Need to do covariant D on Vector & Principal bundle. The associated bundle gives a way to do both.

CONNECTIONS

Defining feature of fiber bundle is that locally it is a product of base space M and fiber F. But, globally is not (generally). New question is does the tangent space split? Can I write

$$TP = TM \times TF \qquad \text{where } P \overset{loc}{=} M \times F$$

That is can we write $TP = H \oplus V$ where H & V are sub-bundles of TP

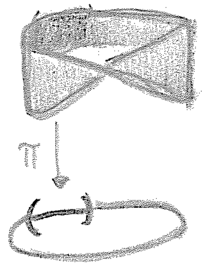
$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr] & TP \arrow[d] \\ & P \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, hook] \arrow[dr] & TP \arrow[d] \\ & P \end{tikzcd}$$

natural, canonical choice.

The vertical vectors are for free, but the horizontal vectors need a choice.

Assumption is that we can find smooth vector fields in nbhd. of point that span H.

Connections

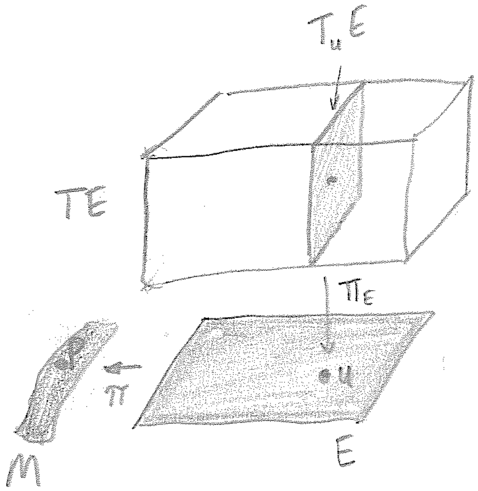


a connection splits the tangent space at each point in a systematic manner.

$$T_u E = H_u \oplus V_u$$

Even though $E \neq M \times F$. So we can decompose at the point level, but we can at the vector level and higher

We have a natural definition for vertical vectors in terms of $d\pi$ but we need a choice to make decomposition above.



$\boxed{\text{Def}^{n}/ \text{ Let } (E, M, \pi, F) \text{ be a fiber bundle. For } u \in E \text{ a vector } v \in T_u E \text{ is vertical iff } d_u\pi(v) = 0 \text{ where } d_u \pi : T_u E \longrightarrow T_{\pi(u)} M. \text{ We denote the set of } \underline{\text{all}} \text{ vertical vectors at } u \in E \text{ by } VT_uE \text{ (vertical tangent vectors at } u)}$

<u>Remark</u>: If $U \subset M$ is open and $\pi_U : U \times F \longrightarrow U$ is the cannonical projection then

or $\oplus$

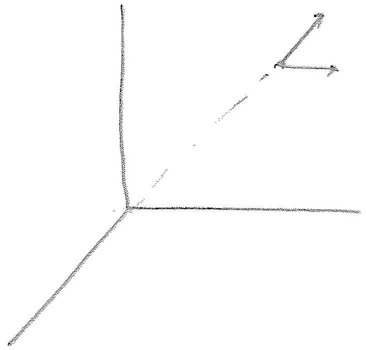
$$d_{(x,f)}\pi_U : T_{(x,f)}(U \times F) = (T_x U) \times (T_f F) \longrightarrow T_x U$$

$$d_{(x,f)}\pi_U (V_x + W_f) = V_x$$

So $V = V_x + W_f$ is vertical iff $d_{(x,f)}\pi_u(V) = 0$ iff $V_x = 0$ iff $V = W_f$ iff V tangent to $\{x\} \times F$ at (x, f)

For example $T(\mathbb{R}^+ \times S^2)$

$$\mathbb{R}^3/\{0\} = \{(r, \varphi, \theta) \mid r > 0 \text{ and } (\varphi, \theta) \in S^2\}$$



$$V = a\frac{\partial}{\partial r} + b\frac{\partial}{\partial \varphi} + c\frac{\partial}{\partial \theta}$$

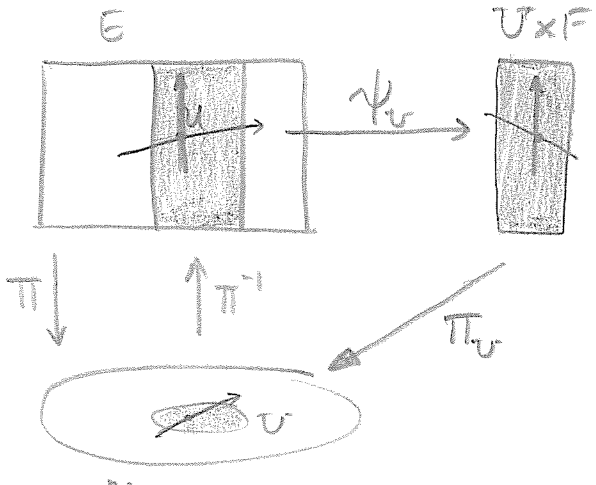
$$\begin{array}{c} \mathbb{R}^+ \times S^2 \\ \Big\downarrow \; \pi(p) = r(p) \\ \mathbb{R}^+ \end{array}$$

then horizontal part is $\frac{\partial}{\partial r}$ and the vertical part is $b\frac{\partial}{\partial \varphi} + c\frac{\partial}{\partial \theta}$.

$d\pi = dr$ so $d\pi\left(b\frac{\partial}{\partial \varphi} + c\frac{\partial}{\partial \theta}\right) = 0$.

Proposition: Using a local trivializing map for an arbitrary fiber bundle (E, M, π, F) say $\psi_U : \pi^{-1}(U) \longrightarrow U \times F$ for $U \subseteq M$, open, we can show that $V \in T_u E$ is vertical iff it is tangent to the fiber of E thru u.

Sketch



We know $\pi_U \circ \psi_U = \pi$

$d_{\psi_U(u)}\pi_U \circ d_u\psi_U = d_u\pi$

then we see

$d_u\pi(V) = 0$

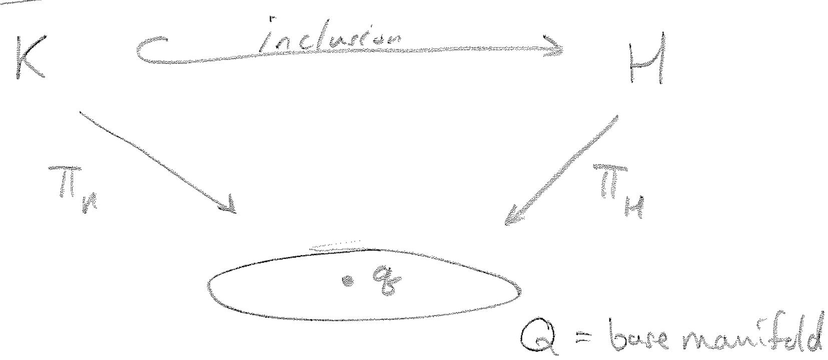
$\iff d\pi_U(d_u\psi_U(V)) = 0$

$\iff d_u\psi_U(V)$ is tangent to fiber of $U \times F$ at $\psi_U(u)$

(Since ψ_U maps fibers to fibers!) $\longrightarrow$ $\iff V$ is tangent to the fiber of E thru u.

