac{5\sqrt{3}}{2} \right\rangle$.



Example 3.8. *Suppose a vector* $\vec{v}$ *has a length* $v = 2$ *at* $\theta = 5\pi/6$ *then* $v\cos\theta = 2\cos(5\pi/6) = -\sqrt{3}$ *and* $v\sin\theta = 2\sin(5\pi/6) = 1$. *Therefore,* $\vec{v} = \langle -\sqrt{3}, 1\rangle$. *Notice,* $\theta = 5\pi/6$ *is in Quadrant II and our result is consistent with the figure above.*

In general, for $\vec{v} = \langle v_1, v_2\rangle \neq 0$ we can describe $\vec{v}$ in terms of its magnitude $v = \sqrt{v_1^2 + v_2^2}$ and standard angle θ. Place $\vec{v}$ at the origin then we know from our discussion of polar coordinates that

$$v_1 = v\cos\theta \qquad \& \qquad v_2 = v\sin\theta$$

Consequently, $\vec{v} = \langle v\cos\theta, v\sin\theta\rangle = v\langle\cos\theta, \sin\theta\rangle$. However, we also know $\vec{v} = v\hat{v}$ hence we find:

$$\boxed{\hat{v} = \langle\cos\theta, \sin\theta\rangle}$$

Notice, $\|\langle\cos\theta, \sin\theta\rangle\| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$. Thus, $\langle\cos\theta, \sin\theta\rangle$ is a unit-vector. Of course, this should not be surprising since we began this course with the unit-circle which is nothing more than the collection of all points a distance of one-unit from the origin. We already know $(\cos\theta, \sin\theta)$ is a typical point on the unit-circle. Now we're simply observing $\langle\cos\theta, \sin\theta\rangle$ is the vector of length one which points from the origin to the point $(\cos\theta, \sin\theta)$.

Example 3.9. *Suppose* $\vec{v}$ *has length* 37 *and is directed at the standard angle* $\theta = 295^o$. *Then the unit-vector in the direction of* $\vec{v}$ *is simply* $\hat{v} = \langle\cos(295^o), \sin(295^o)\rangle$.

4 The Dot Product

The dot-product of two vectors gives a number which relates to whether the given pair of vectors is parallel or perpendicular or somewhere in-between. We will soon discover that the dot-product allows us an elegant algebraic means to calculate the angle between vectors in Euclidean space[4].

Definition 4.1. *dot product.*

The **dot-product** is a useful operation on vectors. In $\mathbb{R}^2$ we define,

$$\langle V_1, V_2\rangle \bullet \langle W_1, W_2\rangle = V_1W_1 + V_2W_2.$$

In $\mathbb{R}^3$ we define,

$$\langle V_1, V_2, V_3\rangle \bullet \langle W_1, W_2, W_3\rangle = V_1W_1 + V_2W_2 + V_3W_3.$$

It is important to notice that the dot-product takes in two *vectors* and outputs a *scalar*. It has a number of interesting properties which we will often use:

Example 4.2. *Let* $\vec{A} = \langle 3, 4\rangle$ *and* $\vec{B} = \langle 7, -2\rangle$. *We calculate,*

$$\vec{A} \bullet \vec{B} = \langle 3, 4\rangle \bullet \langle 7, -2\rangle = (3)(7) + (4)(-2) = 13.$$

Example 4.3. *Let* $\vec{A} = \langle 1, 2, 3\rangle$ *and* $\vec{B} = \langle 1, -1, 5\rangle$. *We calculate,*

$$\vec{A} \bullet \vec{B} = \langle 1, 2, 3\rangle \bullet \langle 1, -1, 5\rangle = 1 - 2 + 15 = 14.$$

[4] I'll limit our discussion to two or three dimensional vectors, but we can easily extend the discussion to $\mathbb{R}^n$ for $n \geq 4$

Proposition 4.4. *properties of the dot-product.*

let $\vec{A}, \vec{B}, \vec{C} \in \mathbb{R}^n$ be vectors and $c \in \mathbb{R}$

(1.) commutative: $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$,

(2.) distributive: $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$,

(3.) distributive: $(\vec{A} + \vec{B}) \bullet \vec{C} = \vec{A} \bullet \vec{C} + \vec{B} \bullet \vec{C}$,

(4.) scalars factor out: $\vec{A} \bullet (c\vec{B}) = (c\vec{A}) \bullet \vec{B} = c\vec{A} \bullet \vec{B}$,

(5.) non-negative: $\vec{A} \bullet \vec{A} \geq 0$,

(6.) no null-vectors: $\vec{A} \bullet \vec{A} = 0$ iff $\vec{A} = 0$.

Proofs of the above claims are not difficult. Let me illustrate two for you:

Example 4.5. *Why is $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$ for two dimensional vectors ? Simple, notice that since real numbers commute we can make the following calculation:*

$$\vec{A} \bullet \vec{B} = A_1 B_1 + A_2 B_2 = B_1 A_1 + B_2 A_2 = \vec{B} \bullet \vec{A}.$$

We find the dot-product is commutative.

Example 4.6. *Why is $\vec{A} \bullet \vec{A} \geq 0$ for three dimensional vectors ? Simple, notice that*

$$\vec{A} \bullet \vec{A} = A_1 A_1 + A_2 A_2 + A_3 A_3 = A_1^2 + A_3^2 + A_3^2 \geq 0$$

since the sum of squares of real numbers is non-negative.

Notice the formula in the example above is familar, we can write: $\boxed{||\vec{A}|| = \sqrt{\vec{A} \bullet \vec{A}}}$ since $||\vec{A}|| = A = \sqrt{A_1^2 + A_2^2 + A_3^2}$. Notice that the dot-product of a vector with itself is the square of the magnitude of the vector; $\vec{A} \bullet \vec{A} = A^2$. This observation works in two or three dimensions[5]

Proposition 4.7. *properties of the norm (also known as length of vector).*

Suppose $\vec{A}, \vec{B} \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

(1.) absolute value of scalar factors out: $||c\vec{A}|| = |c|||\vec{A}||$,

(2.) triangle inequality: $||\vec{A} + \vec{B}|| \leq ||\vec{A}|| + ||\vec{B}||$,

(3.) Cauchy-Schwarz inequality: $|\vec{A} \bullet \vec{B}| \leq ||\vec{A}|| \, ||\vec{B}||$.

(4.) non-negative: $||\vec{A}|| \geq 0$,

(5.) only zero vector has zero length: $||\vec{A}|| = 0$ iff $\vec{A} = 0$.

[5] ok, so to be honest, this concept works in far greater generality than just $\mathbb{R}^2$ or $\mathbb{R}^3$ or even $\mathbb{R}^n$, we can look at music in terms of its Fourier decomposition and a given musical note has a distance which can be understood in terms of Pythagorean like formula which sums all the lengths of the harmonics forming the musical note.

