PROBLEM A13) Let $\gamma(t) = \langle \cos t, t, \sin t \rangle$. Calculate T, N, B curvature and torsion.

Can either use chain rule to calculate κ & τ or can reparametrize with respect to arclength from outset. I choose to work in arclength s

$$s(t) = \int_0^t \|\gamma'(u)\|\, du$$

$$= \int_0^t \sqrt{(-\sin u)^2 + 1 + \cos^2 u}\, du$$

$$= t\sqrt{2} \quad \therefore \quad t(s) = s/\sqrt{2} \quad (t \text{ as fnct. of } s)$$

$$\tilde{\gamma}(s) = \gamma(t(s)) = \gamma(s/\sqrt{2}) = \langle \cos(s/\sqrt{2}), s/\sqrt{2}, \sin(s/\sqrt{2}) \rangle$$

$$\frac{d\tilde{\gamma}}{ds} = \frac{1}{\sqrt{2}} \langle -\sin(s/\sqrt{2}), 1, \cos(s/\sqrt{2}) \rangle$$

Since $\|\tilde{\gamma}'(s)\| = \frac{ds}{ds} = 1$ we have

$$\boxed{\tilde{T}(s) = \frac{1}{\sqrt{2}} \langle -\sin(s/\sqrt{2}), 1, \cos(s/\sqrt{2}) \rangle}$$

Next calculate $\frac{d\tilde{T}}{ds} = \frac{1}{2} \langle -\cos(s/\sqrt{2}), 0, -\sin(s/\sqrt{2}) \rangle$ and observe $\left\|\frac{d\tilde{T}}{ds}\right\| = \frac{1}{2}$ hence $\tilde{N}(s) = \frac{\tilde{T}'(s)}{\|\tilde{T}'(s)\|}$ simplifies to $\boxed{\tilde{N}(s) = \langle -\cos(s/\sqrt{2}), 0, -\sin(s/\sqrt{2}) \rangle}$. The binormal vector.

$$\tilde{B}(s) = \tilde{T} \times \tilde{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{-1}{\sqrt{2}}\sin(s/\sqrt{2}) & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cos(s/\sqrt{2}) \\ -\cos(s/\sqrt{2}) & 0 & -\sin(s/\sqrt{2}) \end{vmatrix}$$

$$\boxed{\tilde{B}(s) = \left\langle -\frac{1}{\sqrt{2}}\sin(s/\sqrt{2}), \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos(s/\sqrt{2}) \right\rangle}$$

PROBLEM A13 Continued

We have T, N, B in terms of arclength which I denote $\tilde{T}, \tilde{N}, \tilde{B}$ to emphasize the special choice of parameter. Recall def$^{\underline{n}}$ 4.3.2,

$$\kappa(s) = \left\| \frac{d\tilde{T}}{ds} \right\| = \boxed{\frac{1}{2} = \kappa} \leftarrow \text{constant curvature.}$$

$$\tau(s) = -\frac{d\tilde{B}}{ds} \cdot \tilde{N}(s)$$

$$= \left\langle \frac{-1}{2}\cos(s/\sqrt{2}),\ 0,\ \frac{-1}{2}\sin(s/\sqrt{2}) \right\rangle \cdot \langle -\cos(s/\sqrt{2}), 0, -\sin(s/\sqrt{2}$$

$$= \boxed{\frac{1}{2} = \tau} \leftarrow \text{constant torsion.}$$

Neat, this helix has constant torsion and curvature which are equal. Generally torsion & curvature of linear helixes are constant but I didn't set out to make them equal.

Remark: You can work these things out in t w/o s and the $\sim$'d functions. But, you need to insert $\frac{ds}{dt}$ in certain places. See my notes following Def$^{\underline{n}}$ 4.3.2 for the formulas.

PROBLEM A14 Edwards problem 1.10 from pg. 63.

Suppose particle has position $\gamma(t)$ and moves in circle of radius r at origin with constant speed v. Show $\gamma''(t) = \frac{-v^2}{r^2}\gamma(t)$

1.) $\underbrace{\gamma(t)\cdot\gamma(t) = r^2}_{\text{circle at origin, radius } r} \Rightarrow \gamma'\cdot\gamma + \gamma\cdot\gamma' = 0$

$\Rightarrow \gamma(t)\cdot\frac{d\gamma}{dt} = 0$. (I)

2.) $\underbrace{\frac{d\gamma}{dt}\cdot\frac{d\gamma}{dt} = v^2}_{\text{constant speed}} \Rightarrow \gamma''\cdot\gamma' + \gamma'\cdot\gamma'' = 0$

constant speed $\left\|\frac{d\gamma}{dt}\right\| = v$ was given.

$\Rightarrow \gamma''\cdot\frac{d\gamma}{dt} = 0$. (II)

Notice $\gamma(t)$ and $\gamma''(t)$ are both orthogonal to $\gamma'(t)$ and since these are two dimensional vectors it follows $\gamma''(t) = k(t)\gamma(t)$ for some, possibly nonconstant, scale factor $k(t)$. Differentiate (I) once more to reveal,

$$\frac{d\gamma}{dt}\cdot\frac{d\gamma}{dt} + \gamma\cdot\frac{d^2\gamma}{dt^2} = 0 \Rightarrow \gamma(t)\cdot\gamma''(t) = -v^2$$

$$\Rightarrow \gamma(t)\cdot[k(t)\gamma(t)] = -v^2$$

$$\Rightarrow k(t)\gamma(t)\cdot\gamma(t) = -v^2$$

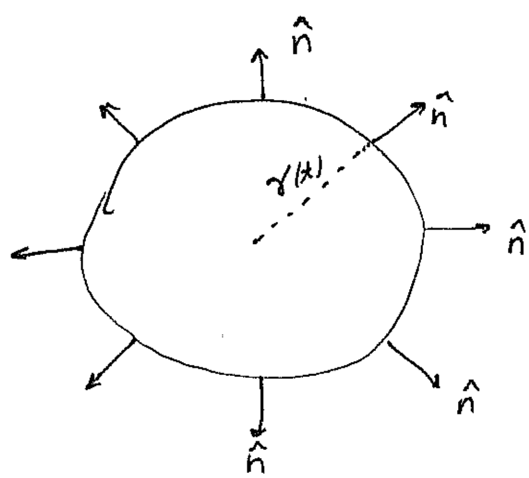
$$\Rightarrow k(t)r^2 = -v^2$$

$$\Rightarrow k(t) = -v^2/r^2$$

$$\therefore \boxed{\gamma''(t) = \frac{-v^2}{r^2}\gamma(t)}$$

Note we can write $\gamma(t) = r\hat{n}(t)$ for an outward pointing unit radial vector then we find the boxed eq$^{\underline{n}}$ becomes

$$\boxed{\vec{a}(t) = \frac{-v^2}{r}\hat{n}(t)}$$



PROBLEM A14 Continued problem 1.11 Edwards, pg. 63. Given a particle with position $\gamma(t)$ we define angular momentum $\vec{L}(t) = \gamma(t) \times (m\gamma'(t))$ and torque $\vec{\tau}(t) = \gamma(t) \times (m\gamma''(t))$.

a.) show $\frac{d\vec{L}}{dt} = \vec{\tau}(t)$

b.) show if the particle is subject to a <u>central</u> force then its motion is planar (central force $\Rightarrow \gamma''(t) = k(t)\gamma(t)$).

a.) $$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left(\gamma \times \left(m\frac{d\gamma}{dt}\right)\right)$$
$$= m\frac{d\gamma}{dt} \times \frac{d\gamma}{dt} + \gamma \times \left(m\frac{d^2\gamma}{dt}\right)$$
$$= \gamma \times (m\gamma'')$$
$$= \vec{\tau}.$$

note $\frac{d\gamma}{dt} \times \frac{d\gamma}{dt} = 0$ because the cross product is antisymmetric hence $\gamma' \times \gamma' = -\gamma' \times \gamma'$ $\Rightarrow 2\gamma' \times \gamma' = 0$ $\Rightarrow \gamma' \times \gamma' = 0$.