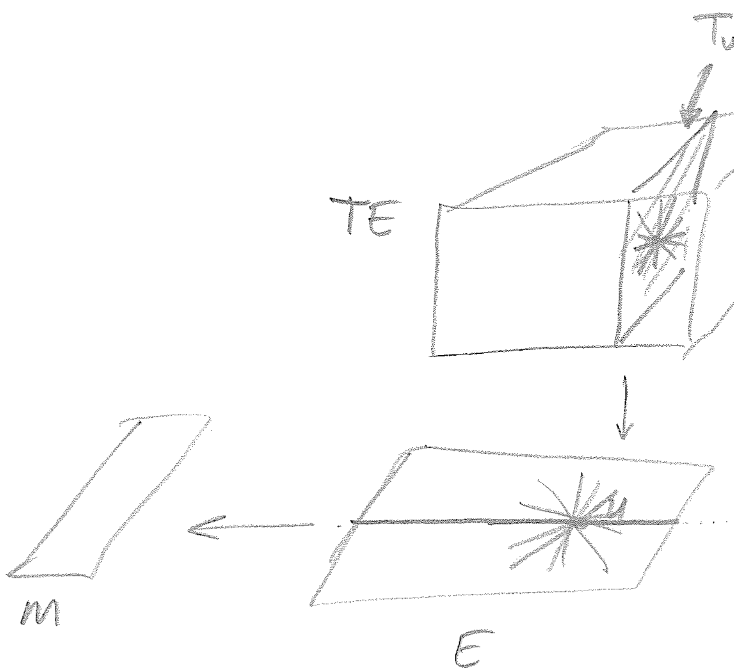
Corr: A vector $v \in T_u E$ is vertical iff it is tangent to $\pi^{-1}(\pi(u))$ at u.

Def^n (SUBBUNDLE) if $H \longrightarrow Q$ is a vector bundle then a bundle $K \longrightarrow Q$ is a subbunde of H iff K_q is a subspace of H_q $\forall q \in Q$. Moreover it is required that for every $q \in Q$ there exist local sections $l_1, l_2, \dots, l_k$ from $U \longrightarrow K$ of $K \longrightarrow Q$ such that $\{l_{i_p}\}$ is a basis of K_p for all $p \in U$.



$$K_q = \pi_K^{-1}(q) \quad \text{whereas} \quad H_q = \pi_H^{-1}(q)$$

- the inclusion map is a vector bundle morphism.



only part of the vectors in $T_u E$ are tangent to fiber

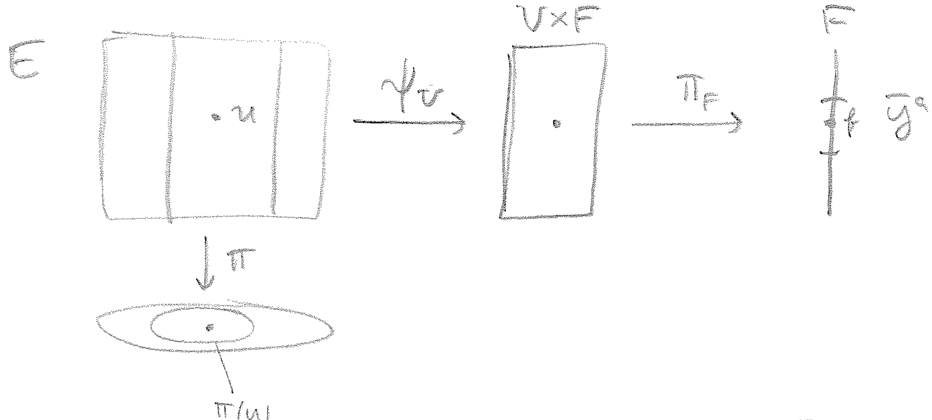
Proposition:
If (E, M, π, F) is a fiber bundle, then $VTE \to E$ is a subbundle of $TE \to E$.

Sketch

Let (x^μ, y^a) be an adapted chart of E in a nbhd of $u \in E$. Choose $\psi_U : \pi^{-1}(U) \to U \times F$ such that $u \in \pi^{-1}(U)$. Choose coordinates

$$(\bar{x}^\mu) \text{ on } U \text{ in } M$$

And choose a chart $(\bar{y}^a)$ of F at $f = (\pi_F \circ \psi_U)(u)$



Then let $x^\mu = \bar{x}^\mu \circ \pi$ and $y^a = \bar{y}^a \circ \pi_F \circ \psi_U$ then (x^μ, y^a) is a chart of $\pi^{-1}(U) \subset E$ at u. This is the adapted chart. Now look at partials of these, they'll span T_uE, $\left\langle \frac{\partial}{\partial x^\mu}\Big|_w, \frac{\partial}{\partial y^a}\Big|_w \right\rangle$ span T_wE for all w near u. Clearly then choosing this same notation,

$$V = a^\mu \frac{\partial}{\partial x^\mu} + b^a \frac{\partial}{\partial y^a}$$

is vertical iff $\underline{a^\mu = 0}$. Thus

$$VT_uE = \left\{ b^a \frac{\partial}{\partial y^a}\Big|_u \;\middle|\; b^a \in \mathbb{R} \right\}$$

You can see this is a subspace spanned by local sections.

Now $VT_uE \subset T_uE$ but what complements VT_uE and is a subspace. More than that so that the complements form a subbundle.

Def^{n}/ A connection on the fiber bundle (E, M, π, F) is a subbundle H of the tangent bundle $TE \to E$ such that for every $u \in E$

$$T_uE = H_u \oplus VT_uE$$

$$\begin{tikzcd} H \arrow[r, hook] \arrow[dr] & TE \arrow[d] \\ & E \end{tikzcd} \qquad \begin{tikzcd} VTE \arrow[r, hook] \arrow[dr] & TE \arrow[d] \\ & E \end{tikzcd}$$

Remark: $\operatorname{Ker}(d_u\pi) = VT_uE$

yet $d_u\pi : T_uE \longrightarrow T_{\pi(u)}M$ thus

$(d_u\pi)\big|_{H_u}$ is a 1-1 map $\operatorname{Ker}(d_u\pi|_{H_u}) = 0$.

And $(d_u\pi|_{H_u} : H_u \longrightarrow T_{\pi(u)}M$ is a vect. space isomorphism.

$$\tilde{X}(u) = \left((d_u\pi|_{H_u} \right)^{-1} \left(X_{\pi(u)} \right) \longleftarrow \text{Horizontal Lift of vector field.}$$

- vertical vectors lie tangent to the fibers

$$VT_uE = \{ v \in T_uE \mid d_u\pi(v) = 0 \}$$
$$= \ker(d\pi)$$
$$= T_u(\pi^{-1}(\pi(u)))$$

- Now infact $d_u\pi$ is nonsingular restricted to H.

$$(d_u\pi|_{H_u} : H_u \longrightarrow T_{\pi(u)}M$$

This map is an injection (manifold dimension, $\pi^{-1}(\pi(u))$ = submanifold)

$$\dim(VT_uE) = \dim(\pi^{-1}(\pi(u))) = \dim(F)$$
$$\dim(T_uE) = \dim(M \times F) = \dim(M) + \dim(F)$$

Therefore,

$$\dim(H_u) = \dim T_uE - \dim VT_uE$$
$$= \dim(M) + \dim(F) - \dim(F)$$
$$= \dim(M)$$
$$= \dim(T_{\pi(u)}M) \quad \therefore d_u\pi|_{H_u} \text{ is injective.}$$

Have linear map from vector space to vector space $\Rightarrow$ must be a bijection. Hence $d_u\pi|_{H_u}$ gives the isomorphism of $H_u \cong T_{\pi(u)}M$.

Now this approach is algebraically clear but pragmatically we employ differential forms for connections, lets make the connection next.