Dot-products of the coordinate unit-vectors in $\mathbb{R}^3$ are very easy to remember:

$$\widehat{x} \bullet \widehat{x} = 1, \qquad \widehat{y} \bullet \widehat{y} = 1, \qquad \widehat{z} \bullet \widehat{z} = 1,$$

$$\widehat{x} \bullet \widehat{y} = 0, \qquad \widehat{x} \bullet \widehat{z} = 0, \qquad \widehat{y} \bullet \widehat{z} = 0.$$

We can calculate dot-products by using the properties of the dot-product paired with the results above. For example:

Example 4.8. *Let* $\vec{A} = \widehat{x} - 2\,\widehat{y} + 3\,\widehat{z}$ *and* $\vec{B} = 5\,\widehat{x} + 9\,\widehat{z}$

$$\begin{aligned}
\vec{A} \bullet \vec{B} &= (\,\widehat{x} - 2\,\widehat{y} + 3\,\widehat{z}) \bullet (5\,\widehat{x} + 9\,\widehat{z}) \\
&= 5\,\widehat{x} \bullet \widehat{x} - 10\,\widehat{y} \bullet \widehat{x} + 15\,\widehat{z} \bullet \widehat{x} + 9\,\widehat{x} \bullet \widehat{z} - 18\,\widehat{y} \bullet \widehat{z} + 27\,\widehat{z} \bullet \widehat{z} \\
&= 5(1) - 10(0) + 15(0) + 9(0) - 18(0) + 27(1) \\
&= 32.
\end{aligned}$$

It is easier to use the $\langle a, b, c\rangle$ notation for examples such as the one above, but the notation $\widehat{x}$, $\widehat{y}$, $\widehat{z}$ (or the equivalent $\widehat{i}$, $\widehat{j}$, $\widehat{k}$ used in many other texts) is often used to emphasize that the object in consideration is a **vector**.

Definition 4.9. *orthogonal vectors.*

We say $\vec{A}$ is **orthogonal** to $\vec{B}$ iff $\vec{A} \bullet \vec{B} = 0$. A set of vectors which is both orthogonal and all of unit length is said to be an **orthonormal set** of vectors.

Orthonormality makes for nice formulas. Let $\vec{V} = \langle V_1, V_2\rangle \in \mathbb{R}^2$ and calculate,

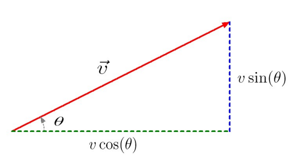
$$\vec{V} \bullet \widehat{x} = (V_1\,\widehat{x}_1 + V_2\,\widehat{x}_2) \bullet \widehat{x}_1 = V_1\,\widehat{x}_1 \bullet \widehat{x}_1 + V_2\,\widehat{x}_2 \bullet \widehat{x}_1 = \delta_{11}V_1 + \delta_{12}V_2 = V_1$$

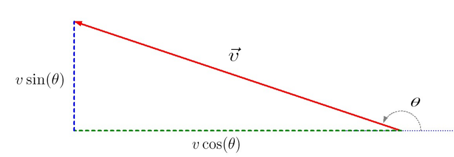
$$\vec{V} \bullet \widehat{x}_2 = (V_1\,\widehat{x}_1 + V_2\,\widehat{x}_2) \bullet \widehat{x}_2 = V_1\,\widehat{x}_1 \bullet \widehat{x}_2 + V_2\,\widehat{x}_2 \bullet \widehat{x}_2 = \delta_{12}V_1 + \delta_{22}V_2 = V_2$$

This means we can use the dot-product to select the scalar components of a given vector.

$$\vec{V} = \langle \vec{V} \bullet \widehat{x}_1,\ \vec{V} \bullet \widehat{x}_2\rangle = \left(\ \vec{V} \bullet \widehat{x}_1\ \right)\widehat{x}_1 + \left(\ \vec{V} \bullet \widehat{x}_2\ \right)\widehat{x}_2.$$

Let's pause to make a connection to the standard angle θ and the cartesian components[6].





Note that $\vec{V} = \cos(\theta)\,\widehat{x} + \sin(\theta)\,\widehat{y}$ and $\vec{V} = \left(\ \vec{V} \bullet \widehat{x}\ \right)\widehat{x} + \left(\ \vec{V} \bullet \widehat{y}\ \right)\widehat{y}$. It follows that:

$$\cos(\theta) = \vec{V} \bullet \widehat{x} \qquad \text{and} \qquad \sin(\theta) = \vec{V} \bullet \widehat{y}.$$

You could use these equations to define the standard angle in retrospect. Alternatively, we can use the standard angle for a two-dimensional vector to derive its unit-vector: observe

$$\vec{A} = \langle A\cos\theta,\ A\sin\theta\rangle = A\langle \cos\theta,\ \sin\theta\rangle \ \ \&\ \ \vec{A} = A\widehat{A} \ \ \Rightarrow \ \ \boxed{\widehat{A} = \langle \cos\theta,\ \sin\theta\rangle}.$$

[6] I did discuss this earlier, but it probably doesn't hurt to cover it again

Example 4.10. *If $\vec{v}$ has length 10 and $\theta = -\pi/6$ then $\widehat{v} = \langle \cos(\pi/6), -\sin(\pi/6) \rangle = \langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \rangle$.*

Notice, I did not need to use $v = 10$ to find the unit-vector in the $\vec{v}$-direction. The standard angle and the direction-vector are equivalent in the **two**-dimensional context.

Example 4.11. *If $\vec{v}$ has length 10 and $\theta = -\pi/6$ then $\widehat{v} = \langle \cos(\pi/6), -\sin(\pi/6) \rangle = \langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \rangle$.*

4.1 Angle Measure via Dot Products

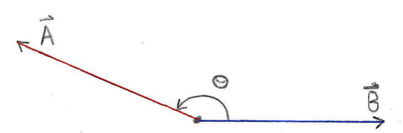
The study of geometry involves lengths and angles of shapes. We have all the tools we need to define the angle θ between nonzero vectors

Definition 4.12. *angle between a pair of vectors.*

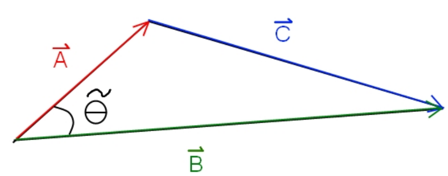
Let $\vec{A}, \vec{B}$ be nonzero vectors in $\mathbb{R}^n$. We define the angle between $\vec{A}$ and $\vec{B}$ by

$$\theta = \cos^{-1}\left[\frac{\vec{A} \bullet \vec{B}}{||\vec{A}||\,||\vec{B}||}\right].$$