Thus $\vec{\tau} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{constant}$.

b.) If we can show $\frac{d\vec{B}}{dt} = 0$ then that proves the motion is planar since the osculating planes at different times will all have the same normal. This is equivalent to proving torsion is zero since $\frac{d\vec{B}}{dt} = -\frac{ds}{dt}\tau\vec{N}$ (third Frenet Seret Eq^{s})

We're given $\gamma''(t) = k(t)\gamma(t)$ hence $\vec{\tau}(t) = \gamma \times m\gamma'' = \gamma \times (mk\gamma)$
$= mk(\gamma \times \gamma)$
$= 0.$

Thus, $$\frac{d}{dt}\left(\gamma \times \left(\frac{d\gamma}{dt}\right)\right) = 0$$

It follows $\exists\, \vec{C}_o = \gamma \times \gamma'$ where $\vec{C}_o$ is constant vector. Calculate,

$$\vec{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} = \frac{1}{v}\frac{d\gamma}{dt}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \frac{1}{\|\vec{T}'(t)\|}\left(\frac{-1}{v^2}\frac{dv}{dt}\frac{d\gamma}{dt} + \frac{1}{v}\frac{d^2\gamma}{dt^2}\right) = A\gamma' + \vec{B}_o\gamma''$$

Sorry, no relation to "$\vec{B}(t)$"

Hence $$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \frac{1}{v}\gamma' \times (A\gamma' \times B\gamma'')$$
$$= \frac{\vec{B}_o}{v}\gamma' \times \gamma''$$
$$= \frac{k\vec{B}_o}{v}\gamma' \times \gamma = \frac{k\vec{B}_o}{v}\vec{C}_o = \frac{k(t)}{v^2\|\vec{T}'(t)\|}\vec{C}_o$$

Problem A14, part 6 of 1.11 pg. 63 continued

We found $\vec{B}(t) = \underbrace{\dfrac{k(t)}{V^2 \, \|\vec{T}'(t)\|}}_{g(t)} \vec{C}_0$

$\nwarrow$ just a scalar function.

Here $\vec{C}_0 = \gamma \times \dfrac{d\gamma}{dt}$ is constant. Calculate then,

$$\frac{d\vec{B}}{dt} = \frac{d}{dt}\left(g\vec{C}_0\right) = \frac{dg}{dt}\vec{C}_0 = -\frac{ds}{dt}\tau \vec{N}(t)$$

$$= -V\tau\left(A\gamma' + B_0 k(t)\gamma(t)\right)$$

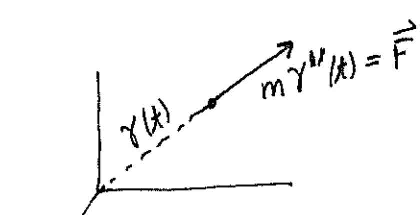
just scalars.

But, $C_0 = \gamma \times \dfrac{d\gamma}{dt}$ is $\perp$ to both γ' and γ because the cross product is perpendicular to both inputs. Consequently it must be that $\tau = 0$ (assuming $V \neq 0$ and so forth).

We conclude $\dfrac{d\vec{B}}{dt} = 0 \quad \therefore \quad \gamma(t)$ travels a planar curve. //

Comment: $\vec{F}(r) = f(r)\,\hat{r}$

central force depends only on distance to some origin, we just proved this causes planar motion.



$$\vec{F} = m\vec{a} = \hat{k}(t)\gamma(t)$$

$$\Rightarrow \gamma''(t) = k(t)\gamma(t)$$

(I realize it may not be obvious why $\gamma = k\gamma''$ desribes a "central force", need to think about F=ma to make it clear.

$\boxed{\text{PROBLEM A15}}$ + 25 for all, see notes at end of chapter 5.

$\boxed{\text{PROBLEM A16}}$ Calculate derivative of mappings below:

1.) $F(x,y,z) = x^2 + y^2 + z^2$

Note, $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ so I expect F' is 1×3.

$$F'(x,y,z) = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right] = \underline{\left[2x, 2y, 2z \right]}.$$

2.) $\gamma(t) = (t, t^2, t^3)$

Note, $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3$ so I expect γ' is 3×1.

$$\gamma'(t) = \begin{bmatrix} \frac{d\gamma_1}{dt} \\ \frac{d\gamma_2}{dt} \\ \frac{d\gamma_3}{dt} \end{bmatrix} = \underline{\begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix}}.$$

3.) $$\left.\begin{aligned} (F \circ \gamma)'(t) &= F'(\gamma(t))\,\gamma'(t) \\ &= \left[2t, 2t^2, 2t^3 \right] \begin{bmatrix} 1 \\ 2t \\ 3t^2 \end{bmatrix} \end{aligned}\right\} \text{sol}^{\underline{n}} \text{ via chain rule.}$$

$$= \underline{2t + 4t^3 + 6t^5}.$$

Alternatively,

$$F(\gamma(t)) = F(t, t^2, t^3) = t^2 + t^4 + t^6 \quad \therefore \quad \underline{(F \circ \gamma)'(t) = 2t + 4t^3 + 6t^5}.$$

4.) $f(x,y) = (x, y, 1/x + 1/y)$

Let's see $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ hence $f'(x,y) \in \mathbb{R}^{3 \times 2}$.

$$f'(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/x^2 & -1/y^2 \end{bmatrix}}.$$

PROBLEM A16 continued

5.) $g(s,t) = (st,\ s+t)$

$$g' = \begin{bmatrix} \frac{\partial}{\partial s}(st) & \frac{\partial}{\partial t}(st) \\ \frac{\partial}{\partial s}(s+t) & \frac{\partial}{\partial t}(s+t) \end{bmatrix} = \underline{\begin{bmatrix} t & s \\ 1 & 1 \end{bmatrix}}.$$

6.) $(f \circ g)'(s,t) = f'(g(s,t))\, g'(s,t)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/s^2t^2 & -1/(s+t)^2 \end{bmatrix} \begin{bmatrix} t & s \\ 1 & 1 \end{bmatrix}$$

$$= \underline{\left[\begin{array}{c|c} t & s \\ 1 & 1 \\ \frac{-1}{s^2 t} - \frac{1}{(s+t)^2} & \frac{-1}{s t^2} - \frac{1}{(s+t)^2} \end{array}\right]}.$$

Alternatively we can calculate $h(s,t) = (f \circ g)(s,t) = f(st,\ s+t)$ or $h(s,t) = \left(st,\ s+t,\ \frac{1}{st} + \frac{1}{s+t}\right)$. Then the Jacobian matrix $h' = \begin{bmatrix} \partial_s h_1 & \partial_t h_1 \\ \partial_s h_2 & \partial_t h_2 \\ \partial_s h_3 & \partial_t h_3 \end{bmatrix}$ yields same result.

7.) $h(r, \theta, z) = (r\cos\theta,\ r\sin\theta,\ z)$

$$h' = \left[\begin{array}{c|c|c} \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \theta} & \frac{\partial h}{\partial z} \end{array}\right] = \underline{\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

typo in Problem Statement, no big deal.