Define a mapping (<u>Ehresmann Connection</u>)

$$\gamma_u : T_u E \longrightarrow V(T_u E)$$

by projecting $H_u \oplus V(T_u E)$ to $V T_u E$

$$\begin{tikzcd} TE \arrow[rr, "\gamma"] \arrow[dr] & & VTE \subseteq TE \arrow[dl] \\ & E & \end{tikzcd}$$

Where we require that its a vector bundlemorphism with,

$$\ker(\gamma_u) = H_u$$

$$\gamma_u / V T_u E = id_{V T_u E}$$

$$\left.\begin{array}{l} \gamma_u^2 = \gamma_u \\ \gamma_u(\gamma_u(v)) = \gamma_u(v) \end{array}\right\} \text{these follow from above, although can reverse it.}$$

This provides an alternative way of spliting the tangent space to E. Namely,

<u>Remark:</u> If $\gamma : TE \longrightarrow VTE$ is a smooth mapping such that

$$\gamma_u^2 = \gamma_u$$

then we can find that $\gamma_u \big/_{\gamma_u(TE)} = id$. So then we can define in this view

$$H_u = \ker(\gamma_u)$$

$$\boxed{\underline{\text{Matrix Theory}}: L : V \longrightarrow V \text{ has } L(V) \oplus \ker(L) = V}$$

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- Now on a <u>principal connection</u> is a bit more structure & we'll need it to construct the connection on a principal bundle.

Def/ If (P, M, π, G) is a principal fiber bundle and $\xi \in \mathfrak{g}$ is the Lie algebra $\mathfrak{g}$ of G we can define a vector field $\xi^{\#}$ on P by (for all u in P)

$$\xi^{\#}_u = \frac{d}{dt}\left(u \cdot \exp(t\xi)\right)\Big|_{t=0}$$

geometrically nice.

(we used δ in fiber bundle notes & Kobayashi uses an # for this)

- Notice here that $\exp(t\xi) \in G$ and u acts on the left by the group action implicit in (P, M, π, G).

- Note that if $u \in P$ and $\sigma_u : G \longrightarrow P$ is defined by $\sigma_u(g) = u \cdot g$ then

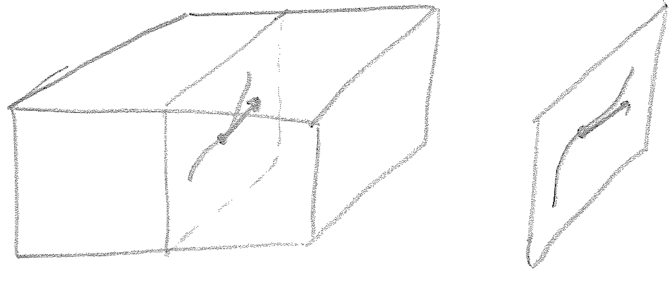
$$\xi^{\#}_u = \frac{d}{dt}\left[\sigma_u(\exp(t\xi))\right]\Big|_{t=0}$$

$$= d_e\sigma_u\left(\frac{d}{dt}\left[\exp(t\xi)\right|_{t=0}\right)$$

$$= d_e\sigma_u(\xi) \quad \text{(algebraically pleasing)}$$

identifying $\mathfrak{g} = T_eG$

Now its obvious this is linear in ξ if we fix u.



Consider then,

$$d_u\pi(\xi_u^{\#}) = d_u\pi(d_e\sigma_u(\xi))$$

$$= d_u(\pi\circ\sigma_u)(\xi) \qquad \text{but } (\pi\circ\sigma_u)(g) = \pi(u\cdot g) = \pi(u)$$

constant mapping.

$$= 0$$

Might be easier to see in exponential notation.

$\therefore$ the mapping from

$$\mathfrak{g} \longrightarrow V(T_u P)$$

defined by $\xi \longrightarrow \xi_u^{\#}$, $u \in P$ is a <u>linear map</u>

> Th$^{\underline{m}}$/ The mapping from $\mathfrak{g} \longrightarrow V(T_u P)$ defined by $\xi \xmapsto{d_e\sigma_u} \xi_u^{\#}$ is an isomorphism of vector spaces for every $u \in P$. Thus we say
>
> $$V\,T_u P = \{\xi_u^{\#} \mid \xi \in \mathfrak{g}\}$$

(Now $u \longmapsto \xi_u^{\#}$ is a vertical vector field, but not every vert. vect. field is of that form. this is a $C^\infty(P)$ module, we can multiply vector fields by functions on P and get back vector fields)

<u>Proof</u>: Recall that the action of G on P is free, thus $(ug = u \Rightarrow g = e)$

$$ug_1 = ug_2 \implies g_1 = g_2 \qquad \sigma_u : G \to P$$

and thus σ_u is one-one for each $u \in P$. Further noting $\sigma_u(g) \equiv u\cdot g$

$$\sigma_u : G \longrightarrow \pi^{-1}(\pi(u)) = u\cdot G \text{ (is onto \& its a submanifold)}$$

in fact this is a diffeomorphism. Thus

$$d_e\sigma_u : T_e G \longrightarrow T_u(\pi^{-1}(\pi(u))) = V(T_u P)$$

is a linear isomorphism & $T_e G = \mathfrak{g}$ $\therefore \xi \longrightarrow \xi_u^{\#} = d_e\sigma_u(\xi)$ //

Vector bundles have no group action, so the last Th^m doesn't make sense directly. We'de have to dig into the trivializing maps for those. Whereas the PFB the connection lives at the tangent space level, where we have a product...

Def^n/ Let (P, M, π, G) be a PFB connection $H \hookrightarrow P$ (a subbundle of $TP \to P$) is a principal connection iff $\forall u \in P$ and $g \in G$

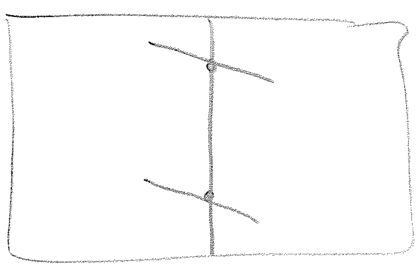
$$\boxed{d_u R_g (H_u) = H_{ug}}$$

Where $R_g : P \to P$ is the right translation (was $\mathcal{R}_g$ before in fib. bundles) on P defined by $R_g(u) = u \cdot g = \sigma_u(g)$.

$$\begin{tikzcd} H \arrow[rr, hook] \arrow[dr] & & TP \arrow[dl] \\ & P & \end{tikzcd}$$

$$\underline{T_u P = H_u \oplus V T_u P}$$

Remark: one can prove $dR_g V(T_u P) = V T_{ug} P$!



- Because $H \hookrightarrow TP$ and $VTP \hookrightarrow TP$ are subbundles every vector field X on P can be written as

$$X = hX + vX$$

Where hX is a smooth section of H over P and vX is a smooth section of VTP over P. Locally since H a sub.bund. $\exists$ local cob down in $u \in P$ then $\{e_a\}$ .. take and write X locally using basis elements which are smooth... sum of smooth is smooth.

$$\begin{array}{ccc} H & & VTP \\ \downarrow & & \downarrow \\ P & & P \end{array}$$

write $h\,\overline{X}$ using smooth local vect. fields from H
& $v\,\overline{X}$ using smooth local vect fields from VTP
they span $\Rightarrow$ $\overline{X} = h\,\overline{X} + v\,\overline{X}$.

Th^{n}/ P bundle over contractible space is automatically trivial.
but connections can be ...