Note nonzero vectors $\vec{A}, \vec{B}$ have $||\vec{A}|| \neq 0$ and $||\vec{B}|| \neq 0$ thus the Cauchy-Schwarz inequality $|\vec{A} \bullet \vec{B}| \leq ||\vec{A}||\,||\vec{B}||$ implies $\frac{\vec{A} \bullet \vec{B}}{||\vec{A}||\,||\vec{B}||} \leq 1$. It follows that the argument of the inverse cosine is within its domain. Moreover, since the standard inverse cosine has range $[0, \pi]$ it follows the angle which is given by the formula above is the smallest angle *between* the vectors. Of course, if θ is the angle between $\vec{A}, \vec{B}$ then geometry clearly indicates $2\pi - \theta$ is the angle on the other side of the θ vertex. I think a picture helps:



The careful reader will question how I know the formula really recovers the idea of angle that we have previously used in our studies of trigonometry. All I have really argued thus far is that the formula for θ is reasonable. Examine the triangle formed by $\vec{A}, \vec{B}$ and $\vec{C} = \vec{B} - \vec{A}$. Notice that $\vec{A} + \vec{C} = \vec{B}$. Picture $\vec{A}$ and $\vec{B}$ as adjacent sides to an angle $\tilde{\theta}$ which has opposite side $\vec{C}$. Let the lengths of $\vec{A}, \vec{B}, \vec{C}$ be A, B, C respective.



Applying[7] the **Law of Cosines** to the triangle above yields

$$C^2 = A^2 + B^2 - 2AB\cos(\tilde{\theta}).$$

[7] if you had Math 131 with me then you proved the Law of Cosines in one of your first Problem Sets.

Solve for $\tilde{\theta}$,

$$\tilde{\theta} = \cos^{-1}\left[\frac{A^2 + B^2 - C^2}{2AB}\right]$$

Is this consistent, does $\theta = \tilde{\theta}$? Choose coordinates[8] which place the vectors $\vec{A}, \vec{B}, \vec{C}$ are in the xy-plane and let $\vec{A} = \langle A_1, A_2\rangle, \vec{B} = \langle B_1, B_2\rangle$ hence $\vec{C} = \langle B_1 - A_1, B_2 - A_2\rangle$ we calculate

$$C^2 = (B_1 - A_1)^2 + (B_2 - A_2)^2 = B_1^2 - 2A_1B_1 + A_1^2 + B_2^2 - 2A_2B_2 + A_2^2$$

Thus, $C^2 = A^2 + B^2 - 2\vec{A} \bullet \vec{B}$ and we find:

$$\tilde{\theta} = \cos^{-1}\left[\frac{2\vec{A} \bullet \vec{B}}{2AB}\right] = \cos^{-1}\left[\frac{\vec{A} \bullet \vec{B}}{||\vec{A}||\,||\vec{B}||}\right] = \theta.$$

Thus, we find the algebraic definition of angle agrees with the two-dimensional geometric concept we've explored throughout this courses. Moreover, we discover a geometrically lucid formula for the dot-product:

$$\boxed{\vec{A} \bullet \vec{B} = ||\vec{A}||\,||\vec{B}|| \cos(\theta)}$$

or if we denote $\vec{A} = A\widehat{A}$ and $\vec{B} = B\widehat{B}$ then

$$\boxed{\vec{A} \bullet \vec{B} = AB\cos(\theta).}$$

The connection between this formula and the definition is nontrivial and is essentially equivalent to the Law of Cosines. This means that this is a powerful formula which allows deep calculation of geometrically non-obvious angles through the machinery of vectors. Notice:

$$\boxed{\text{If } \vec{A}, \vec{B} \text{ are nonzero orthogonal vectors then the angle between them is } \pi/2.}$$

this observation is an immediate consequence of the the definition of orthogonal vectors and the fact $\cos(\pi/2) = 0$. We find that orthogonal vectors are in fact perpendicular (which is a known term from geometry). In addition,

$$\boxed{\text{If } \vec{A}, \vec{B} \text{ are parallel vectors then } \vec{A} \bullet \vec{B} = AB \text{ and } \theta = 0.}$$

likewise,

$$\boxed{\text{If } \vec{A}, \vec{B} \text{ are antiparallel vectors then } \vec{A} \bullet \vec{B} = -AB \text{ and } \theta = \pi.}$$

The dot-product gives us a concrete method to test for whether two vectors point in the same direction, opposite directions or are purely perpendiular.

Example 4.13. *Let $\vec{A} = \langle -5, 3, 7\rangle$ and $\vec{B} = \langle 6, -8, 2\rangle$. Are these vectors parallel, antiparallel or orthogonal? We can calculate the dot-product to answer this question. Observe,*

$$\vec{A} \bullet \vec{B} = \langle -5, 3, 7\rangle \bullet \langle 6, -8, 2\rangle = -30 - 24 + 14 = -40 \neq 0.$$

Thus, we know $\vec{A}$ and $\vec{B}$ are not orthogonal. Furthermore, they cannot be parallel as the dot-product's sign indicates they point in directions more than 90^o oppposed. Are they antiparallel?

$$-AB = -\sqrt{25 + 9 + 49}\sqrt{36 + 64 + 4} = -\sqrt{8932} \neq -40$$

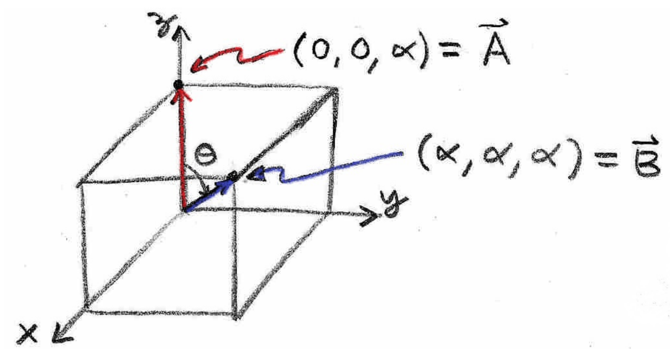
[8] even in the context of $\mathbb{R}^n$ we can place $\vec{A}, \vec{B}$ and $\vec{B} - \vec{A}$ in a particular plane, this argument actually extends to n-dimensions provided you accept the Law of Cosines is known in any plane

Therefore, the given pair of vectors is neither parallel, antiparallel nor orthogonal. Of course, we could have ascertained all these comments by simply calculating the angle between the given vectors:

$$\theta = \cos^{-1}\left(\frac{\vec{A} \bullet \vec{B}}{AB}\right) = \cos^{-1}\left(\frac{-40}{\sqrt{8932}}\right) = 115.5^o.$$

I hope the reader can forgive me for abusing notation and sometimes using radian and other times angle measure. When I use degree measure it is primarily to emphasize geometric content.