8.) $X(s,t) = (R\cos s \sin t,\ R\sin s \sin t,\ R\cos t)$ for constant $R > 0$

$$X' = \left[\begin{array}{c|c} \frac{\partial X}{\partial s} & \frac{\partial X}{\partial t} \end{array}\right] = \underline{\begin{bmatrix} -R\sin s \sin t & R\cos s \cos t \\ R\cos s \sin t & R\sin s \cos t \\ 0 & -R\sin t \end{bmatrix}}.$$

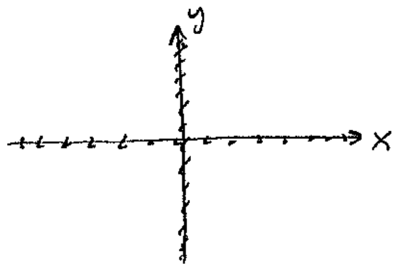
PROBLEM A17 Again consider mappings f & Φ from A16,

$$f(x,y) = \left(x, y, \tfrac{1}{x} + \tfrac{1}{y}\right) \quad \& \quad \Phi(s,t) = \langle R\cos s \sin t, R\sin s \sin t, R\cos t \rangle$$

Find $U, V \subseteq \mathbb{R}^2$ such that $f(U)$ & $\Phi(V)$ are patched manifolds of dimension two in $\mathbb{R}^3$

We simply need rank f' and rank Φ' to be two for the sets U, V, and we need f, Φ injective. Choose

$$\boxed{U = (0,\infty) \times (0,\infty)}$$ to avoid $x = 0$ & $y = 0$



many other choices possible just need to avoid axes.

Since $f'(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1/x^2 & -1/y^2 \end{bmatrix}$

rank $(f') = 2$ is clear, the only issue is f' d.n.e for $x = 0, y = 0$.

Since Φ is the parametrization of sphere based on spherical coordinates $\rho = R$ $\longrightarrow$ $x = R\cos s \sin t$, $y = R \sin s \sin t$, $z = R\cos t$

Here $0 \leq s \leq 2\pi$ & $0 \leq t \leq \pi$ are the usual choices.

We need to avoid doubling up on 0 & 2π and 0 & π so just use $$\boxed{V = (0, 2\pi) \times (0, \pi)}$$

PROBLEM A18 Prove $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$ for all $i,j = 1,2,\dots,n$.

Coordinate x_i is a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(x) = x_i$ for each $x \in \mathbb{R}^n$. Another nice formula is $f(x) = x \cdot e_i$.

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0}\left(\frac{f(a + he_j) - f(a)}{h}\right)$$: def$^{\underline{n}}$ of partial derivative applied to f.

$$= \lim_{h\to 0}\left(\frac{(a + he_j)\cdot e_i - a\cdot e_i}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\cancel{a\cdot e_i} + he_j\cdot e_i - \cancel{a\cdot e_i}}{h}\right) = \lim_{h\to 0}\left(\frac{he_j\cdot e_i}{h}\right) = \underbrace{e_j \cdot e_i}_{\text{standard basis is orthogonal.}} = \delta_{ji}. //$$

$$\left(\delta_{ji} = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases} \quad \underline{\text{Kronecker Delta}}\right)$$

PROBLEM A19 Let $f(x,y) = x^y$. Calculate df.
Let $\gamma(t) = (t,t)$ and use chain rule to calculate $(f\circ\gamma)'(t)$.
This gives us a formula for $\frac{d}{dt}(t^t)$. Now, contrast this method to the calculus I implicit diff. method.

$$df(a,b)\vec{h} = f'(a,b)\vec{h}$$ actually I want $f'(x,y)$.

$$f'(x,y) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] = \left[yx^{y-1}, \ln(x)\,x^y\right]$$

$$\begin{aligned}
(f\circ\gamma)'(t) &= f'(\gamma(t))\,\gamma'(t) \\
&= f'(t,t)\begin{bmatrix}1\\1\end{bmatrix} \\
&= \left[t\,t^{t-1}, \ln(t)\,t^t\right]\begin{bmatrix}1\\1\end{bmatrix} \\
&= t^t + \ln(t)\,t^t \\
&= \boxed{t^t(1+\ln(t)) = \frac{d}{dt}(t^t)}
\end{aligned}$$

<u>CALC. I Soln</u>

$$y = t^t \Rightarrow \ln(y) = t\ln(t)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dt} = \ln(t) + \frac{t}{t}$$

$$\frac{dy}{dt} = \underline{t^t(\ln(t)+1) = \frac{d}{dt}(t^t)}.$$

(quicker, yes, funner? no.)

↑

on purpose.

$\boxed{\text{PROBLEM A20}}$ problem 2.7 pg. 75 Edwards.

If $F : \mathbb{R}^n \to \mathbb{R}^n$ can be written as $F = -\nabla V$ for some potential function $V : \mathbb{R}^n \to \mathbb{R}$ then F is said to be a conservative vector field. Show

(a.) $F(x) = r^n x$ where $r = \|x\|$. Treat $n=2$ & $n \neq 2$ separately.

(b.) $F(x) = \left[\frac{g'(r)}{r}\right] x$ where $g : \mathbb{R} \to \mathbb{R}$ is differentiable fnct.

a.) We seek to solve $\left\langle \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n} \right\rangle = -r^n x = \langle -r^n x_1, -r^n x_2, \dots, -r^n x_n \rangle$

$$\frac{\partial V}{\partial x_1} = -x_1 \left(x_1^2 + x_2^2 + \cdots + x_n^2 \right)^{n/2}$$

$$V(x) = \int -x_1 (\underbrace{x_1^2 + x_2^2 + \cdots + x_n^2}_{u})^{n/2} \, \partial x_1 \qquad : \quad u = r^2$$

$$= \int -\frac{1}{2} u^{n/2} \, du$$

$$= -\frac{1}{2} \frac{u^{\frac{n}{2}+1}}{\frac{n}{2}+1} + C$$

$$= \frac{-1}{n+2} u^{\frac{n+2}{2}} + C$$

$$= \frac{-1}{n+2} r^{n+2} + \underbrace{C(x_2, x_3, \dots, x_n)}_{\text{in principle, but we'll see shortly the first term covers all we need.}}$$

Note $\frac{\partial r}{\partial x_j} = \frac{x_j}{r}$.

Calculate then,

$$-\frac{\partial V}{\partial x_j} = \frac{1}{n+2} \frac{\partial}{\partial x_j} \left(r^{n+2} \right) = \frac{n+2}{n+2} r^{n+1} \frac{\partial r}{\partial x_j} = r^{n+1} \frac{x_j}{r} = r x_j$$

$$\boxed{\text{Thus } \quad F = r^n x = -\nabla V \quad \text{if we choose} \quad V = \frac{-1}{n+2} r^{n+2}}$$

Remark: I do not need a separate $n=2$ argument here. However, $n = -2$ is troubling! I suspect there is a typo in Edwards here, maybe $F = r^{-n} x$ was the intended function.

PROBLEM A20 Continued

If $F(x) = r^{-n} x$ where $x = (x_1, x_2, \dots, x_n)$ and $r = \sqrt{x \cdot x}$

then $V = \dfrac{-1}{-n+2} r^{-n+2} = \dfrac{-1}{2-n} \dfrac{1}{r^{n-2}}$ provides a potential

for $F(x)$ as

$$-\frac{\partial V}{\partial x_j} = \frac{1}{2-n} \frac{\partial}{\partial x_j}\left[\frac{1}{r^{n-2}}\right]$$

$$= \frac{1}{2-n} \frac{-(n-2)}{r^{n-1}} \frac{x_j}{r}$$

$$= \frac{x_j}{r^{n}} \Rightarrow \underline{F = -\nabla V \text{ for } V = \frac{-1}{-n+2} r^{2-n}}.$$

(again supposing $F = r^{-n} x$)

Now if $n=2$, $r^{-2} = \frac{1}{r^2} = \frac{1}{x^2+y^2}$,

$$F(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

We need $\dfrac{\partial V}{\partial x} = \dfrac{-x}{x^2+y^2}$ & $\dfrac{\partial V}{\partial y} = \dfrac{-y}{x^2+y^2}$

Hmm... $V = \tan^{-1}(y/x)$ perhaps?

