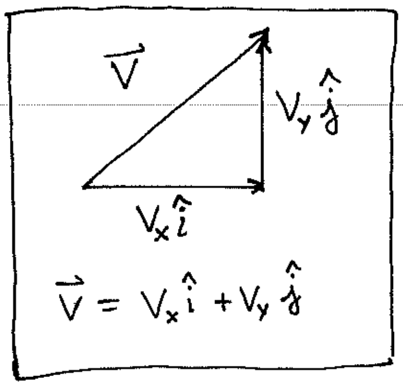
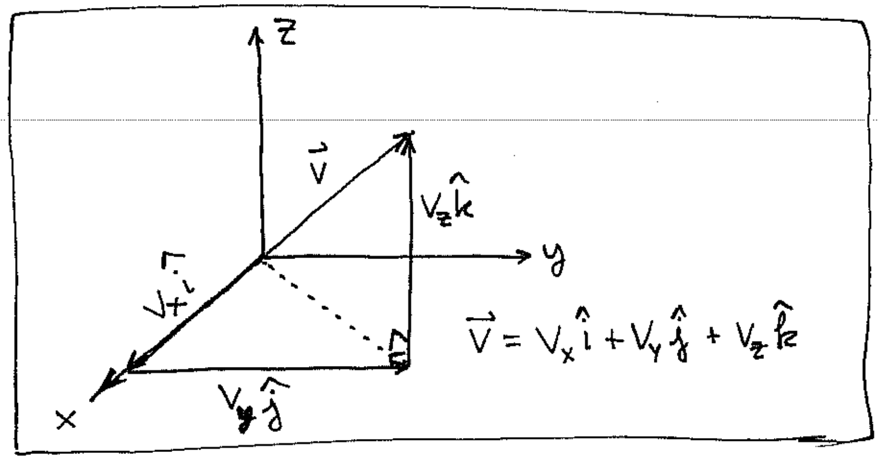
A vector is a directed line segment.

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k} = \langle V_x, V_y, V_z \rangle$$

The <u>components</u> are V_x, V_y, V_z. These are numbers or functions. In contrast, the <u>vector components</u> or <u>component vectors</u> are $V_x\hat{i}$, $V_y\hat{j}$, $V_z\hat{k}$.

Let me attempt a two & three dimensional sketch





You can see from this picture that the length of $\vec{V}$ is $\sqrt{V_x^2 + V_y^2 + V_z^2}$. We <u>define</u>

Def^n/ $\boxed{|\vec{V}| = \sqrt{V_x^2 + V_y^2 + V_z^2} \quad \text{the length or magnitude of the vector } \vec{V}}$

<u>Unit-vectors</u>: a vector $\vec{u}$ with length one is called a <u>unit-vector</u>. Any nonzero vector can be written as its magnitude multiplying a unit-vector.

<u>Observation</u>: $\vec{A} \neq 0 \Rightarrow |\vec{A}| \neq 0 \quad \therefore \quad \vec{A} = |\vec{A}|\frac{1}{|\vec{A}|}\vec{A}$

thus $\vec{A} = |\vec{A}|\hat{A}$ where $\hat{A} = \frac{1}{|\vec{A}|}\vec{A}$.

<u>Notation</u>: we often use A to denote <u>the magnitude of</u> $\vec{A}$. In this notation we may write $\boxed{\vec{A} = A\hat{A}}$

(this is consistent with the $\hat{i}, \hat{j}, \hat{k}$ notation)

Dot Products:

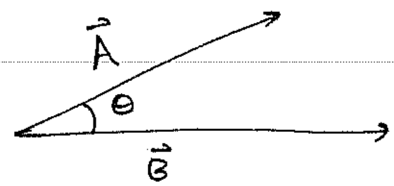
If $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$ then the <u>dot product</u> of $\vec{A}$ & $\vec{B}$ is the <u>real</u> <u>number</u> $\vec{A}\cdot\vec{B}$ defined below,

$$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$$

It can be shown that

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

where θ is the angle between $\vec{A}$ & $\vec{B}$



Ex| If $\vec{A} = \langle 1, 0, 2 \rangle$ and $\vec{B} = \langle 3, 4, 0 \rangle$

then $\vec{A}\cdot\vec{B} = \langle 1,0,2\rangle \cdot \langle 3,4,0\rangle = 3 + 0 + 0 = 3.$