Example 4.14. *Consider a cube of side-length α. What is the angle between the interior diagonal of the cube and the edge of the cube? We place the cube at the origin and envision the diagonal from $(0,0,0)$ to (α,α,α). The edge goes from $(0,0,0)$ to $(0,0,\alpha)$. Let us label the diagonal and edge by $\vec{B}$ and $\vec{A}$ respectively:*



Observe $A = \alpha$ and $B = \alpha\sqrt{3}$ whereas $\vec{A} \bullet \vec{B} = \alpha^2$. We find $\frac{\vec{A} \bullet \vec{B}}{AB} = \frac{\alpha^2}{\alpha^2\sqrt{3}} = \frac{1}{\sqrt{3}}$. Thus $\cos\theta = \frac{1}{\sqrt{3}}$ hence $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approxeq 54.74^o$.

The reason the angle is not 45^o in the example above is that the vectors $\vec{A}$ and $\vec{B}$ lie on the edge and diagonal of a nonsquare-rectangle. The larger point here: **use vectors** to escape wrong intuition in three-dimensional geometry. The mathematics of vectors allows us to solve problems step-by-step which defy direct geometric methods.

Example 4.15. *Judging the colinearity of two vectors is important to physics. The work done by a force is maximized when the force is applied over a displacement which is precisely parallel to the force. On the other hand, the work done by a perpendicular force is zero. The dot-product captures all these concepts in a nice neat formula: the work W done by a constant force $\vec{F}$ applied to an object undergoing a displacement $\triangle\vec{r}$ is given by $\boxed{W = \vec{F} \bullet \triangle\vec{r}}$. For example, if a force $\vec{F} = \langle 1,1,1\rangle\ N$ is applied to a particle displaced under $\triangle\vec{r} = \langle 1,-2,4\rangle m$ then the work done is:*

$$W = \vec{F} \bullet \triangle\vec{r} = \langle 1,1,1\rangle\ N \bullet \langle 1,-2,4\rangle m = 3\,Nm = 3\,J.$$

Here N is the unit of force called a Newton, m is the unit of distance called a meter and J is the unit of energy called a Joule.

The formula in the example above only works because the force is constant. If the force varies with position then we need methods of calculus to calculate the work.

Example 4.16. *Let $\vec{F} = \langle 10, 18, -6\rangle$ be a constant force field. Find the work done by the given force field on an object which moves from $(2,3,0)$ to $(4,9,15)$. It turns out[9] that the same work is*

[9]for reasons we only complete understand towards the conclusion of this course!

done by the given force no matter which path is taken from $(2,3,0)$ *to* $(4,9,15)$*. So, we assume a linear path for our convenience and note* $\triangle\vec{r} = (4,9,15) - (2,3,0) = \langle 2,6,15\rangle$*. The dot-product of the force and displacement give the work done by the force:*

$$W = \vec{F} \bullet \triangle\vec{r} = \langle 10, 18, -6\rangle \bullet \langle 2, 6, 15\rangle = 20 + 108 - 90 = 38.$$

Naturally, we could assume the points are given in terms of meters and the force in Newtons then our answer above would indicate $38\,J$ *of work done. Of course, you could use other units. I leave further discussion of this matter for your physics course(s).*

There are many dot-products in basic physics.

Example 4.17. *If* $\vec{v}$ *is the velocity of a mass* m *then the kinetic energy is given by* $K = \frac{1}{2}m\vec{v} \bullet \vec{v}$*.*

Example 4.18. *Or, if* $\vec{v}$ *is the velocity of a mass* m *and* $\vec{F}$ *is the net-force on* m *then the power developed by* $\vec{F}$ *is given by* $P = \vec{v} \bullet \vec{F}$*.*

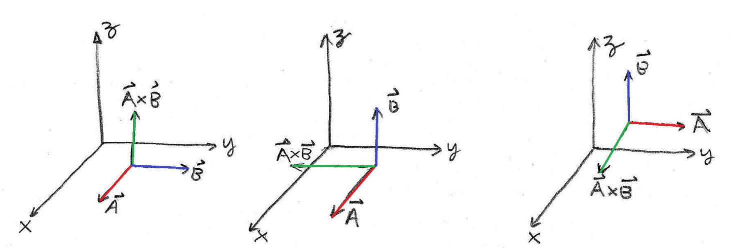
Example 4.19. *If* $\vec{J}$ *is a constant current density then* $\vec{J} \bullet (A\widehat{n})$ *gives the current flowing through an area* A *with unit-normal* $\widehat{n}$*.*

Example 4.20. *If* $\vec{E}$ *is the electric field then* $\vec{E} \bullet (A\widehat{n})$ *gives the electric flux through an area* A *with unit-normal* $\widehat{n}$*.*

Example 4.21. *If* $\vec{B}$ *is the magnetic field then* $\vec{B} \bullet (A\widehat{n})$ *gives the magnetic flux through an area* A *with unit-normal* $\widehat{n}$*.*

5 The Cross Product

We saw that the dot-product gives us a natural way to check if a pair of vectors is orthogonal. You should remember: $\vec{A}, \vec{B}$ are orthogonal iff $\vec{A} \bullet \vec{B} = 0$. We turn to a slightly different goal in this section: given a pair of nonzero, nonparallel vectors $\vec{A}, \vec{B}$ how can we find another vector $\vec{A} \times \vec{B}$ which is perpendicular to both $\vec{A}$ and $\vec{B}$? Geometrically, in $\mathbb{R}^3$ it's not too hard to picture it:



My intent in this section is to motivate the standard formula for this product and to prove some of the standard properties of this cross product. These calculations are special to $\mathbb{R}^3$. The material from here to Definition 5.1 is simply to give some insight into where the mysterious formula for the cross product arises. If you insist on remaining unmotivated, feel free to skip to the definition.