$$\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1+y^2/x^2} \frac{-y}{x^2} = \frac{-y}{x^2+y^2} \quad \text{nope.}$$

Instead use $V = \tan^{-1}(x/y)$

$$\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{1}{1+x^2/y^2} \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{1}{1+x^2/y^2} \frac{-x}{y^2} = \frac{-x}{x^2+y^2}$$

Oh well, guess I'll just have to $\int$.

$$\int \frac{-x\,dx}{x^2+y^2} = \int \frac{-du/2}{u} = -\frac{1}{2}\ln(x^2+y^2) = \ln\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$

Thus,

$$\boxed{V(x,y) = -\ln\left(\sqrt{x^2+y^2}\right) = -\ln(r)}$$

With my modification you might notice $n=3$ gives the Coulomb field $\vec{F} = \dfrac{kq}{r^2}\hat{r} = \dfrac{kq}{r^3}\vec{r} = \dfrac{kq}{r^3} x$. (in our notation here)

PROBLEM A20 Continued

(b.) find potential $V(x)$ such that $\left\langle \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \dots, \frac{\partial V}{\partial x_n} \right\rangle = -\left[\frac{g'(r)}{r}\right]\langle x_1, \dots, x_n\rangle$

$$\frac{\partial V}{\partial x_1} = \frac{-x_1}{r}\frac{dg}{dr} = -\frac{\partial r}{\partial x_1}\frac{dg}{dr}$$

Choose $\boxed{V(x_1, x_2, \dots, x_n) = -g\left(\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\right).}$ Note

$$\frac{\partial V}{\partial x_j} = -\frac{\partial}{\partial x_j} g\left(\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\right)$$

$$= -g'(r)\frac{\partial}{\partial x_j}\left(\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}\right)$$

$$= -g'(r)\frac{x_j}{r}.$$

For example, the Coulomb potential $U(r) = \frac{-k}{r}$ has $U'(r) = \frac{+k}{r^2}$ then $F = \frac{-k}{r^3}\langle x, y, z\rangle$ gives Coulomb field.

PROBLEM A21 problem 3.13 pg. 89 of Edwards

If $f(x, y, z) = \frac{1}{r} g(t - r/c)$ where c = constant and $r^2 = x^2 + y^2 + z^2$ then show $\nabla^2 f = \frac{1}{c^2}\partial_t^2 f$ where $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$

We will use $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$ & $\frac{\partial r}{\partial z} = \frac{z}{r}$ many times in what follows.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{r} g(t - r/c)\right]$$

$$= \frac{-1}{r^2}\frac{x}{r} g + \frac{1}{r} g'(t - r/c)\frac{\partial}{\partial x}\left(t - \frac{r}{c}\right)$$

$$= \frac{-x}{r^3} g\left(t - \frac{r}{c}\right) - \frac{x}{cr^2} g'(t - r/c)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{r^3} g + \frac{3x^2}{r^5} g + \frac{x^2}{cr^4} g' - \frac{1}{cr^2} g' + \frac{2x^2}{cr^4} g' + \frac{x^2}{c^2 r^3} g''$$

PROBLEM A21 Continued

By symmetry,

$$f_{xx} = \frac{-1}{r^3} g + \frac{3x^2}{r^5} g + \frac{3x^2}{cr^4} g' - \frac{1}{cr^2} g' + \frac{x^2}{c^2 r^3} g''$$

or cleaning up a bit

$$f_{xx} = -\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right) g + \left(\frac{3x^2}{r^2} - 1\right)\frac{g'}{cr^2} + \frac{x^2}{r^2}\frac{1}{c^2 r} g''$$

$$f_{yy} = -\left(\frac{1}{r^3} - \frac{3y^2}{r^5}\right) g + \left(\frac{3y^2}{r^2} - 1\right)\frac{g'}{cr^2} + \frac{y^2}{r^2}\frac{1}{c^2 r} g''$$

$$f_{zz} = -\left(\frac{1}{r^3} - \frac{3z^2}{r^5}\right) g + \left(\frac{3z^2}{r^2} - 1\right)\frac{g'}{cr^2} + \frac{z^2}{r^2}\frac{1}{c^2 r} g''$$

Thus,

$$f_{xx} + f_{yy} + f_{zz} = -\cancelto{0}{\left(\frac{3}{r^3} - \frac{3r^2}{r^5}\right)} g + \cancelto{0}{\left(\frac{3r^2}{r^2} - 3\right)} g' + \frac{1}{c^2 r} g''$$

On the other hand

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[g(t - r/c)\frac{1}{r}\right] = \frac{1}{r} g'(t - \tfrac{r}{c})\frac{\partial}{\partial t}\left(t - \frac{r}{c}\right) = \frac{1}{r} g'(t - \tfrac{r}{c})$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{1}{r} g''(t - r/c) \quad \therefore \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{c^2 r} g''(t - r/c)$$

$$\therefore \quad \underline{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}}.$$

PROBLEM A22

Problem 3.12 pg. 195

Suppose $f(x,y,z) = 0$ can be solved for x, y or z as fncts of the remaining variables (y,z), (x,z) or (x,y) respective. Prove that

$$\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1$$

Verify this relation for ideal gas eq^n $PV = RT$ where P, V, T are variables and R is the gas constant.

$$\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1$$

is a more careful version of the requested identity.

We're given $f_x, f_y, f_z \neq 0$. Note then $\exists\, h_1$ such that (I) $f(h_1(y,z), y, z) = 0$ and $\exists\, h_2$ such that (II) $f(x, h_2(x,z), z) = 0$ and $\exists\, h_3$ such that (III) $f(x,y,h_3(x,y)) = 0$. Differentiate,

$$\text{I} \Rightarrow \frac{\partial f}{\partial x}\frac{\partial h_1}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \quad \therefore\; f_x\frac{\partial h_1}{\partial y} = -f_y$$

$$\text{II} \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial h_2}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial z} = 0 \quad \therefore\; f_y\frac{\partial h_2}{\partial z} = -f_z$$

$$\text{III} \Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial h_3}{\partial x} = 0 \quad \therefore\; f_z\frac{\partial h_3}{\partial x} = -f_x$$

Therefore,

$$\left(\frac{\partial h_1}{\partial y}\right)\left(\frac{\partial h_2}{\partial z}\right)\left(\frac{\partial h_3}{\partial x}\right) = \left(\frac{-f_y}{f_x}\right)\left(\frac{-f_z}{f_y}\right)\left(\frac{-f_x}{f_z}\right) = -1$$

← all non zero, cancellation ok.

Hence, as $h_1 = x$, $h_2 = y$ and $h_3 = z$,

$$\underline{\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y = -1}.$$

Let's calculate this for $f(P,V,T) = PV - RT = 0$ ⟷ ideal gas Law.

$$P = \frac{RT}{V}, \quad V = \frac{RT}{P}, \quad T = \frac{1}{R}PV$$

Use these for calculati $\left.\frac{\partial P}{\partial V}\right|_T$ etc.

$$\left.\frac{\partial P}{\partial V}\right|_T \left.\frac{\partial V}{\partial T}\right|_P \left.\frac{\partial T}{\partial P}\right|_V = \left(-\frac{RT}{V^2}\right)\left(\frac{R}{P}\right)\left(\frac{V}{R}\right)$$

$$= \frac{-RT}{VP}$$

$$= -\frac{PV}{VP}$$

$$= -1.$$

PROBLEM A22 continued / Edwards problem 3.17 pg. 195

Suppose pressure P, volume V, temperature T, and internal energy U of a gas satisfy the equations:

$$f(P, V, T, U) = 0 \quad \& \quad g(P, V, T, U) = 0$$

and that these two eq^{n}'s can be solved for any two of these variables for the remaining two variables. Show that

$$\left(\frac{\partial U}{\partial P}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V = \left(\frac{\partial U}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V + \left(\frac{\partial U}{\partial P}\right)_T$$

$$df = f_P\,dP + f_V\,dV + f_T\,dT + f_U\,dU = 0$$

$$dg = g_P\,dP + g_V\,dV + g_T\,dT + g_U\,dU = 0$$

To calculate $\left(\frac{\partial U}{\partial P}\right)_V \Rightarrow P \,\&\, V$ are functions of $T \,\&\, U$.