The <u>angle</u> between $\vec{A}$ & $\vec{B}$ can be calculated as follows,

$$\vec{A}\cdot\vec{B} = AB\cos\theta \longrightarrow 3 = \sqrt{5}\sqrt{25}\cos\theta$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{3}{\sqrt{125}}\right)$$

Def^n/ Two vectors $\vec{A}, \vec{B}$ are <u>orthogonal</u> iff $\vec{A}\cdot\vec{B} = 0$

Observation: $\hat{i}$, $\hat{j}$ and $\hat{k}$ are pairwise orthogonal

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

whereas $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1.$

Remark: in retrospect we see $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$ or we could write $A = \sqrt{\vec{A}\cdot\vec{A}}$.

Cross Products

Given two vectors $\vec{A} = \langle A_x, A_y, A_z \rangle$ and $\vec{B} = \langle B_x, B_y, B_z \rangle$ we may construct a <u>perpendicular</u> or <u>orthogonal</u> vector $\vec{A}\times\vec{B}$ via the rule

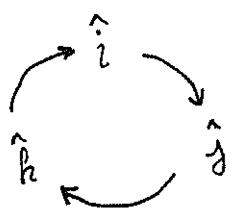
$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}\left(A_y B_z - A_z B_y\right) - \hat{j}\left(A_x B_z - A_z B_x\right) + \hat{k}\left(A_x B_y - A_y B_x\right)$$

$$= \underline{\langle A_y B_z - A_z B_y,\ A_z B_x - A_x B_z,\ A_x B_y - A_y B_x \rangle}$$

this is the def^{n}, the determinant formula is just a tool to remember this formula.

<u>Alternatively we can use $\hat{i}, \hat{j}, \hat{k}$ and distributive properties</u>



$\hat{i}\times\hat{j} = \hat{k}$	$\hat{j}\times\hat{i} = -\hat{k}$	$\hat{i}\times\hat{i} = 0$
$\hat{j}\times\hat{k} = \hat{i}$	$\hat{k}\times\hat{j} = -\hat{i}$	$\hat{j}\times\hat{j} = 0$
$\hat{k}\times\hat{i} = \hat{j}$	$\hat{i}\times\hat{k} = -\hat{j}$	$\hat{k}\times\hat{k} = 0$

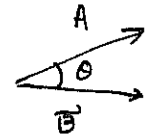
$\boxed{\text{Remark: } \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}}$

<u>Ex)</u> $(3\hat{i} + \hat{k})\times(\hat{j} + \hat{i}) = 3\hat{i}\times\hat{j} + 3\hat{i}\times\hat{i} + \hat{k}\times\hat{j} + \hat{k}\times\hat{i}$

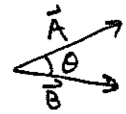
$$= \underbrace{3\hat{k} - \hat{i} + \hat{j}}.$$

this vector is perpendicular to both $3\hat{i} + \hat{k}$ and $\hat{j} + \hat{i}$.

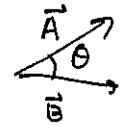
Observation:

$\vec{A}$ is parallel to $\vec{B}$ $\iff$ $\vec{A} = kB$ for some $k > 0$ $\iff$ $\vec{A}\cdot\vec{B} = AB$ $\iff$ 

, $\underline{\theta = 0}$.

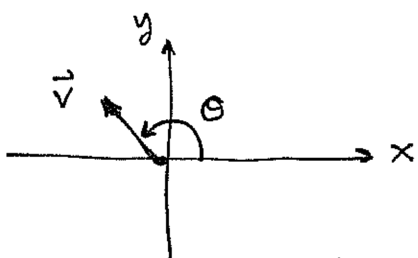
$\vec{A}$ is antiparallel to $\vec{B}$ $\iff$ $\vec{A} = kB$ for some $k < 0$ $\iff$ $\vec{A}\cdot\vec{B} = -AB$ $\iff$ 

, $\underline{\theta = 180°}$.

$\vec{A} \perp \vec{B}$ (perpendicular) $\iff$ $\vec{A}\cdot\vec{B} = 0$ $\iff$ $|\vec{A}\times\vec{B}| = AB$ $\iff$ 

, $\theta = 90°$ or $\theta = 270°$

In <u>two-dimensions</u> we can use the standard angle Θ defined by: angle measured counter-clockwise (CCW) from x-axis ($x > 0$)



In the examples below I show how to find standard angle from the given Cartesian coordinates.