Suppose $\vec{A}, \vec{B}$ are nonzero, nonparallel vectors in $\mathbb{R}^3$. I'll calculate conditions on $\vec{A} \times \vec{B}$ which insure it is perpendicular to both $\vec{A}$ and $\vec{B}$. Let's denote $\vec{A} \times \vec{B} = \vec{C}$. We should expect $\vec{C}$ is some function of the components of $\vec{A}$ and $\vec{B}$. I'll use $\vec{A} = \langle A_1, A_2, A_3\rangle$ and $\vec{B} = \langle B_1, B_2, B_3\rangle$ whereas $\vec{C} = \langle C_1, C_2, C_3\rangle$

$$0 = \vec{C} \bullet \vec{A} = C_1A_1 + C_2A_2 + C_3A_3$$

$$0 = \vec{C} \bullet \vec{B} = C_1 B_1 + C_2 B_2 + C_3 B_3$$

Suppose $A_1 \neq 0$, then we may solve $0 = \vec{C} \bullet \vec{A}$ as follows,

$$C_1 = -\frac{A_2}{A_1} C_2 - \frac{A_3}{A_1} C_3$$

Suppose $B_1 \neq 0$, then we may solve $0 = \vec{C} \bullet \vec{B}$ as follows,

$$C_1 = -\frac{B_2}{B_1} C_2 - \frac{B_3}{B_1} C_3$$

It follows, given the assumptions $A_1 \neq 0$ and $B_1 \neq 0$,

$$\frac{A_2}{A_1} C_2 + \frac{A_3}{A_1} C_3 = \frac{B_2}{B_1} C_2 + \frac{B_3}{B_1} C_3$$

Multiply by $A_1 B_1$ to obtain:

$$B_1 A_2 C_2 + B_1 A_3 C_3 = A_1 B_2 C_2 + A_1 B_3 C_3$$

Thus,

$$(A_1 B_2 - B_1 A_2) C_2 + (A_1 B_3 - B_1 A_3) C_3 = 0$$

One solution is simply $C_2 = A_3 B_1 - A_1 B_3$ and $C_3 = A_1 B_2 - B_1 A_2$ and it follows that $C_1 = A_2 B_3 - B_2 A_3$. Of course, generally we could have vectors which are nonzero and yet have $A_1 = 0$ or $B_1 = 0$. The point of the calculation is not to provide a general derivation. Instead, my intent is simply to show you how you might be led to make the following definition:

Definition 5.1. *cross product.*

Let $\vec{A}, \vec{B}$ be vectors in $\mathbb{R}^3$. The vector $\vec{A} \times \vec{B}$ is called the **cross product** of $\vec{A}$ with $\vec{B}$ and is defined by

$$\vec{A} \times \vec{B} = \langle\ A_2 B_3 - A_3 B_2,\ A_3 B_1 - A_1 B_3,\ A_1 B_2 - A_2 B_1\ \rangle.$$

We say $\vec{A}$ cross $\vec{B}$ is $\vec{A} \times \vec{B}$.

It is a simple exercise to verify that

$$\vec{A} \bullet (\vec{A} \times \vec{B}) = 0 \qquad \text{and} \qquad \vec{B} \bullet (\vec{A} \times \vec{B}) = 0.$$

Both of these identities should be utilized to check your calculation of a given cross product. Let's think about the formula for the cross product a bit more. We have

$$\vec{A} \times \vec{B} = (A_2 B_3 - A_3 B_2)\, \widehat{x}_1 + (A_3 B_1 - A_1 B_3)\, \widehat{x}_2 + (A_1 B_2 - A_2 B_1)\, \widehat{x}_3$$

distributing,

$$\vec{A} \times \vec{B} = A_2 B_3\, \widehat{x}_1 - A_3 B_2\, \widehat{x}_1 + A_3 B_1\, \widehat{x}_2 - A_1 B_3\, \widehat{x}_2 + A_1 B_2\, \widehat{x}_3 - A_2 B_1\, \widehat{x}_3$$

The pattern is clear. Each term has indices $1, 2, 3$ without repeat and we can generate the signs via the antisymmetric symbol ϵ_{ijk} which is defined be zero if any indices are repeated and

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1 \qquad \text{whereas} \qquad \epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1.$$

With this convenient shorthand we find the nice formula for the cross product that follows:

$$\boxed{\vec{A} \times \vec{B} = \sum_{i,j,k=1}^{3} A_i B_j \epsilon_{ijk} \, \widehat{x}_k}$$

Interestingly the Cartesian unit-vectors $\widehat{x}_1$, $\widehat{x}_2$, $\widehat{x}_3$ satisfy the simple relation:

$$\boxed{\widehat{x}_i \times \widehat{x}_j = \sum_{k=1}^{3} \epsilon_{ijk} \, \widehat{x}_k,}$$

which is just a fancy way of saying that

$$\boxed{\widehat{x} \times \widehat{y} = \widehat{z}, \qquad \widehat{y} \times \widehat{z} = \widehat{x}, \qquad \widehat{z} \times \widehat{x} = \widehat{y}}$$

There are many popular mnemonics to remember these. The basic properties of the cross product together with these formula allow us to quickly calculate some cross products (see Example 5.7)

Proposition 5.2. *basic properties of the cross product.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$

(1.) anticommutative: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$,

(2.) distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$,

(3.) distributive: $(\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}$,

(4.) scalars factor out: $\vec{A} \times (c\vec{B}) = (c\vec{A}) \times \vec{B} = c\vec{A} \times \vec{B}$,

Remark: I left these proofs here to help you understand why I care about the funny ϵ_{ijk} notation. I omitted the more sophisticated proofs later in this section for the sake of brevity. You can look at my Calculus III notes for all the missing details if you're curious.

Proof: once more, the proof is easy with the right notation. Begin with (1.),

$$\vec{A} \times \vec{B} = \sum_{i,j,k=1}^{3} A_i B_j \epsilon_{ijk} \, \widehat{x}_k = -\sum_{i,j,k=1}^{3} A_i B_j \epsilon_{jik} \, \widehat{x}_k = -\sum_{i,j,k=1}^{3} B_j A_i \epsilon_{jik} \, \widehat{x}_k = -\vec{B} \times \vec{A}.$$

The key observation was that $\epsilon_{ijk} = -\epsilon_{jik}$ for all i, j, k. If you don't care for this argument then you could also give the brute-force argument below:

$$\begin{aligned} \vec{A} \times \vec{B} &= \langle\ A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1\ \rangle \\ &= -\langle\ A_3B_2 - A_2B_3,\ A_1B_3 - A_3B_1,\ A_2B_1 - A_1B_2\ \rangle \\ &= -\langle\ B_2A_3 - B_3A_2,\ B_3A_1 - B_1A_3,\ B_1A_2 - B_2A_1\ \rangle \\ &= -\vec{B} \times \vec{A}. \end{aligned}$$

Next, to prove (2.) we once more use the compact notation,

$$\begin{aligned}
\vec{A} \times (\vec{B} + \vec{C}) &= \sum_{i,j,k=1}^{3} A_i(B_j + C_j)\epsilon_{ijk}\,\widehat{x}_k \\
&= \sum_{i,j,k=1}^{3} (A_i B_j \epsilon_{ijk}\,\widehat{x}_k + A_i C_j \epsilon_{ijk}\,\widehat{x}_k) \\
&= \sum_{i,j,k=1}^{3} A_i B_j \epsilon_{ijk}\,\widehat{x}_k + \sum_{i,j,k=1}^{3} A_i C_j \epsilon_{ijk}\,\widehat{x}_k \\
&= \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.
\end{aligned}$$

The proof of (3.) follows naturally from (1.) and (2.), note:

$$(\vec{A} + \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} + \vec{B}) = -\vec{C} \times \vec{A} - \vec{C} \times \vec{B} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C}.$$

I leave the proof of (4.) to the reader. □

The properties above basically say that the cross product behaves the same as the usual addition and multiplication of numbers with the caveat that the order of factors matters. If we switch the order then we must include a minus due to the anticommutivity of the cross product.