$$f_P\,dP + f_V\,dV = -f_T\,dT - f_U\,dU$$

$$g_P\,dP + g_V\,dV = -g_T\,dT - g_U\,dU$$

$$\begin{bmatrix} f_P & f_V \\ g_P & g_V \end{bmatrix} \begin{bmatrix} dP \\ dV \end{bmatrix} = \begin{bmatrix} -f_T\,dT - f_U\,dU \\ -g_T\,dT - g_U\,dU \end{bmatrix}$$

$$dP = \frac{\det \begin{bmatrix} -f_T\,dT - f_U\,dU & f_V \\ -g_T\,dT - g_U\,dU & g_V \end{bmatrix}}{\det \begin{bmatrix} f_P & f_V \\ g_P & g_V \end{bmatrix}}$$

$$= \frac{-f_T g_V\,dT - f_U g_V\,dU + f_V g_T\,dT + g_U f_V\,dU}{f_P g_V - f_V g_P}$$

$$= \underbrace{\frac{(f_V g_T - f_T g_V)}{(f_P g_V - f_V g_P)}}_{\left(\frac{\partial P}{\partial T}\right)_U} dT + \underbrace{\left(\frac{f_V g_U - f_U g_V}{f_P g_V - f_V g_P}\right)}_{\left(\frac{\partial P}{\partial U}\right)_T} dU$$

oops! this is not what I wanted to calculate, but I leave it to illustrate a typical calculation.

Continuing Problem A22, Edwards 3.17 pg. 195

$$f_T\,dT + f_u\,du = -f_P\,dP - f_V\,dV$$
$$g_T\,dT + g_u\,du = -g_P\,dP - g_V\,dV$$

$$\begin{bmatrix} f_T & f_u \\ g_T & g_u \end{bmatrix}\begin{bmatrix} dT \\ du \end{bmatrix} = \begin{bmatrix} -f_P\,dP - f_V\,dV \\ -g_P\,dP - g_V\,dV \end{bmatrix}$$

Kramer's Rule

$$dT = \frac{\det\begin{bmatrix} -f_P\,dP - f_V\,dV & f_u \\ -g_P\,dP - g_V\,dV & g_u \end{bmatrix}}{\det\begin{bmatrix} f_T & f_u \\ g_T & g_u \end{bmatrix}}$$

$$= \frac{-g_u(f_P\,dP + f_V\,dV) + f_u(g_P\,dP + g_V\,dV)}{f_T g_u - f_u g_T}$$

$$= \underbrace{\left(\frac{f_u g_P - f_P g_u}{f_T g_u - f_u g_T}\right)}_{\left(\frac{\partial T}{\partial P}\right)_V \ \text{I}} dP + \underbrace{\left(\frac{f_u g_V - f_V g_u}{f_T g_u - f_u g_T}\right)}_{\left(\frac{\partial T}{\partial V}\right)_P} dV$$

Likewise, Kramer's Rule for du,

$$du = \frac{1}{f_T g_u - f_u g_T}\det\begin{bmatrix} f_T & -f_P\,dP - f_V\,dV \\ g_T & -g_P\,dP - g_V\,dV \end{bmatrix}$$

$$= \frac{1}{f_T g_u - f_u g_T}\left(-f_T g_P\,dP - f_T g_V\,dV + f_P g_T\,dP + f_V g_T\,dV\right)$$

$$= \underbrace{\left(\frac{f_P g_T - f_T g_P}{f_T g_u - f_u g_T}\right)}_{\left(\frac{\partial u}{\partial P}\right)_V \ \text{II}} dP + \underbrace{\left(\frac{f_V g_T - f_T g_V}{f_T g_u - f_u g_T}\right)}_{\left(\frac{\partial u}{\partial V}\right)_P} dV$$

We still need to calculate $\left(\frac{\partial u}{\partial T}\right)_V$, $\left(\frac{\partial u}{\partial T}\right)_P$ & $\left(\frac{\partial u}{\partial P}\right)_T$

PROBLEM 22 continued

Note,

$$f_P\,dP + f_u\,du = -f_v\,dv - f_T\,dT$$
$$g_P\,dP + g_u\,du = -g_v\,dv - g_T\,dT$$

$$dP = \frac{\det\begin{bmatrix} -f_v\,dv - f_T\,dT & f_u \\ -g_v\,dv - g_T\,dT & g_u \end{bmatrix}}{f_P g_u - f_u g_P} = -f_v g_u\,dV - f_T g_u\,dT$$

$$= \frac{-f_v g_u\,dv - f_T g_u\,dT + f_u g_v\,dV + f_u g_T\,dT}{f_P g_u - f_u g_P}$$

$$= \underbrace{\left(\frac{f_u g_T - f_T g_u}{f_P g_u - f_u g_P}\right)}_{\left(\frac{\partial P}{\partial T}\right)_V} dT + \underbrace{\left(\frac{f_u g_v - f_v g_u}{f_P g_u - f_u g_P}\right)}_{\left(\frac{\partial P}{\partial V}\right)_T} dV$$

Likewise,

$$du = \frac{\det\begin{bmatrix} f_P & -f_v\,dv - f_T\,dT \\ g_P & -g_v\,dv - g_T\,dT \end{bmatrix}}{f_P g_u - f_u g_P}$$

$$= \frac{-f_P g_v\,dV - f_P g_T\,dT + g_P f_v\,dV + g_P f_T\,dT}{f_P g_u - f_u g_P}$$

$$= \underbrace{\left(\frac{f_T g_P - f_P g_T}{f_P g_u - f_u g_P}\right)}_{\left(\frac{\partial u}{\partial T}\right)_V} dT + \underbrace{\left(\frac{f_V g_P - f_P g_V}{f_P g_u - f_u g_P}\right)}_{\left(\frac{\partial u}{\partial V}\right)_T} dV$$

III

PROBLEM 22 Continued

More of the same,

$$f_u\,du + f_v\,dV = -f_P\,dP - f_T\,dT$$
$$g_u\,du + g_v\,dV = -g_P\,dP - g_T\,dT$$

$$du = \frac{\det\begin{bmatrix} f_u & -f_P\,dP - f_T\,dT \\ g_u & -g_P\,dP - g_T\,dT \end{bmatrix}}{f_u g_v - f_v g_u}$$

$$= \frac{-f_u g_P\,dP - f_u g_T\,dT + f_T g_u\,dT + f_P g_u\,dP}{f_u g_v - f_v g_u}$$

$$= \underbrace{\left(\frac{f_P g_u - f_u g_P}{f_u g_v - f_v g_u}\right)}_{\left(\frac{\partial u}{\partial P}\right)_T} dP + \underbrace{\left(\frac{f_T g_u - f_u g_T}{f_u g_v - f_v g_u}\right)}_{\left(\frac{\partial u}{\partial T}\right)_P \ \text{IV}} dT$$

Consider then,

$$\left(\frac{\partial u}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V = \underbrace{\left(\frac{f_T g_P - f_P g_T}{f_P g_u - f_u g_P}\right)}_{\text{see III}} \underbrace{\left(\frac{f_u g_P - f_P g_u}{f_T g_u - f_u g_T}\right)}_{\text{see I}} = \underbrace{\frac{f_P g_T - f_T g_P}{f_T g_u - f_u g_T}}_{\text{see II}} = \left(\frac{\partial u}{\partial P}\right)_V$$

Also,

$$\left(\frac{\partial u}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V + \left(\frac{\partial u}{\partial P}\right)_T = \underbrace{\left(\frac{\cancel{f_T g_u - f_u g_T}}{f_u g_v - f_v g_u}\right)}_{\text{IV}} \underbrace{\left(\frac{f_u g_P - f_P g_u}{\cancel{f_T g_u - f_u g_T}}\right)}_{\text{I}} + \underbrace{\frac{f_P g_u - f_u g_P}{f_u g_v - f_v g_u}}_{\text{II}} = 0$$

So, unless I'm missing something,

$$\boxed{\begin{aligned} \left(\frac{\partial u}{\partial P}\right)_T &= -\left(\frac{\partial u}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V \\ \left(\frac{\partial u}{\partial P}\right)_V &= \left(\frac{\partial u}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V \end{aligned}}$$

← seems to disagree with Edwards.