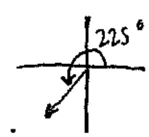
Ex] $\vec{V} = \langle 1, 1 \rangle$

$$\tan\theta = \frac{V_y}{V_x} = \frac{1}{1} = 1$$

$$\left.\begin{array}{l} V_x = V\cos\theta = \sqrt{2}\cos\theta = 1 \longrightarrow \cos\theta = \frac{1}{\sqrt{2}} \\ V_y = V\sin\theta = \sqrt{2}\sin\theta = 1 \longrightarrow \sin\theta = \frac{1}{\sqrt{2}} \end{array}\right\} \longrightarrow \underline{\theta = \pi/4 = 45°}.$$

magnitude of $\vec{V}$ is $|\vec{V}| = V = \sqrt{2}$.

Ex] $\vec{V} = \langle -1, -1 \rangle$ has $|\vec{V}| = \sqrt{2}$



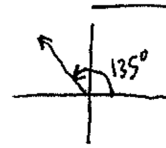
$$\left.\begin{array}{l} V_x = \sqrt{2}\cos\theta = -1 \longrightarrow \cos\theta = -1/\sqrt{2} \\ V_y = \sqrt{2}\sin\theta = -1 \longrightarrow \sin\theta = -1/\sqrt{2} \end{array}\right\} \longrightarrow \underline{\theta = 225°}.$$

$\&\ V = \sqrt{2}$

Ex] $\vec{V} = \langle -1, 1 \rangle$ has $|\vec{V}| = \sqrt{2}$

$$\left.\begin{array}{l} V_x = \sqrt{2}\cos\theta = -1 \longrightarrow \cos\theta = -1/\sqrt{2} \\ V_y = \sqrt{2}\sin\theta = 1 \longrightarrow \sin\theta = 1/\sqrt{2} \end{array}\right\} \longrightarrow \underline{\theta = 135°}.$$

$\&\ \underline{V = \sqrt{2}}$.



CALCULUS OF SPACE CURVES

Given $\vec{r} : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ for $m = 2, 3$. (in this course) we can calculate derivatives, integrals, dot-products, cross-products etc...

Def^n/ If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ then

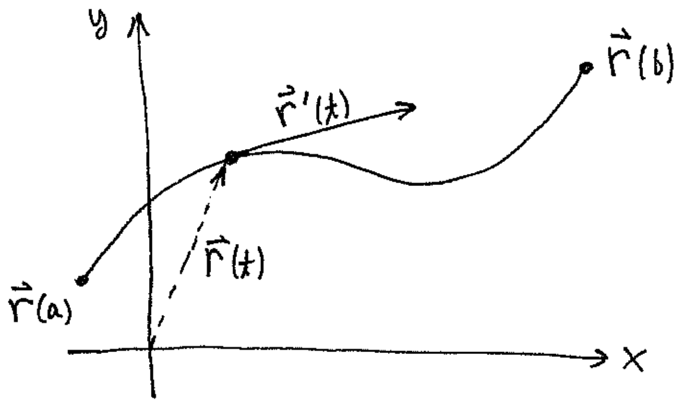
$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle, \quad \vec{r}\,''(t) = \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right).$$

$$\int \vec{r}(t)\,dt = \left\langle \int x\,dt, \int y\,dt, \int z\,dt \right\rangle$$

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b x\,dt, \int_a^b y\,dt, \int_a^b z\,dt \right\rangle$$

for all t, a, b such that the component formulas exist... I'm omitting some mathematical fine-print here.

I can draw a picture to explain the geometric meaning of $\frac{d\vec{r}}{dt}$,



$\vec{r}\,'(t) = \frac{d\vec{r}}{dt}$ is called the <u>tangent</u> <u>vector</u>. It points in the direction of the tangent line.

endpoints $\vec{r}(a)$ & $\vec{r}(b)$ are called terminal points of the path $\vec{r} : [a, b] \longrightarrow \mathbb{R}^2$

Ex/ $\vec{r}(t) = \langle t, t^3 \rangle$ find <u>tangent</u> <u>line</u> at $\vec{r}(1)$,

$$\vec{r}\,'(t) = \frac{d\vec{r}}{dt} = \langle 1, 3t^2 \rangle$$

(see next page for def^n of line in $\mathbb{R}^3$)

$\Rightarrow \vec{r}\,'(1) = \langle 1, 3 \rangle$ direction vector of tangent line.

$$\Rightarrow \vec{l}(t) = \vec{r}(1) + \vec{r}\,'(1)t = \langle 1, 1 \rangle + t\langle 1, 3 \rangle = \vec{l}(t)$$

$$\therefore \langle 1 + t, 1 + 3t \rangle = \vec{l}(t)$$

The tangent line has $x = 1 + t$ and $y = 1 + 3t$.