Example 5.3. *Consider,* $\vec{A} \times \vec{A} = -\vec{A} \times \vec{A}$ *hence* $2\vec{A} \times \vec{A} = 0$. *Consequently,* $\vec{A} \times \vec{A} = 0$.

We often use the result of the example above in future work. For example:

Example 5.4. *Let* $\vec{A}, \vec{B}$ *be two three dimensional vectors. Simplify* $(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B})$.

$$\begin{aligned}
(\vec{A} - \vec{B}) \times (\vec{A} + \vec{B}) &= \vec{A} \times (\vec{A} + \vec{B}) - \vec{B} \times (\vec{A} + \vec{B}) \\
&= \vec{A} \times \vec{A} + \vec{A} \times \vec{B} - \vec{B} \times \vec{A} - \vec{B} \times \vec{B} \\
&= 2\vec{A} \times \vec{B}.
\end{aligned}$$

There are a number of popular tricks to remember the rule for the cross-product. Let's look at a particular example a couple different ways:

Example 5.5. *Let* $\vec{A} = \langle 1, 2, 3 \rangle$ *and* $\vec{B} = \langle 4, 5, 6 \rangle$. *Calculate* $\vec{A} \times \vec{B}$ *directly from the definition:*

$$\begin{aligned}
\vec{A} \times \vec{B} &= \langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle \\
&= \langle 2(6) - 3(5),\ 3(4) - 1(6),\ 1(5) - 2(4) \rangle \\
&= \langle -3,\ 6,\ -3 \rangle.
\end{aligned}$$

There are at least 6 opportunities to make an error in the calculation of a cross product. It is important to check our work before we continue. A simple check is that $\vec{A}$ *and* $\vec{B}$ *must be orthogonal to the cross product. We can easily calculate that* $\vec{A} \bullet (\vec{A} \times \vec{B}) = 0$ *and* $\vec{B} \bullet (\vec{A} \times \vec{B}) = 0$. *This almost guarantees we have correctly calculated the cross product.*

The other popular method to calculate the cross product is based on an abuse of notation with the **determinant**. A determinant can be calculated for any $n \times n$ matrix A. The significance of the determinant is that it gives the signed-volume of the n-piped with edges taken as the rows or columns of A. A simple formula for the determinant in general is given by:

$$\det(A) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1 i_2 \dots i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

Ok, I jest. This formula takes a bit of work to really appreciate. So, typically we introduce the determinant in terms of the **expansion by minors** due to Laplace. We begin with a 2×2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Next, a 3×3 can be calculated by an expansion across the top-row,

$$\begin{aligned} \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} &= a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg). \end{aligned}$$

The minus sign in the middle term is part of the structure of the expansion. It is also one of the most common places where students make an error in their computation of a determinant [10]. We can express the cross product by following the patterns introduced for the 3×3 case. In particular,

$$\begin{aligned} \langle A_1, A_2, A_3 \rangle \times \langle B_1, B_2, B_3 \rangle &= \det \begin{bmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix} \\ &= \widehat{x}(A_2B_3 - A_3B_2) - \widehat{y}(A_1B_3 - A_3B_1) + \widehat{z}(A_1B_2 - A_2B_1) \\ &= (A_2B_3 - A_3B_2)\,\widehat{x} + (A_3B_1 - A_1B_3)\,\widehat{y} + (A_1B_2 - A_2B_1)\,\widehat{z}. \end{aligned}$$

I invite the reader to verify this aligns perfectly with Definition 5.1.

Example 5.6. *Let $\vec{A} = \langle 1, 2, 3 \rangle$ and $\vec{B} = \langle 4, 5, 6 \rangle$. Calculate $\vec{A} \times \vec{B}$ via the determinant formula:*

$$\begin{aligned} \langle 1, 2, 3 \rangle \times \langle 4, 5, 6 \rangle &= det \begin{bmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \\ &= \widehat{x}(2(6) - 3(5)) - \widehat{y}(1(6) - 3(4)) + \widehat{z}(1(5) - 2(4)) \\ &= -3\,\widehat{x} + 6\,\widehat{y} - 3\,\widehat{z}. \end{aligned}$$

This result matches $\vec{A} \times \vec{B} = \langle -3, 6, -3 \rangle$ as we found in Example 5.5.

Technically, this formula is not really a determinant since genuine determinants are formed from matrices filled with objects of the same type. In the hybrid expression above we actually have one row of vectors and two rows of scalars. That said, I include it here since many people use it and

[10] If we go on, a 4×4 matrix breaks into a signed-weighted-sum of 4 determinants of 3×3 submatrices. More generally, an $n \times n$ martrix has a determinant which requires on the order of $n!$ arithmetic steps. You'll learn more in your linear algebra course, I merely initiate the discussion here. Fortunately, we only need $n = 2$ and $n = 3$ for the majority of the topics in this course.

I also have found it useful in past calculations. If nothing else at least it helps you learn what a determinant is. That is a calculation which is worthwhile since determinants have application far beyond mere cross products. We can also use the basic relations:

$$\widehat{x} \times \widehat{y} = \widehat{z}, \qquad \widehat{y} \times \widehat{z} = \widehat{x}, \qquad \widehat{z} \times \widehat{x} = \widehat{y}$$

and the properties of cross products to work out cross products algebraically:

Example 5.7. *Let* $\vec{A} = \widehat{x} + 2\,\widehat{y} + 3\,\widehat{z}$ *and* $\vec{B} = 4\,\widehat{x} + 5\,\widehat{y} + 6\,\widehat{z}$. *Calculate* $\vec{A} \times \vec{B}$ *as follows:*

$$\begin{aligned}
\vec{A} \times \vec{B} &= \widehat{x} \times (4\,\widehat{x} + 5\,\widehat{y} + 6\,\widehat{z}) + 2\,\widehat{y} \times (4\,\widehat{x} + 5\,\widehat{y} + 6\,\widehat{z}) + 3\,\widehat{z} \times (4\,\widehat{x} + 5\,\widehat{y} + 6\,\widehat{z}) \\
&= \widehat{x} \times (5\,\widehat{y} + 6\,\widehat{z}) + 2\,\widehat{y} \times (4\,\widehat{x} + 6\,\widehat{z}) + 3\,\widehat{z} \times (4\,\widehat{x} + 5\,\widehat{y}) \\
&= 5\,\widehat{x} \times \widehat{y} + 6\,\widehat{x} \times \widehat{z} + 8\,\widehat{y} \times \widehat{x} + 12\,\widehat{y} \times \widehat{z} + 12\,\widehat{z} \times \widehat{x} + 15\,\widehat{z} \times \widehat{y} \\
&= 5\,\widehat{z} + 6(-\widehat{y}) + 8(-\widehat{z}) + 12\,\widehat{x} + 12\,\widehat{y} + 15(-\widehat{x}) \\
&= -3\,\widehat{x} + 6\,\widehat{y} - 3\,\widehat{z}.
\end{aligned}$$

This agrees with the conclusion of the previous pair of examples.