← Edwards agrees.

Remark: It is certainly possible I've made a mistake somewhere in the last 4 pages to solve this problem... however, I hope you get the idea, the interplay between differentials & partials is very neat.

PROBLEM 23 Edwards Problems 4.1, 4.2 & 4.3 of pg. 99

4.1) Find shortest distance from $(1,0)$ to the parabola $y^2 = 4x$.

Objective function : $f(x,y) = (x-1)^2 + y^2$

Constraint function : $g(x,y) = 4x - y^2$

Lagrange Multiplier Method :

$$\nabla f = \lambda \nabla g \longrightarrow \langle 2(x-1), 2y \rangle = \lambda \langle 4, -2y \rangle$$

$$2(x-1) = 4\lambda$$

$$2y = -2\lambda y \Rightarrow y(1+\lambda) = 0$$

Thus $y = 0$ or $\lambda = -1$. Note that $\lambda = -1 \Rightarrow 2(x-1) = -4$ but then $2x - 2 = -4 \Rightarrow 2x = -2 \Rightarrow x = -1$ but $y^2 = 4x$ does not possess $x = -1$ as a sol$^{\underline{n}}$. Thus $y = 0$ gives only interesting sol$^{\underline{n}}$ and $y^2 = 4x \Rightarrow 0 = 4x \Rightarrow x = 0$

Therefore, $\underline{(0,0) \text{ is the closest point}}$.

4.2) Find points on ellipse $x^2/9 + y^2/4 = 1$ which are closest/furthest from $(1,0)$.

Objective function : $f(x,y) = (x-1)^2 + y^2$

Constraint fnct : $g(x,y) = 4x^2 + 9y^2 - 36$

Lagrange Multiplier : $\nabla f = \lambda \nabla g$

$$\langle 2(x-1), 2y \rangle = \lambda \langle 8x, 18y \rangle$$

Thus, $\begin{matrix} 2x - 2 = 8\lambda x \\ 2y = 18\lambda y \end{matrix} \rightarrow \lambda = \dfrac{x-1}{4x} = \dfrac{y}{9y} = \dfrac{1}{9} \Rightarrow 9x - 9 = 4x$

$\star$

$$5x = 9$$

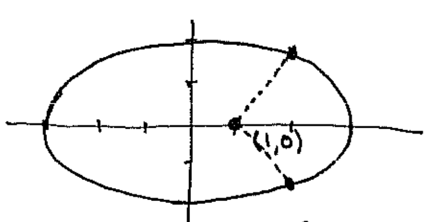
$$\underline{x = 9/5}.$$

Note $x = \frac{9}{5} \Rightarrow \underbrace{\frac{x^2}{9}}+\frac{y^2}{4} = 1$

$$\frac{81}{25(9)} + \frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = 1 - \frac{9}{25} = \frac{16}{25} \rightarrow y^2 = \frac{64}{25} \therefore y = \pm \frac{8}{5}$$

$\underline{\text{Apparently, } (9/5, \pm 8/5) \text{ are closest pts.}}$



Note $y = 0$ is lost in our algebra at $\star$

$y = 0 \Rightarrow 4x^2 = 36 \Rightarrow x^2 = 9 \Rightarrow \underline{x = \pm 3}$

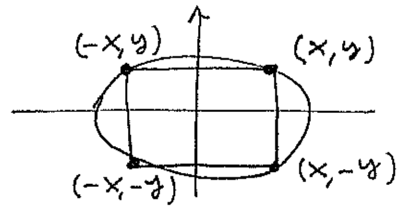
Clearly $\boxed{(-3,0) \text{ is furthest point}}$

(Note that $f(\frac{9}{5}, \frac{8}{5}) = \left(\frac{4}{5}\right)^2 + \frac{64}{25} = \frac{80}{25}$ but $f(3,0) = 4 = \frac{100}{25} \therefore (3,0)$ not extremal.)

PROBLEM 23 continued

4.3) Find max. area of a rectangle, placed horizontal/vertical, inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.

Objective function: ~~$\theta(x,y)$~~ = Area = lw



using symmetry of this particular ellipse.

$$l = 2x \quad \& \quad w = 2y$$

Hence $f(x,y) = 4xy$ ← objective fnct.

$$g(x,y) = \underbrace{b^2x^2 + a^2y^2 - a^2b^2 = 0}_{\text{constraint fnct gives ellipse}}$$

$$\nabla f = \lambda \nabla g$$

$$\langle 4y, 4x \rangle = \lambda \langle 2b^2x, 2a^2y \rangle$$

$$\left.\begin{array}{l} 4y = 2\lambda b^2 x \\ 4x = 2\lambda a^2 y \end{array}\right\} \to \lambda = \frac{4y}{2b^2x} = \frac{4x}{2a^2y}$$

$$a^2y^2 = b^2x^2$$

$$\hookrightarrow 2a^2y^2 = a^2b^2$$

$$y^2 = b^2/2 \quad \therefore \quad y = \frac{\pm b}{\sqrt{2}}$$

$$\Rightarrow x^2 = \frac{a^2y^2}{b^2} = \frac{a^2}{2}$$

$$\Rightarrow x = \frac{\pm a}{\sqrt{2}}$$

By our set-up, $x > 0$, $y > 0$ only interesting sol'n's hence $\boxed{x = \frac{a}{\sqrt{2}} \ \& \ y = \frac{b}{\sqrt{2}}}$

When $a = b$ the ellipse becomes a circle and the maximum area inscribed rectangle is a square.

$\boxed{\text{PROBLEM A24}}$ 4.10, 4.12 Edwards, Pg. 99

4.10 Show that the rectangular solid of largest volume inscribed in unit-sphere is a cube

Let (x,y,z) be a corner of the solid in octant 1 then it follows the other corners are at $(-x,y,z), (x,-y,z)$ etc... hence $V = (2x)(2y)(2z) = 8xyz$.

Objective fnct.: $f(x,y,z) = 8xyz$

Constraint: $g(x,y,z) = x^2+y^2+z^2-1 = 0$

$$\nabla f = \lambda \nabla g$$

$$\langle 8yz, 8xz, 8xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\left.\begin{aligned} 8yz &= 2x\lambda \\ 8xz &= 2y\lambda \\ 8xy &= 2z\lambda \end{aligned}\right\} \longrightarrow \frac{\lambda}{4} = \frac{yz}{x} = \frac{xz}{y} = \frac{xy}{z}$$

$$\underbrace{\underbrace{x^2 = y^2}\quad \underbrace{y^2 = z^2}}$$

$$x^2 = y^2 = z^2$$

$$\Rightarrow 3x^2 = 1 \therefore x = \pm 1/\sqrt{3}$$

Clearly $x=0$, $y=0$ or $z=0$ does not yield any interesting sol^n.

The interesting sol^n is $x = y = z = \frac{1}{\sqrt{3}}$ (a cube).