Given a principal connection on a PFB (P, M, π, G) one has a corresponding Ehresmann connection $\gamma : TP \to VTP$

One also has, for each $u \in P$, an isomorphism $\mathfrak{g} \longrightarrow V(T_u P)$ by

$$\xi \longmapsto \xi_u^{\#}$$

Let φ_u be the inverse of this map and define

$$\omega : TP \longrightarrow \mathfrak{g}$$

By $\omega_u : T_u P \longrightarrow \mathfrak{g}$ with $\omega_u = \varphi_u \circ \gamma_u$

$$T_u P \xrightarrow{\gamma_u} VT_u P \xrightarrow{\varphi_u} \mathfrak{g}$$

Note ~~then~~ that we have some properties,

(1.) $\omega_u(H_u) = \varphi_u(\gamma_u(H_u)) = \varphi_u(0) = 0$

(2.) $\omega_u(\xi_u^{\#}) = \varphi_u(\gamma_u(\xi_u^{\#})) = \varphi_u(\xi_u^{\#}) = \xi$

It projects horizontal vectorfields to zero & vertical vector fields to the Lie algebra element that induced it,

It turns out that (not immediately obvious)

$$\boxed{\omega_u(d_u R_g(X_u)) = Ad(g^{-1})(\omega_u(X_u))}$$

Where $u \in P$, $X_u \in T_u P$ and $g \in G$. And the adjoint,

$$i_g : G \longrightarrow G$$

$$i_g(x) = g x g^{-1}$$

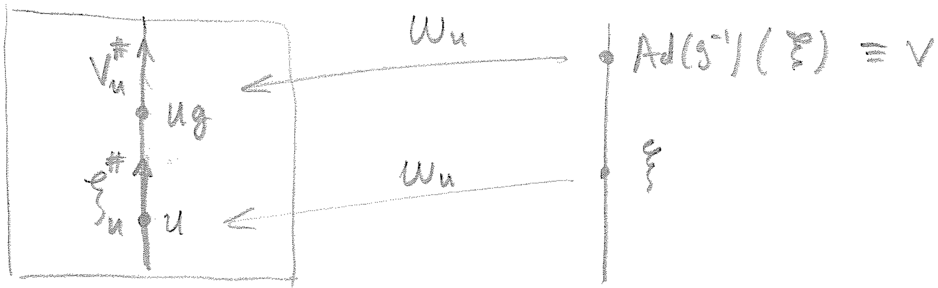
$$Ad(g) = d_e i_g : T_e G \longrightarrow T_e G$$

$$Ad : G \longrightarrow End(\mathfrak{g})$$

Lemma: If (P, M, π, G) is a PFB then

$$d_u R_g(\xi^\#_u) = \left[Ad(g^{-1})(\xi)\right]^\#_{ug}$$

Proof on pg. 256 & 257 of the old notes



Th^m/ If (P, M, π, G) is a PFB with $H \hookrightarrow P$ a principal connection, then $\exists!$ smooth $\mathfrak{g}$-valued one-form $\omega : TP \longrightarrow \mathfrak{g}$ such that

(1.) $\forall u \in P \quad H_u = \ker(\omega_u)$

(2.) $\forall u \in P \quad \omega_u(\xi^\#_u) = \xi$

(3.) $R^*_g \omega = Ad(g^{-1})\omega \quad \forall g \in G$ (need Lemma to prove this, its easy)

Conversely given a 1-form $\omega : TP \longrightarrow \mathfrak{g}$ satisfying (1.) $\longrightarrow$ (3.) it follows that

$$\ker(\omega) \hookrightarrow TP$$

is a principal connection, where $\ker(\omega) = \bigcup_{u \in P} \ker(\omega_u)$

pg. 258 $\rightarrow$ 260 proof

If ω_1 & ω_2 are principal connections (in view of Th^{m} on 52) on P then $\tau = \omega_1 - \omega_2$ has certain properties

(1.) $\tau_u(H_u) = 0$

(2.) $\tau_u(\xi_u^{\#}) = 0$

(3.) $R_g^* \tau = Ad(g^{-1})\tau$

People say that a mapping satisfying conditions (1)–(3) of the Th^{m} is pseudo-tensorial. For example

$$\Gamma = \Gamma^{\lambda}_{\mu\nu} dx^{\mu} \quad \& \quad \Gamma : TM \longrightarrow gl(n)$$

well Γ doesn't transform as a tensor, we say it transforms as a "connection" however their difference of two connections is actually a tensor, the above is the generalization of that.

<u>Remark</u>: Maps like τ are in 1-1 correspondance with sections of the bundle (Lie Algebra Bundle)

$$\begin{tikzcd} P \times_G \mathfrak{g} \arrow[d, shift right] \\ M \arrow[u, shift right, "\tau"'] \end{tikzcd}$$

Where G acts on $\mathfrak{g}$ by the adjoint representation, meaning $g \cdot \xi = Ad(g)(\xi)$

- fix nonphysical background connection ω_0 and vary $\omega_0 + \tau$ with fields, matter... all down on M then solve for τ

$$\begin{array}{c} E \\ \downarrow \\ M \end{array} \qquad \mathcal{L} : J^1 E \longrightarrow \mathbb{R}$$

integrate over space of sections

Jet Bundles

$$\begin{tikzcd} E \arrow[d, shift right] \\ M \arrow[u, shift right, "\varphi"'] \end{tikzcd}$$

$$\varphi(x) = \varphi^a(x)\, e_a(x)$$
$$\bar\varphi(x) = \bar\varphi^a(x)\, e_a(x)$$

$$\bar\varphi \underset{x}{\sim} \varphi \quad \text{if} \quad \begin{aligned} \varphi^a(x) &= \bar\varphi^a(x) \\ \partial_\mu \varphi^a(x) &= \partial_\mu \bar\varphi^a(x) \end{aligned}$$

(1st jet equivalent)

it looks coordinate dependent but could say $\varphi \sim \bar\varphi$ $\varphi(x) = \bar\varphi(x)$ & $d_x \varphi = d_x \bar\varphi$. Now physicists

$$\mathcal{L}(\varphi^a, \partial_\mu \varphi^a) : J^1 E \longrightarrow \mathbb{R}$$

that's it in a nutshell.

$$\begin{tikzcd} P \times_G \mathfrak{g} \arrow[d, shift right] \\ M \arrow[u, shift right, "\tau"'] \end{tikzcd}$$

$$\mathcal{L}(\underbrace{\omega, \partial\omega}_{\text{bosons}}, \underset{\substack{\uparrow \\ \text{fermions}}}{\varphi})$$

Bleeker does gauge theory on PFB, but typically physicists work on base manifold. In physics literature on encounters fields A^a_μ

$$A^a_\mu : M \longrightarrow \mathbb{R}$$

$$\mathcal{L}(A^a_\mu, \partial_\nu A^a_\mu)$$

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In physics one encounters $A^a_\mu : M \longrightarrow \mathbb{R}$. The index "a" is a Lie algebra index & μ is a space-time index. Thus

$$A = A^a_\mu (dx^\mu \otimes e_a)$$

Interpreted as follows, $p \in M$ & $X_p \in T_pM$

$$A(X_p) = A^a_\mu \, dx^\mu (X_p) \, e_a = A^a_\mu X^\mu_p \, e_a$$

$$\Rightarrow \quad A\left(\frac{\partial}{\partial x^\nu}\right) = A^a_\mu \, dx^\mu(\partial_\nu) \, e_a = A^a_\nu \, e_a$$

The field strengths are related to A^a_μ by eq's such as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{bc} A^b_\mu A^c_\nu$$

$\leftarrow$ structure constants of $\mathfrak{g}$ its just $[A_\mu, A_\nu]^a$

$A^a_\mu \leftarrow$ gauge fields

$F^a_{\mu\nu} \leftarrow$ gauge field strengths

$\mathfrak{g} \subset gl(n)$

Free space pure-gauge Lagrangian, using normalization in Bertlemann.

$$\boxed{\mathcal{L}(A^a_\mu, \partial_\nu A^a_\mu) \equiv -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = \frac{1}{2} tr(F_{\mu\nu} F^{\mu\nu})}$$

A <u>gauge transformation</u> (for physicsist who work on M) is a fnct. $g : M \longrightarrow \mathfrak{g}$ of the Lie group G of the theory. where $\{e_a\}$ is a basis of $\mathfrak{g}$. $g \in Maps(M, \mathfrak{g})$ its an ∞ dim'l Lie algebra. The gauge transformation acts on fields

$$\bar{A}_\mu = g^{-1} A_\mu g + g^{-1}(\partial_\mu g)$$

Meaning at $p \in M$,

$$\bar{A}_\mu(p) = g^{-1}(p) A_\mu(p) g(p) + g^{-1}(p)(\partial_\mu g)(p)$$

Just multiply like matrices.