The calculation above is probably not the quickest for the example at hand here, but it is faster for other computations. For example:

Example 5.8. *Suppose* $\vec{A} = \langle 1, 2, 3 \rangle$ *and* $\vec{B} = \widehat{x}$ *then*

$$\begin{aligned}
\vec{A} \times \vec{B} &= (\widehat{x} + 2\,\widehat{y} + 3\,\widehat{z}) \times \widehat{x} \\
&= 2\,\widehat{y} \times \widehat{x} + 3\,\widehat{z} \times \widehat{x} \\
&= -2\,\widehat{z} + 3\,\widehat{y}.
\end{aligned}$$

Example 5.9. *Let* $\vec{A} = \langle 3, 2, 4 \rangle$ *and* $\vec{B} = \langle 1, -2, -3 \rangle$. *We calculate,*

$$\begin{aligned}
\vec{A} \times \vec{B} &= det \begin{bmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{bmatrix} \\
&= \widehat{x}(-6 + 8) - \widehat{y}(-9 - 4) + \widehat{z}(-6 - 2) \\
&= 2\,\widehat{x} + 13\,\widehat{y} - 8\,\widehat{z}.
\end{aligned}$$

As a check on our computation, note that $\vec{A} \bullet (\vec{A} \times \vec{B}) = 0$ *and* $\vec{B} \bullet (\vec{A} \times \vec{B}) = 0$.

There are a number of identities which connect the dot and cross products. These formulas require considerable effort if you choose to use brute-force proof methods.

Proposition 5.10. *nontrivial properties of the cross product.*

Let $\vec{A}, \vec{B}, \vec{C}$ be vectors in $\mathbb{R}^3$

(1.) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \bullet \vec{C})\vec{B} - (\vec{A} \bullet \vec{B})\vec{C}$

(2.) Jacobi Identity: $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0,$

(3.) cyclicity of triple product: $\vec{A} \bullet (\vec{B} \times \vec{C}) = \vec{B} \bullet (\vec{C} \times \vec{A}) = \vec{C} \bullet (\vec{A} \times \vec{B})$

(4.) Lagrange's identity: $||\vec{A} \times \vec{B}||^2 = ||\vec{A}||^2\,||\vec{B}||^2 - \left[\ \vec{A} \bullet \vec{B}\ \right]^2$

Use Lagrange's identity together with $\vec{A} \bullet \vec{B} = AB\cos(\theta)$,

$$||\vec{A} \times \vec{B}||^2 = A^2B^2 - [AB\cos(\theta)]^2 = A^2B^2(1 - \cos^2(\theta)) = A^2B^2\sin^2(\theta)$$

It follows there exists some unit-vector $\hat{\mathbf{n}}$ such that

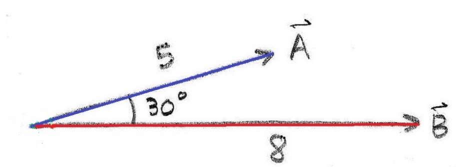
$$\boxed{\vec{A} \times \vec{B} = AB\sin(\theta)\hat{\mathbf{n}}}$$

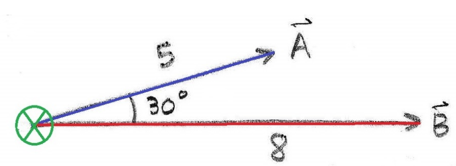
The direction of the unit-vector $\hat{\mathbf{n}}$ is conveniently indicated by the **right-hand-rule**. I typically perform the rule as follows:

1. point fingers of **right hand** in direction $\vec{A}$
2. cross the fingers into the direction of $\vec{B}$
3. the direction your thumb points is the approximate direction of $\hat{\mathbf{n}}$

I say *approximate* because $\vec{A} \times \vec{B}$ is strictly perpendicular to both $\vec{A}$ and $\vec{B}$ whereas your thumb's direction is a little ambiguous. But, it does pick one side of the plane in which the vectors $\vec{A}$ and $\vec{B}$ reside.

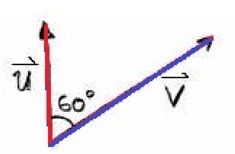
Example 5.11. . *Consider $\vec{A}$ and $\vec{B}$ pictured below. Find the magnitude of $\vec{A} \times \vec{B}$ and describe its direction. We produce the right picture by the right hand rule:*

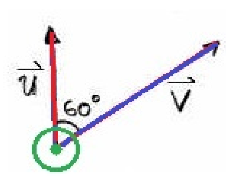




Note $||\vec{A} \times \vec{B}|| = AB\sin\theta = 40\sin 30^o = 20$. By the right hand rule, we find the direction of $\vec{A} \times \vec{B}$ is into the page. The $\otimes$ symbol intends we visualize the vector as an arrow pointing into the page.

Example 5.12. *Let $\vec{u}$ and $\vec{v}$ be as pictured below with $u = 5$ and $v = 4\sqrt{3}$. Find the magnitude and direction vector of $\vec{v} \times \vec{u}$: we use the right hand rule to produce the diagram on the right:*



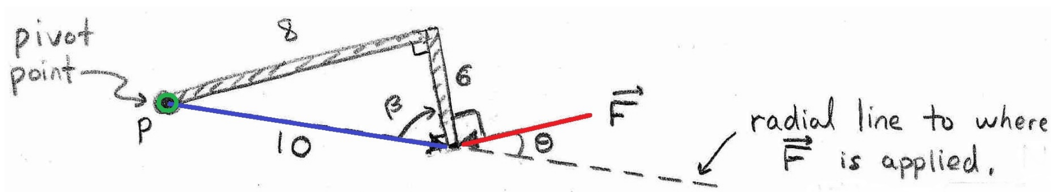


Note $||\vec{v}\times\vec{u}|| = vu\sin\theta = 20\sqrt{3}\sin 60^o = 30$. *By the right hand rule, we find the direction of* $\vec{v}\times\vec{u}$ *is out of the page. The* $\odot$ *symbol indicates a vector pointing out of the page.*

The cross product is also found in many physical applications. I give two common examples.