4.12 Find max product of sines of three angles in triangle. Show such a triangle is equilateral

Objective: $f(x,y,z) = \sin x \sin y \sin z$

Constraint: $x+y+z = \pi$ a.k.a. $g(x,y,z) = x+y+z-\pi$

$$\nabla f = \lambda \nabla g$$

$$\langle \cos x \sin y \sin z, \sin x \cos y \sin z, \sin x \sin y \cos z \rangle = \lambda \langle 1,1,1 \rangle$$

$$\begin{aligned} \cos x \sin y \sin z &= \lambda \\ \sin x \cos y \sin z &= \lambda \\ \sin x \sin y \cos z &= \lambda \end{aligned} \quad \begin{aligned} &\longrightarrow \tan(x) = \tan(y) \\ &\longrightarrow \tan(y) = \tan(z) \end{aligned}$$

Thus $\tan(x) = \tan(y) = \tan(z)$. This yields $\exists\, m,n,k \in \mathbb{Z}$ s.t. $x + m\pi = y + n\pi = z + k\pi$. However, $x+y+z = \pi$ therefore we obtain sol^n's where $m=n=k$ since otherwise the sum of the angles wouldn't need to be π. Thus $x = y = z$, it's equilateral triangle.

PROBLEM 25] problem 5.4, 5.5, 5.6 of pg. 116

5.4] Find points on ellipse $x^2/9 + y^2/4 = 1$ which are closest and farthest from the pt. $(1,1)$

Objective fnct: $f(x,y) = (x-1)^2 + (y-1)^2$

Constraint: $g(x,y) = 4x^2 + 9y^2 - 36 = 0$

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x-1), 2(y-1) \rangle = \lambda \langle 8x, 18y \rangle$$

$$\left.\begin{array}{l} 2x - 2 = 8\lambda x \\ 2y - 2 = 18\lambda y \end{array}\right\} \rightarrow \frac{y-1}{x-1} = \frac{9y}{4x}$$

$$4xy - 4x = 9xy - 9y$$

$$\hookrightarrow 5xy = 9y - 4x$$

$$\hookrightarrow y = \frac{4x}{9-5x}$$

$$\frac{x^2}{9} + \frac{1}{4}\left(\frac{4x}{9-5x}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{4x^2}{(9-5x)^2} = 1$$

$$x^2(9-5x)^2 + (4x^2)9 = 9(9-5x)^2$$

$$(x^2-9)(9-5x)^2 = -36x^2$$

$$(x^2-9)(81 - 90x + 25x^2) = -36x^2$$

$$81x^2 - 90x^3 + 25x^4 - 9(81) - 810x - 225x^2 = -36x^2$$

5.5] Find maximal volume of closed rectangular box whose total surface area is 54

Objective: $V = xyz$

Constraint: $g(x,y,z) = 2xy + 2yz + 2xz$

$$\nabla V = \lambda \nabla g$$

$$\langle yz, xz, xy \rangle = \lambda \langle 2y + 2z, 2x + 2z, 2y + 2x \rangle$$

$$yz = 2\lambda (y+z)$$
$$xz = 2\lambda (x+z)$$
$$xy = 2\lambda (x+y)$$

$$2\lambda = \frac{yz}{y+z} = \frac{xz}{x+z} = \frac{xy}{x+y}$$

$$y(x+z) = x(y+z) \qquad z(x+y) = y(x+z)$$

$$x(y+z) = z(x+y) = y(x+z)$$

$$xy + xz = zx + zy = yx + yz$$

$$\Rightarrow \underline{xy = zy}. \qquad \underline{zx = yx}.$$

$$\Rightarrow \underline{x = z}. \qquad \underline{z = y}.$$

Then $\underbrace{2xy + 2yz + 2xz}= 54$

$$2x^2 + 2x^2 + 2x^2 = 6x^2 = 54 \quad \therefore x^2 = 9$$

$$x = \pm 3$$

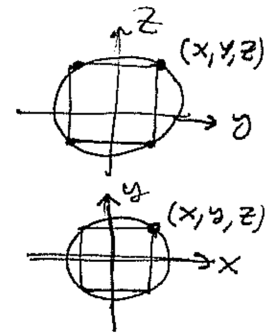
But, $x = 3$

(since I'm using x to denote a length)

$$\boxed{\text{Hence } V(3,3,3) = 27 \text{ cube is answer.}}$$

PROBLEM 25 continued

5.6 Find dimensions of box of maximal volume which can be inscribed in $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.



Objective function: $V = 8xyz$ (draw picture to see this

constraint: $g(x,y,z) = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$

$$\nabla V = \lambda \nabla g$$

$$\langle 8yz,\ 8xz,\ 8xy \rangle = \lambda \left\langle \frac{2x}{a^2},\ \frac{2y}{b^2},\ \frac{2z}{c^2} \right\rangle$$

$$\left.\begin{aligned} 8yz &= \frac{2\lambda x}{a^2} \\ 8xz &= \frac{2\lambda y}{b^2} \\ 8xy &= \frac{2\lambda z}{c^2} \end{aligned}\right\} \quad \frac{2\lambda}{8} = \frac{a^2yz}{x} = \frac{b^2xz}{y} = \frac{c^2xy}{z}$$

$$\Rightarrow a^2y^2z^2 = b^2x^2z^2 = c^2x^2y^2$$

$$\Rightarrow a^2y^2 = b^2x^2 \ \ \& \ \ b^2z^2 = c^2y^2$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} \ \ \& \ \ \frac{z^2}{c^2} = \frac{y^2}{b^2}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\Rightarrow \frac{3x^2}{a^2} = \frac{3y^2}{b^2} = \frac{3z^2}{c^2} = 1$$

$$\Rightarrow x = \frac{a}{\sqrt{3}},\quad y = \frac{b}{\sqrt{3}},\quad z = \frac{c}{\sqrt{3}}.$$

$$\therefore \boxed{V = \frac{8abc}{3\sqrt{3}}}$$

$\boxed{\text{PROBLEM 26}}$ problems 5.10, 5.11, 5.12 & 5.14 of p. 116-117

5.10) Constraints
$$\left.\begin{aligned} g_1(x,y,z) &= x+2y+z-4 \\ g_2(x,y,z) &= 3x+y+2z-3 \end{aligned}\right\} \quad \underbrace{G=(g_1,g_2)=0}$$

gives line of intersection of planes $g_1=0$ & $g_2=0$.

Objective: $f(x,y,z) = x^2+y^2+z^2$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\langle 2x, 2y, 2z\rangle = \lambda_1 \langle 1, 2, +1\rangle + \lambda_2 \langle 3, 1, 2\rangle$$

$$\begin{aligned} 2x &= \lambda_1 + 3\lambda_2 \\ 2y &= 2\lambda_1 + \lambda_2 \\ 2z &= \lambda_1 + 2\lambda_2 \end{aligned}$$

$\Rightarrow \begin{aligned} 4x &= 2\lambda_1 + 6\lambda_2 \\ 2y &= 2\lambda_1 + \lambda_2 \end{aligned} \longrightarrow 4x - 2y = 5\lambda_2$

$\begin{aligned} 4y &= 4\lambda_1 + 2\lambda_2 \\ 2z &= \lambda_1 + 2\lambda_2 \end{aligned} \longrightarrow \underline{4y - 2z = 3\lambda_1}.$

$\begin{aligned} 2x &= \lambda_1 + 3\lambda_2 \\ 6y &= 6\lambda_1 + 3\lambda_2 \end{aligned} \longrightarrow \underline{5\lambda_1 = 6y - 2x}.$

$$\lambda_1 = \frac{6y-2x}{5} = \frac{4y-2z}{3} \Rightarrow \underline{18y - 6x = 20y - 10z} \quad \text{I}.$$