Continuing to decribe <u>physical</u> gauge theory

$$\bar{A}_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

$$\mathcal{L}(\bar{A}^a_\mu, \partial_\nu \bar{A}^a_\mu) = \mathcal{L}(A^a_\mu, \partial_\nu A^a_\mu)$$

$g \in \mathrm{Map}(M, \mathcal{G})$ fixes $\mathcal{L}$ and $\exists$ ∞ly many g's.
there are to many possible gauge fields, how to deal with this... Gribov obstruction...

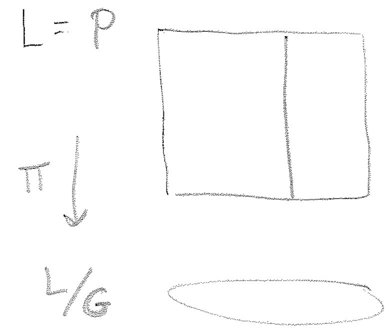
- Next time

$$s^* \omega = A$$

$$\bar{s}^* \omega = \bar{A}$$

$$\bar{s}(x) = s(x) g(x)$$

Let (P, M, π, G) be a PFB and assume that one has a <u>fixed</u> left action $G \times V \longrightarrow V$ which acts linearly on the finite dim'l vector space V.



SU(2) Lie Group $\underset{\text{diff}}{\approx} S^3$

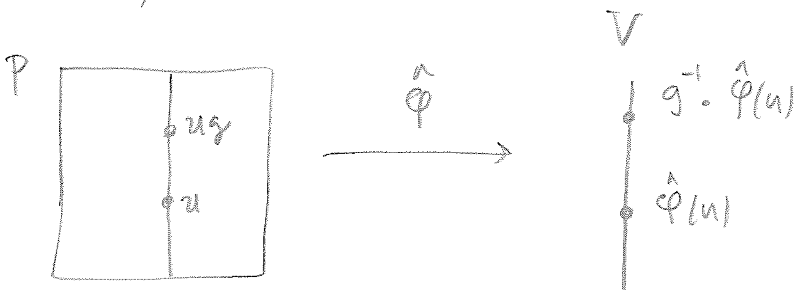
$SU(2)/S_1 \approx S^2$

(for example)

a mapping $\hat{\varphi} : P \longrightarrow V$ is equivariant iff

$$\hat{\varphi}(u \cdot g) = g^{-1} \cdot \hat{\varphi}(u)$$

for all $u \in P$ and $\forall g \in G$. The inverse is because we have left and right action, thanks to linear algebra.



Note that for $u \in P$ there is a mapping

$$\hat{u} : V \longrightarrow \tilde{\pi}^{-1}(\pi(u))$$

defined by

$$\hat{u}(\xi) = (u, \xi)G \qquad \text{for } \xi \in V$$ (not necessarily a Lie Alg. element, careful)

where $\tilde{\pi} : P \times_G V \longrightarrow M$ & $\pi : P \longrightarrow M$.

Recall that $P \times_G V$ is the space of orbits, and its a vector bundle

$$(u, \xi_1)G + (u, \xi_2)G = (u, \xi_1 + \xi_2)G$$

Th^m/ There is a <u>one-one correspondance</u> between the set of all equivariant maps from P to V and the set of all sections of the vector bundle $P \times_G V \xrightarrow{\tilde{\pi}} M$. Explicitly then given an equivariant map $\hat{\varphi} : P \longrightarrow V$

(1.) the corresponding section satisfies the condition

$$\varphi(x) = (u_x, \hat{\varphi}(u_x))G$$

where $u_x \in \pi^{-1}(x)$ is arbitrairy. In particular if $s : U \longrightarrow P$ is a local section of π then

$$\varphi(x) = (s(x), \hat{\varphi}(s(x)))G$$

(2.) Given a section $\varphi : P \times_G V \longrightarrow M$ the corresponding equivariant map $\hat{\varphi} : P \longrightarrow V$ satisfies the equation

$$\hat{\varphi}(u) = \hat{u}^{-1}(\varphi(\pi(u)))$$

for all $u \in P$.

- <u>Remark</u>: this correspondance is really a vector space isomorphism. There are vector space structures hiding here (recall Γ from last course)
- In last class we proved that (1.) & (2.) are inverse. Their composition gives the identity map.

Example

Consider vector-bundle $T^1_1 M$ and take $\tau \in T^1_1 M$

$$\tau(x) = \tau^\mu_\nu \, dx^\nu \otimes \partial_\mu$$

Here $P = \mathcal{F}M = \{ (p, e_i) \mid p \in M, \{e_i\} \text{ basis of } T_p M \}$

$$\hat{\tau} : \mathcal{F}M \longrightarrow ? = T^1_1 \mathbb{R}^m$$

We wanna get rid of the charts and use $\mathbb{R}^m$,

$$T^1_1 M \cong \mathcal{F}M \times_{Gl(m)} T^1_1 \mathbb{R}^m$$

where the group actions are

$$(p, e_i) \cdot g = (p, e_j \, g^j_i)$$

$GL(m)$ acts on $T^1_1 \mathbb{R}^m$ by

$$g \cdot t^i_j e^j \otimes e_i = t^i_j \, g^{-1\,j}_{\ \ l} \, g^k_i \, (e^l \otimes e_k)$$

Which comes from
$$\begin{aligned} g \cdot (e^j \otimes e_i) &= (g \cdot e^j) \otimes (g \cdot e_i) \\ &= (g^{-1\,j}_{\ \ l} e^l) \otimes (g^m_i e_m) \\ &= (g^{-1\,j}_{\ \ l} g^m_i) e^l \otimes e_m \end{aligned}$$

So then what is $\hat{\tau}$

$$u = (p, e_i)$$
$$\hat{u} : T^1_1 \mathbb{R}^m \longrightarrow \tilde{\pi}^{-1}(\pi(u)) \subset T^1_1 M$$

$$\hat{u}(t^i_j e^j \otimes e_i) = t^\mu_\nu \left(dx^\nu \otimes \frac{\partial}{\partial x^\mu} \right)$$

$$\hat{\tau}((p, e_i)) = \left(\widehat{(p, e_i)} \right)^{-1} (\tau(p))$$ continued on next page

$$\hat{\tau}((p, e_i)) = \left(\widehat{(p, e_i)}\right)^{-1} (\tau(p))$$

$$= \left(\widehat{(p, e_i)}\right)^{-1} \tau_\kappa^{\ \lambda} dx^\kappa \otimes \partial_\lambda$$

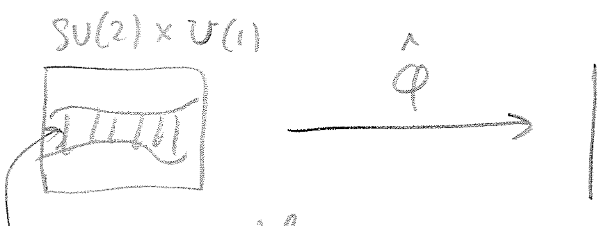
$$= \tau_\nu^{\ \mu} e^\nu \otimes e_\mu$$

And $\hat{\tau} : \mathcal{F}M \longrightarrow T^1_1 \mathbb{R}^m$. We take things from principal bundle $\mathcal{F}M$ into vector bundle $T^1_1 \mathbb{R}^m$.