Example 5.13. *In rotational physics the direction of a rotation is taken to be the axis of the rotation where a counter-clockwise-rotation (CCW) is taken to be positive. To decide which direction is CCW we grip the rotation axis and point our right-hand's thumb in the direction of the positive axis. Once that grip is made the fingers on the right hand encircle the axis in the CCW-rotational sense. A torque on a body allowed to rotate around some axis makes it rotate. In particular, if* $\vec{r}$ *is the* **moment arm** *and* $\vec{F}$ *is the force applied then* $\vec{\tau} = \vec{r}\times\vec{F}$ *is the torque produced by* $\vec{F}$ *relative to the given axis.*

Problem: *Find the torque due to the force* $\vec{F}$ *pictured below. Describe the rotation produced as CCW or CW given the axis of rotation points out of the page*



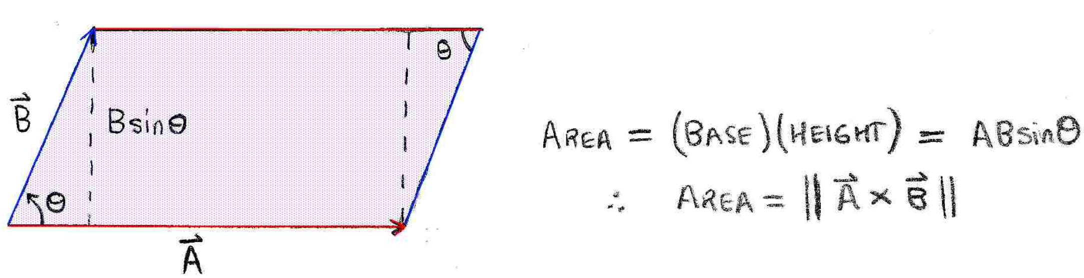
Solution: *Imagine moving* $\vec{F}$ *to* P *while maintaining its direction. This is called* **parallel transport**. *We calculate* $\vec{r}\times\vec{F}$ *as if they are both attached to* P. *The right hand rule reveals the direction is into the page* ($\otimes$) *and we can determine* θ *from trigonometry and the given geometric data. Observe* θ *is also interior to the triangle at* P *hence* $\sin\theta = \frac{6}{10}$. *Also, by pythagorean theorem,* $r = \sqrt{8^2+6^2} = 10$. *Therefore,* $\tau = rF\sin\theta = 6F$. *The direction of the torque is* $\otimes$ *which indicates a CW-rotation relative to the outward pointing axis through* P.

Example 5.14. *Another important application of the cross product to physics is the Lorentz force law. If a charge* q *has velocity* $\vec{v}$ *and travels through a magnetic field* $\vec{B}$ *then the force due to the electromagnetic interaction between* q *and the field is* $\vec{F} = q\vec{v}\times\vec{B}$.

Finally, we should investigate how the dot and cross product give nice formulas for the area of a parallelogram or the volume of a parallel piped. Suppose $\vec{A}, \vec{B}$ give the sides of a parallelogram.

$$Area = ||\ \vec{A}\times\vec{B}\ ||$$

The picture below shows why the formula above is true:



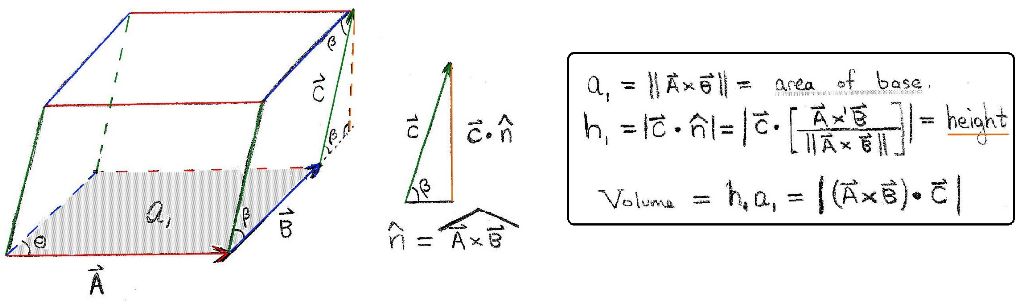
On the other hand, if $\vec{A}, \vec{B}, \vec{C}$ give the corner-edges of a parallelogram then[11]

$$Volume = \left| \ \vec{A} \bullet (\vec{B} \times \vec{C}) \ \right|$$

These formulas are connected by the following thought: the volume subtended by $\vec{A}, \vec{B}$ and the unit-vector $\hat{\mathbf{n}}$ from $\vec{A} \times \vec{B} = AB\sin(\theta)\hat{\mathbf{n}}$ is equal to the area of the parallelogram with sides $\vec{A}, \vec{B}$. Algebraically:

$$\left| \ \hat{\mathbf{n}} \bullet (\vec{A} \times \vec{B}) \ \right| = \left| \ \hat{\mathbf{n}} \bullet (AB\sin(\theta)\hat{\mathbf{n}}) \right| = |AB\sin(\theta)| = ||\vec{A} \times \vec{B}||.$$

The picture below shows why the triple product formula is valid.



Example 5.15. *Find the volume of a parallel-piped with edge-vectors* $\vec{A} = \langle 0, 1, 1 \rangle$ *and* $\vec{B} = \langle 1, 0, 0 \rangle$ *and* $\vec{C} = \langle 0, 1, 0 \rangle$. *We calculate* $\vec{B} \times \vec{C} = \widehat{x} \times \widehat{y} = \widehat{z}$. *Therefore, the volume of the solid is* $V = \vec{A} \bullet (\vec{B} \times \vec{C}) = \langle 0, 1, 1 \rangle \bullet \widehat{z} = 1$.

Moreover, given this geometric interpretation we find a new proof (up to a sign) for the cyclic property. By the symmetry of the edges it follows that $| \ \vec{A} \bullet (\vec{B} \times \vec{C}) \ | = | \ \vec{B} \bullet (\vec{C} \times \vec{A}) \ | = | \ \vec{C} \bullet (\vec{A} \times \vec{B}) \ |$. We should find the same volume no matter how we label width, depth and height.

[11] we could also show that $\det[\vec{A}|\vec{B}|\vec{C}] = \vec{A} \bullet (\vec{B} \times \vec{C})$ thus the determinant of the three edge vectors of a parallel piped yields its signed-volume. We can define the sign of the volume to be positive if the edges are ordered to respect the right hand rule. Respecting the right hand rule means the angle between $\vec{A} \times \vec{B}$ and $\vec{C}$ is less than 90^o.