Note

$\begin{aligned} 2y &= 2\lambda_1 + \lambda_2 \\ 4z &= 2\lambda_1 + 4\lambda_2 \end{aligned} \longrightarrow 3\lambda_2 = 4z - 2y \longrightarrow \lambda_2 = \dfrac{4z-2y}{3}$

$$\lambda_2 = \frac{4z-2y}{3} = \frac{4x-2y}{5} \longrightarrow \underline{20z - 10y = 12x - 6y} \quad \text{II}.$$

Then I $\longrightarrow 2y = 10z - 6x$

$\underline{y = 5z - 3x}.$

Likewise II $\longrightarrow 4y = 20z - 12x \longrightarrow \underline{y = 5z - 3x}.$

Then
$$\begin{aligned} g_1 &= 0 = x + 2y + z - 4 \\ g_2 &= 0 = 3x + y + 2z - 3 \end{aligned}$$

$$0 = \cancel{3x} + (5z - \cancel{3x}) + 2z - 3 \Rightarrow 7z = 3 \Rightarrow \underline{z = 3/7}.$$

But, $z = 3/7 \Rightarrow y = 15/7 - 3x$

$g_1 = 0 \rightarrow x + 2\left(\frac{15}{7} - 3x\right) + \frac{3}{7} = 4$

$-5x + \frac{30}{7} + \frac{3}{7} = \frac{28}{7} \longrightarrow 5x = \frac{5}{7} \quad \therefore \underline{x = 1/7}.$

Then $y = \frac{15}{7} - \frac{3}{7} = \underline{\frac{12}{7} = y}. \quad \therefore \boxed{(1/7,\ 12/7,\ 3/7)}$

PROBLEM 26 continued

S.11)

Constraints $\begin{cases} g_1(x,y,z) = x^2+y^2-1 = 0 \\ g_2(x,y,z) = x+y+z-1 = 0 \end{cases}$

objective function $f(x,y,z) = z$.

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\langle 0, 0, 1 \rangle = \lambda_1 \langle 2x, 2y, 0 \rangle + \lambda_2 \langle 1, 1, 1 \rangle$$

$$\left.\begin{array}{l} 0 = 2\lambda_1 x + \lambda_2 \\ 0 = 2\lambda_1 y + \lambda_2 \\ 1 = \lambda_2 \end{array}\right\} \rightarrow \underbrace{\begin{array}{l} 0 = 2\lambda_1 x + 1 \\ 0 = 2\lambda_1 y + 1 \end{array}}$$

$$-1 = 2\lambda_1 x = 2\lambda_1 y$$

$$\Rightarrow \underline{x = y}.$$

Then $x+y+z-1 = 0 \Rightarrow z = 1-2x$, well that doesn't help!

on the other hand, $x^2+x^2-1 = 0 \Rightarrow x^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2}$

Now it helps, $z = 1 - 2\left(\frac{\pm 1}{\sqrt{2}}\right) = \frac{\sqrt{2} \pm 2}{\sqrt{2}}$ and we

find two points, $\boxed{\left(1/\sqrt{2},\ 1/\sqrt{2},\ \frac{\sqrt{2}+2}{\sqrt{2}}\right) \text{ and } \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{\sqrt{2}-2}{\sqrt{2}}\right)}$

Highest z-value is $\frac{\sqrt{2}+2}{\sqrt{2}}$ and lowest $\frac{\sqrt{2}-2}{\sqrt{2}}$.

~~S.12)~~

~~Constraints $g_1(x,y,z) = x+y-10$~~

~~$g_2(x,y,z) = x^2+2y^2-1$ need to use g_2~~

S.12) Let $x+y = 10$ and $u^2+2v^2 = 1$ so constraints are

$g_1(x,y,u,v) = x+y-10$ and $g_2(x,y,u,v) = u^2+2v^2-1$.

Objective function: $f(x,y,u,v) = (x-u)^2 + (y-v)^2$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

$$\langle 2(x-u), 2(y-v), -2(x-u), -2(y-v) \rangle = \lambda_1 \langle 1, 1, 0, 0 \rangle + \lambda_2 \langle 0, 0, 2u, 4v \rangle$$

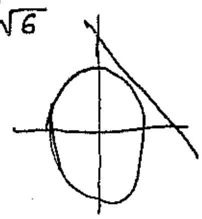
$$\lambda_1 = 2(x-u) = 2(y-v) \Rightarrow \underline{x-u = y-v}$$

$$\lambda_2 = \frac{x-u}{-u} = \frac{y-v}{-2v} \longrightarrow \frac{1}{-u} = \frac{1}{-2v} \longrightarrow u = 2v$$

Hence, $u^2+2v^2 = 1 \Rightarrow 4v^2+2v^2 = 1 \Rightarrow v^2 = 1/6 \Rightarrow v = \frac{\pm 1}{\sqrt{6}}$ & $u = \frac{\pm 2}{\sqrt{6}}$ ★

Also, $x-u = y-v \Rightarrow x-y = v = \pm 1/\sqrt{6}$

$$\left.\begin{array}{l} x-y = \pm 1/\sqrt{6} \\ x+y = 10 \end{array}\right\} \quad \underline{x = \frac{10 \pm 1/\sqrt{6}}{2}} ★ \quad \underline{y = \frac{10 \mp 1/\sqrt{6}}{2}} ★$$



Then you can select the answer from the choices here at ★.

PROBLEM 26 continued

5.14 Find points of ellipsoid $x^2+2y^2+3z^2=1$ which are closest & farthest from $x+y+z=10$

Constraint: $G(x,y,z,u,v,w) = (x^2+2y^2+3z^2-1,\ u+v+w-10)$

Objective: $f(x,y,z,u,v,w) = (x-u)^2+(y-v)^2+(z-w)^2$

$$\nabla f = \lambda_1 \nabla G_1 + \lambda_2 \nabla G_2$$

$$\langle 2(x-u), 2(y-v), 2(z-w), -2(x-u), -2(y-v), -2(z-w) \rangle =$$

$$= \lambda_1 \langle 2x, 4y, 6z, 0,0,0 \rangle + \lambda_2 \langle 0,0,0,1,1,1 \rangle$$

$$x-u = \lambda_1 x$$
$$y-v = 2\lambda_1 y$$
$$z-w = 3\lambda_1 z$$
$$x-u = -\lambda_2/2$$
$$y-v = -\lambda_2/2$$
$$z-w = -\lambda_2/2$$

$$\underbrace{x-u = y-v = z-w}$$

$$\lambda_1 x = 2\lambda_1 y = 3\lambda_1 z$$

$$x = 2y = 3z$$

$$\underbrace{x^2+2y^2+3z^2=1}$$
$$x^2+2\left(\frac{x^2}{4}\right)+3\left(\frac{x^2}{9}\right)=1$$
$$36x^2+18x^2+12x^2=36$$
$$76x^2=36$$
$$x^2=\frac{36}{76}=\frac{18}{38}$$

Thus $x = \pm\frac{6}{\sqrt{76}}$ ∴ $y = \frac{\pm 3}{\sqrt{76}}$ ∴ $z = \frac{\pm 2}{\sqrt{76}}$.

$$\frac{\pm 6}{\sqrt{76}} - u = \frac{\pm 3}{\sqrt{76}} - v = \frac{\pm 2}{\sqrt{76}} - w$$

$$u = \frac{\pm 3}{\sqrt{76}} + v, \quad w = \frac{\pm 1}{-\sqrt{76}} + v$$

$$10 = u+v+w = \frac{\pm 3}{\sqrt{76}} + v + v - \frac{\pm 1}{\sqrt{76}} + v$$

$$3v = 10 \mp \frac{2}{\sqrt{76}} \Rightarrow \underline{v = \frac{1}{3}\left(10 \mp \frac{2}{\sqrt{76}}\right)}$$