Motivation: Differentiation can be done in two ways. And covariant derivatives on section has nice geometry. Whereas in physical theories the group G look at $\mathcal{F}M \hookrightarrow \mathcal{O}M \subset \mathcal{F}M$ just construct mapping from $\mathcal{F}M$

$$\mathcal{F}M \xrightarrow{\ \hat{g}\ } T^0_2 \mathbb{R}^m$$

Then take $\hat{g}^{-1}(\eta) = \mathcal{O}_g M$ this $\hat{g}$ is the "symmetry - breaking" mapping.

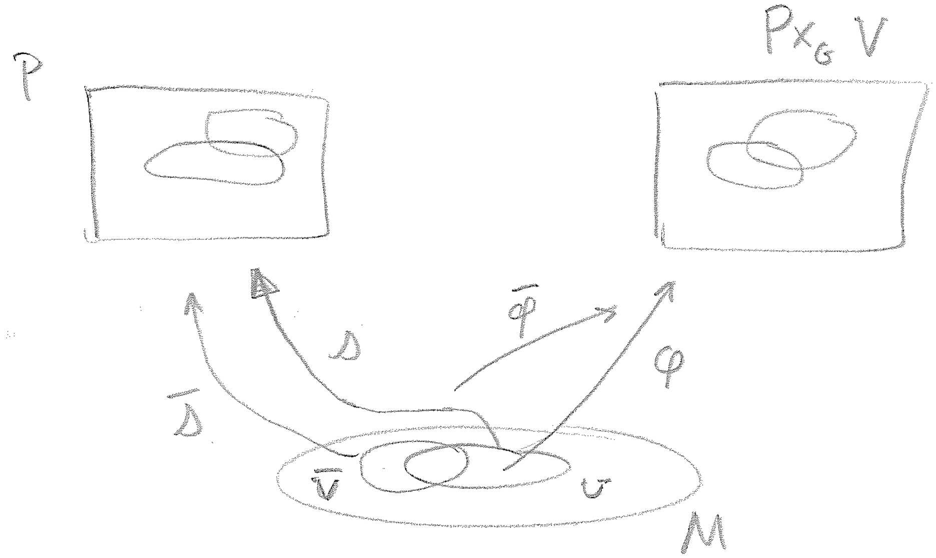


U(1) - bundle
look at components of connections restricted to this subundle & $W^\pm Z$ get mass.

Proof of (1.).

Given $\hat{\varphi} : P \longrightarrow V$ equivariant and a local section $s : U \longrightarrow P$ of $\pi : P \longrightarrow M$. Define

$$\varphi(x) = \varphi_s(x) = (s(x), \hat{\varphi}(s(x)))G \quad \forall x \in G$$



Next assume $\bar{s} : \bar{U} \longrightarrow P$ is another local section such that $U \cap \bar{U} \neq \phi$. Let

$g : \bar{U} \cap U \longrightarrow G$ be defined by $\bar{s}(x) = s(x)g(x)$

Noting such exists as $\bar{s}(x) \in \pi^{-1}(x)$ = group orbit & $s(x) \in \pi^{-1}(x)$ = same orbit $\therefore$ $\bar{s}(x) \in s(x)G$ and $\exists!$ element $g(x)$ of G such that $\bar{s}(x) = s(x)g(x)$ because the group-action is free. Now we have two-definitions of φ

Continuing to prove φ well-defined, 9/20/05 (62)

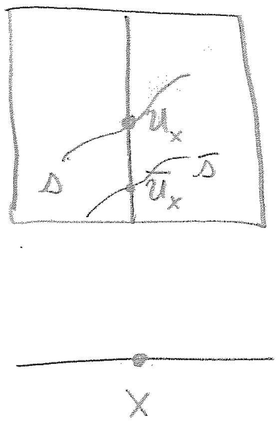
$$\left(\bar{s}(x), \hat{\varphi}(\bar{s}(x))\right)G = \left(s(x)g(x), \hat{\varphi}(s(x)g(x))\right)G$$

$$= \left(s(x)g(x), g^{-1}(x)\hat{\varphi}(s(x))\right)G$$

$$= \left(s(x), \hat{\varphi}(s(x))\right)G$$

from $P\times_G V$ definition of eq classes.

Hence on intersection we have $\bar{\varphi} = \varphi$.



we show φ & $\bar{\varphi}$ are the same independent of choice of u_x, $\bar{u}_x$

• Also $\varphi|_U$ is smooth as it is the composite

$$x \longrightarrow s(x) \longrightarrow \left(s(x), \hat{\varphi}(s(x))\right) \longrightarrow \left(s(x), \hat{\varphi}(s(x))\right)G$$

These maps are smooth thus the composite is smooth.

$$U \xrightarrow{s} P \xrightarrow{id_P \times \hat{\varphi}} P \times V \xrightarrow{\pi_G} P \times_G V$$

Proof of 2.)

Given a section φ of $\tilde{\pi} : P \times_G V \to M$.

Define $\hat{\varphi}$ as before

$$\hat{\varphi}(u) = \hat{u}^{-1}(\varphi(\pi(u)))$$

Need to show its equivariant since its not entirely obvious

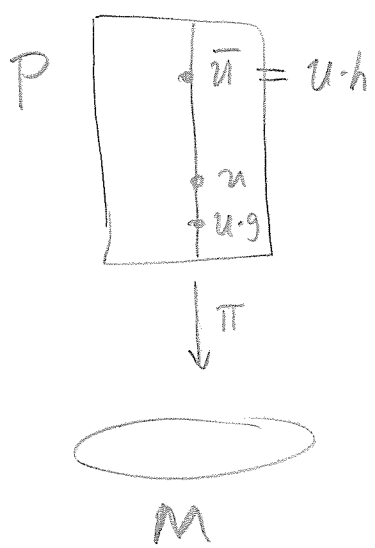
Fix $u \in P$ then let

$$\varphi(\pi(u)) = (\bar{u}, \xi)G \quad \text{for some } (\bar{u}, \xi) \in P \times V$$

no control of these at this stage. We'll pin it down as we go on.

$$\begin{aligned} \text{Then } \pi(u) &= \tilde{\pi}(\varphi(\pi(u))) \\ &= \tilde{\pi}((\bar{u}, \xi)G) \\ &= \pi(\bar{u}) \quad \Rightarrow \quad u, \bar{u} \in \pi^{-1}(u) = u \cdot G \end{aligned}$$

$$\Rightarrow \underline{\bar{u} = u \cdot h \text{ for some } h \in G}$$



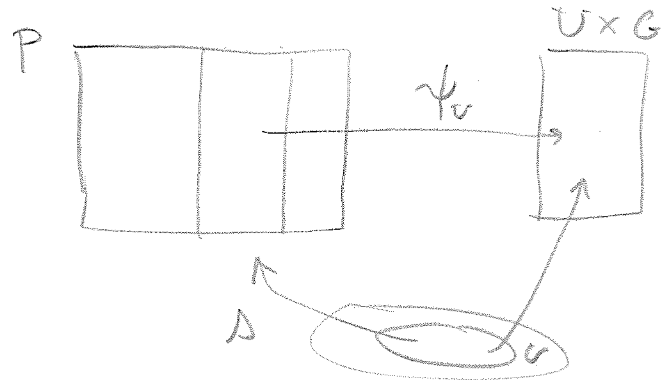
Now lets try to show equivariance,

$$\begin{aligned} \hat{\varphi}(ug) &= \widehat{ug}^{-1}(\varphi(\pi(ug))) \\ &= \widehat{ug}^{-1}(\varphi(\pi(u))) \\ &= \widehat{ug}^{-1}(\bar{u}, \xi)G \\ &= \widehat{ug}^{-1}((uh, \xi)G) \\ &= \widehat{ug}^{-1}((u, h\xi)G) \\ &= \widehat{ug}^{-1}((ug, g^{-1}h\xi)G) \\ &= g^{-1}(h\xi) = g^{-1}\hat{u}^{-1}(u, h\xi)G \\ &= g^{-1} \cdot \hat{\varphi}(u) \quad // \end{aligned}$$

$$\hat{u}(v) = (u, v)G$$
$$v = \hat{u}^{-1}((u, v)G)$$

Claim: $\hat{\varphi}$ is smooth.