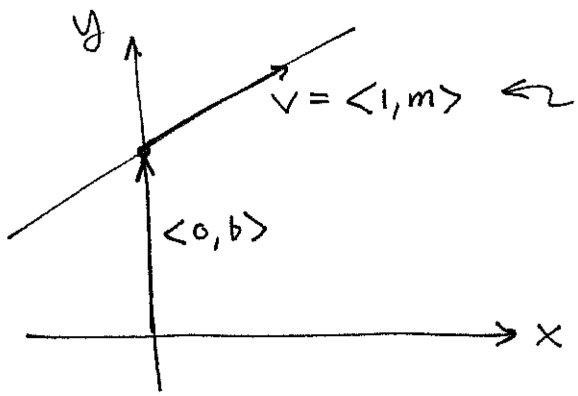
Def^n/ Given a point P and a vector $V \neq 0$ we define the line through point P with direction V,

$$\mathcal{L} = \{P + tV \mid t \in \mathbb{R}\}$$

We say $\vec{r}(t) = P + tV$ gives a <u>parametrization</u> of the line.

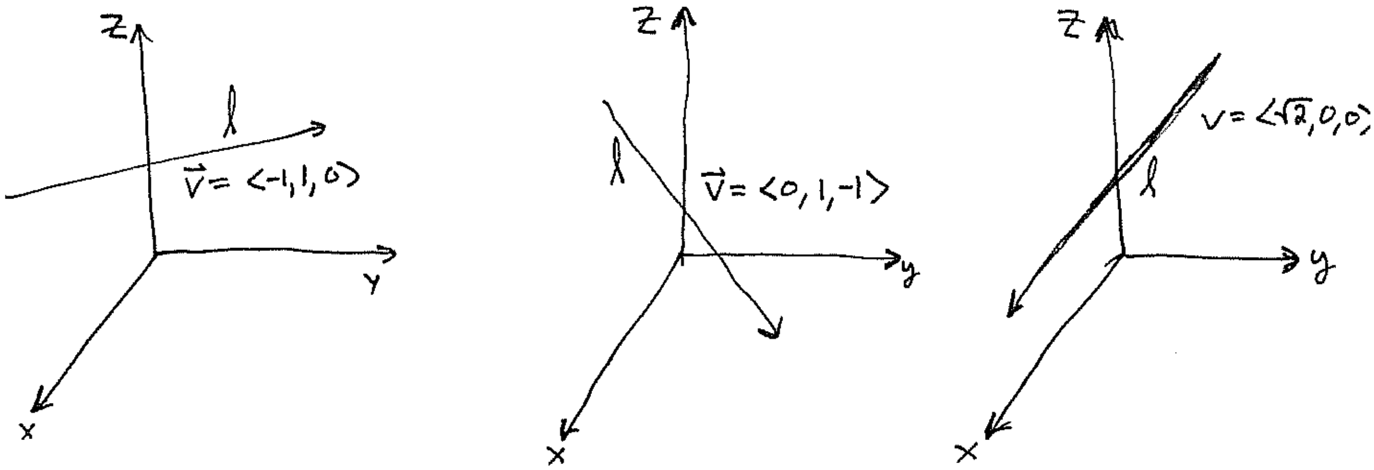
Ex| If $y = mx + b$ we can use x as the parameter and find $\vec{r}(x) = \langle x, mx+b \rangle$ thus

$$\begin{aligned} \vec{r}(x) &= \langle 0, b \rangle + \langle x, mx \rangle \\ &= \langle 0, b \rangle + x \langle 1, m \rangle \end{aligned}$$



$\leftarrow$ direction vector for the line $y = mx + b$

Notice, we have lines in <u>3</u> dimensions so the single slope "m" is not sufficient to desribe the different types of lines one can imagine



Examples:

Ex] $\vec{r}(t) = \langle \cos t, e^t, 10^t \rangle$

$$\vec{r}\,'(t) = \langle -\sin t, e^t, \ln(10)\, 10^t \rangle = \frac{d\vec{r}}{dt}$$

$$\vec{r}\,''(t) = \langle -\cos t, e^t, (\ln(10))^2 10^t \rangle = \frac{d^2\vec{r}}{dt^2}$$

$$\int \vec{r}(t)\,dt = \left\langle \int \cos t\,dt, \int e^t\,dt, \int 10^t\,dt \right\rangle$$

$$= \left\langle \sin t + c_1, e^t + c_2, \frac{10^t}{\ln(10)} + c_3 \right\rangle$$

$$= \left\langle \sin t, e^t, \frac{1}{\ln(10)} 10^t \right\rangle + \vec{C} \qquad \text{where } \vec{C} = \langle c_1, c_2, c_3 \rangle$$

Remark: often we're given initial conditions such as $\vec{r}(0) = (x_0, y_0, z_0)$ we can incoorporate these in vector form by exploiting the rules from calculus I repeatedly,

Ex] Given $\vec{v} = \frac{d\vec{r}}{dt}$ and $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ we can show that $\vec{r}(t) = (x_0, y_0, z_0) + \int_0^t \vec{v}(\bar{t})\,d\bar{t}$.

$$\int_0^t \vec{v}(\bar{t})\,d\bar{t} = \int_0^t \frac{d\vec{r}}{d\bar{t}}\,d\bar{t}$$

$$= \int_0^t \left\langle \frac{dx}{d\bar{t}}, \frac{dy}{d\bar{t}}, \frac{dz}{d\bar{t}} \right\rangle d\bar{t}$$

$$= \left\langle \int_0^t \frac{dx}{d\bar{t}}\,d\bar{t}, \int_0^t \frac{dy}{d\bar{t}}\,d\bar{t}, \int_0^t \frac{dz}{d\bar{t}}\,d\bar{t} \right\rangle$$

$$= \langle x(t) - x(0), y(t) - y(0), z(t) - z(0) \rangle$$

$$= \langle x(t), y(t), z(t) \rangle - \langle x_0, y_0, z_0 \rangle$$

$$\Rightarrow \boxed{\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + \int_0^t \vec{v}(\bar{t})\,d\bar{t}}$$

Ex] Given $\vec{a} = \frac{d\vec{V}}{dt}$ and $\vec{V} = \frac{d\vec{r}}{dt}$ with $\vec{V}(0) = \langle V_{ox}, V_{oy}, V_{oz} \rangle$ and $\vec{r}(0) = \langle x_o, y_o, z_o \rangle$ we can show that

$$\vec{V}(t) = \langle V_{ox}, V_{oy}, V_{oz} \rangle + \int_0^t \vec{a}(\bar{t})\, d\bar{t}$$

and also,

$$\vec{r}(t) = \langle x_o, y_o, z_o \rangle + t \langle V_{ox}, V_{oy}, V_{oz} \rangle + \int_0^t \left(\int_0^\lambda \vec{a}(\bar{t})\, d\bar{t} \right) d\lambda$$

I will prove this in a special case of constant acceleration. I leave the calculation for that time.

Ex] Given $\vec{a} = \langle 12t, 12t^2 \rangle$ and $\vec{V}(0) = \langle 1, 1 \rangle$ and $\vec{r}(0) = \langle 2, 2 \rangle$ where $\vec{a} = \frac{d\vec{V}}{dt}$ & $\vec{V} = \frac{d\vec{r}}{dt}$ we can calculate,

$$\vec{a} = \langle 12t, 12t^2 \rangle$$

$$\int \vec{a}\, dt = \vec{V} = \langle 6t^2, 4t^3 \rangle + \vec{C} \quad \text{but } \vec{V}(0) = \langle 1, 1 \rangle$$

$$\Rightarrow \vec{C} = \langle 1, 1 \rangle.$$

$$\therefore \underline{\vec{V}(t) = \langle 6t^2 + 1, 4t^3 + 1 \rangle}.$$

$$\int \vec{V}\, dt = \vec{r} = \langle 2t^3 + t, t^4 + t \rangle + \vec{C}$$

but $\vec{r}(0) = \langle 2, 2 \rangle = \vec{C}$

$$\therefore \underline{\vec{r}(t) = \langle 2t^3 + t + 2, t^4 + t + 2 \rangle}.$$