Let $\psi_U : \pi^{-1}(U) \longrightarrow U \times G$ be a local trivializing map. And let $s(x) = \psi_U^{-1}(x, e)$.



Now then $(\hat{\varphi} \circ \psi_U^{-1})(x, g) = \hat{\varphi}(\psi_U^{-1}(x, g))$

$$= \hat{\varphi}(\psi_U^{-1}(x,e) \cdot g)$$
$$= \hat{\varphi}(s(x) \cdot g)$$
$$= g^{-1} \hat{\varphi}(s(x))$$

Then $\hat{\varphi} \circ \psi_U^{-1}$ is smooth if the map

$$x \longmapsto \hat{\varphi}(s(x))$$

from U to V is smooth. Now we've assumed φ is smooth but $\hat{\varphi}(u) = \hat{u}^{-1}(\varphi(\pi(u)))$ maps into linear maps... a local trivializing map

$$\Psi_U^{-1}(x, \xi) = (s(x), \xi)G \quad \text{for} \quad P \times_G V \xrightarrow{\pi} M$$

Write then $\varphi(x) = (s(x), \xi(x))G$ for some $\xi(x) \in V$ don't know much about $\xi(x)$ except that it exists. Notice

$$\underbrace{x \longmapsto \Psi_U(\varphi(x))}_{\text{assumed to be smooth}} = \Psi_U(s(x), \xi(x))G = (x, \xi(x))$$

$\therefore$ $x \longmapsto \xi(x)$ is smooth.

Thus $\hat{\varphi}(s(x)) = \widehat{s(x)}(\varphi(x)) = \widehat{s(x)}(s(x), \xi(x))G = \xi(x) \leftarrow$ smooth

Th^{m}/ Assume (P, M, π, G) is a PFB and G acts linearly on the left of a f-dim'l vector space V. If Ξ is a vector field on M and $\hat{\varphi} : P \to V$ is equivariant then $\tilde{\Xi}(\hat{\varphi})$ is equivariant, where $\tilde{\Xi}$ is a horizontal lift (we assume some connection here)

Proof: $u \in P$, $a \in G$. Note $\hat{\varphi}$ is vector valued, $\hat{\varphi}(\bar{u}) = \hat{\varphi}^b(\bar{u}) V_b$
this basis just rides along.

$$\tilde{\Xi}(\hat{\varphi})(ua) = d_{ua}\hat{\varphi}(\tilde{\Xi}(ua))$$

$$= d_{ua}\hat{\varphi}(d_u R_a(\tilde{\Xi}(u)))$$ ← Some fine print here, proved last time.

$$= d_u(\hat{\varphi} \circ R_a)(\tilde{\Xi}(u))$$: $\hat{\varphi} \circ R_a = a^{-1}\hat{\varphi}$ by equivariance's def$^{\underline{n}}$

$$= d_u(a^{-1}\hat{\varphi})(\tilde{\Xi}(u))$$: $L_a : V \to V$, $L_a(x) = a \cdot x$

$$= d_u(L_{a^{-1}} \circ \hat{\varphi})(\tilde{\Xi}(u))$$: now see how to differentiate

$$= (d_{\varphi(u)} L_{a^{-1}} \circ d_u\hat{\varphi})(\tilde{\Xi}(u))$$

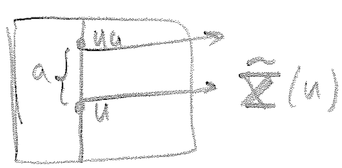
$$= L_{a^{-1}}(d_u\hat{\varphi}(\tilde{\Xi}(u)))$$

$$= a^{-1} d\hat{\varphi}(\tilde{\Xi}(u))$$

$$= a^{-1}\tilde{\Xi}(\hat{\varphi})(u)$$ //

Goal: covariant derivative of metric tensor.

picture



$$\boxed{\begin{array}{l}\text{Th}^{\underline{m}}/ \text{ Notation as before then} \\ D\hat{\varphi} = d\hat{\varphi} + \omega \cdot \hat{\varphi} \\ (D_u\hat{\varphi})(Y_u) = d_u\hat{\varphi}(Y_u) + \omega_u(Y_u)\cdot\hat{\varphi}(u) \in V \\ \forall u \in P \text{ and } Y_u \in T_uP\end{array}}$$

<u>Remark</u>: If $\hat{\varphi} : P \longrightarrow V$ is equivariant then

$D\hat{\varphi} : TP \longrightarrow V$ is defined by

$$D_u\hat{\varphi}(Y_u) \equiv d_u\hat{\varphi}(hY_u)$$

Covariant derivatives have to do with the horizontal part of the bundle. While efficient, this definition isn't very pragmatical. (the statement in $\text{Th}^{\underline{m}}$ is the way to do it.)

<u>Remark</u>: Since G acts linearly on V $\exists$ induced action of $\mathfrak{g}$ on V, $\mathfrak{g} \times V \longrightarrow V$ defined by

$$\xi \cdot W = \frac{d}{dt}\Big[\exp(t\xi)\cdot W\Big]_{t=0}$$

$$= W^n \frac{d}{dt}\Big[\exp(t\xi)\cdot V_n\Big]_{t=0} \qquad W = W^n \underline{\underline{V_n}}$$

basis.

Proof of Th^{m} on 74: $D\hat{\varphi} = d\hat{\varphi} + \omega \cdot \hat{\varphi}$

Let $u \in P$ and $Y_u \in T_uP$. Since $vY_u \in V(T_uP)$

$$vY_u = \xi^{\#}(u) \quad \text{for some } \xi \in \mathfrak{g}$$

$$\omega_u(Y_u) = \omega(hY_u + vY_u) = \omega(\xi^{\#}(u)) = \xi$$

$$\begin{aligned} d\hat{\varphi}(vY) &= d_u\hat{\varphi}(\xi^{\#}(u)) \\ &= d_u\hat{\varphi}\left(\frac{d}{dt}\left[u \cdot \exp(t\xi)\right]_{t=0}\right) \\ &= \frac{d}{dt}\left(\hat{\varphi}(u \cdot \exp(t\xi))\right)\Big|_{t=0} && \text{chain-rule.} \\ &= \frac{d}{dt}\left[\exp(-t\xi) \cdot \hat{\varphi}(u)\right|_{t=0} && : \hat{\varphi}(ua) = a^{-1}\hat{\varphi}(u),\ a = \exp(t\xi),\ a^{-1} = \exp(-t\xi) \\ &= -\xi \cdot \hat{\varphi}(u) && : \text{By definition of } \mathfrak{g} \times V \to V \text{ action (Lie algebra action)} \\ &= -\omega_u(Y_u) \cdot \hat{\varphi}(u) \end{aligned}$$

$$\begin{aligned} d_u\hat{\varphi}(Y_u) &= d_u\hat{\varphi}(hY_u) + d\hat{\varphi}_u(vY) \\ &= D_u\hat{\varphi}(Y_u) - \omega_u(Y_u) \cdot \hat{\varphi}(u) \end{aligned}$$

$$\therefore \quad \underline{D_u\hat{\varphi}(Y_u) = d_u\hat{\varphi}(Y_u) + \omega_u(Y_u) \cdot \hat{\varphi}(u) \in V} \quad //$$

This could likewise be taken as the def^{n} but the $D\hat{\varphi}(Y_u) = d_u\hat{\varphi}(hY_u)$ is a very general notion of covariant differentiation, for many objects...

Consider the following metric tensor

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu \in T^0_2 M$$

Now $\hat{g} : \mathcal{F}M \longrightarrow T^0_2 \mathbb{R}^m$ is defined by

$$\hat{g}(p, e_i) = g_{\mu\nu} A^\mu_i A^\nu_j\, e^i \otimes e^j$$

$$= \boxed{g_{\mu\nu} A^\mu_i A^\nu_j\, \theta^i \otimes \theta^j = \hat{g}(p, e_i)}$$

Where $A^\mu_i,\ A^\nu_j \in GL(m)$. Well what about $\tilde{X}\hat{g} = ?$

Consider,

$$\tilde{\partial}_\mu \hat{g} = g_{\mu\nu} A^\mu_i A^\nu_j\, \tilde{\partial}_\lambda (e^i \otimes e^j)$$

hmm....

!

We'll come back

Recall that,

$$\nabla_{\partial_\lambda}(g_{\mu\nu}) = -\Gamma^\kappa_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa_{\lambda\nu} g_{\mu\kappa}$$

$$\partial_\lambda(\partial_\mu) = \Gamma^\delta_{\lambda\mu}\, \partial_\delta$$

Hmm. weird that he found right answer from !.

Def^n/ (P, M, π, G) a PFB with

$G \times V \longrightarrow V$ a linear action

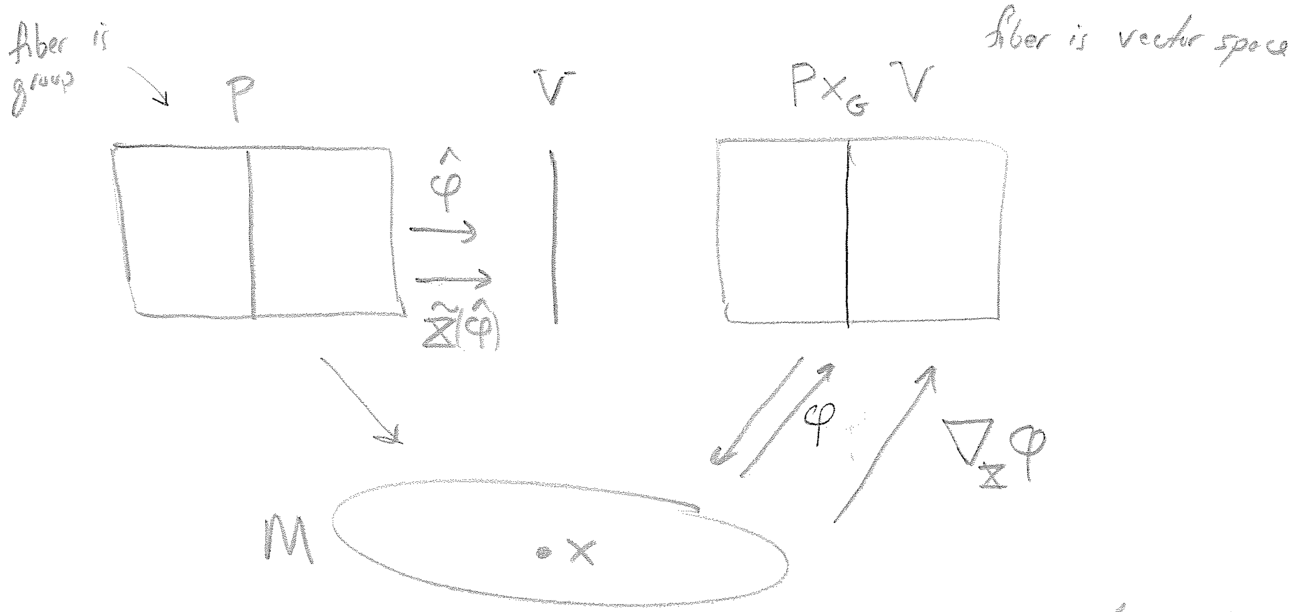
φ a section of associated bundle $P \times_G V \longrightarrow M$

$\hat{\varphi}$ is the corresponding equivariant map from $P \longrightarrow V$

Meaning $\varphi(x) = (u_x, \hat{\varphi}(u_x))G$ for any $u_x \in \pi^{-1}(x), x \in M$.

The covariant derivative of φ along a vector field X on M is the section of $P \times_G V \longrightarrow M$ defined by

$$\left(\nabla_X \varphi\right)(x) = \left(u_x, \tilde{X}(\hat{\varphi})(u_x)\right)G$$



Works for any bundle where you have any action of any sort. eg. SU(3) QCD acts on $\mathbb{C}^3$

take PFB with group SU(3) & connection on ..

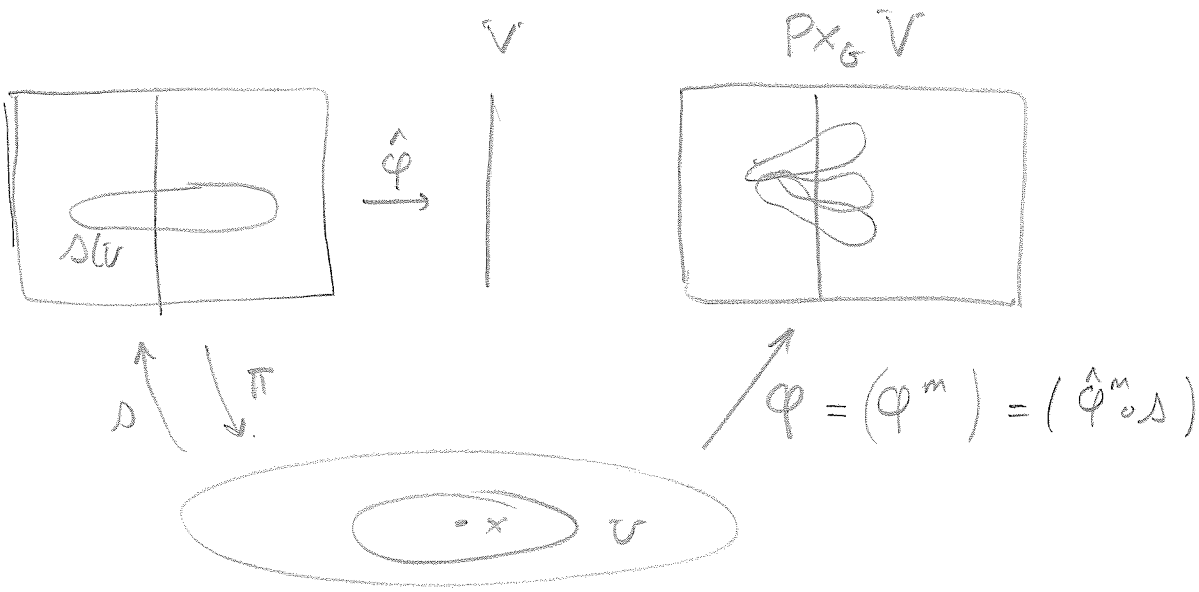
Th^m/ (Assume same notation)

Let $\{e_a\}$ be a basis of $\mathfrak{g}$ and let $\{V_m\}$ be basis of the vector space V. Define <u>numbers</u> $h_{an}^{\ \ m}$ by

$$e_a \cdot V_n = h_{an}^{\ \ m} V_m$$

Let $s : U \to P$ be a local section of $\pi : P \to M$. Let $\{\tilde{V}_n\}$ be the basis of local sections of the vector bundle $P \times_G V$ defined by for $x \in U$

$$\tilde{V}_m(x) \equiv (s(x), V_m)G$$



One section of P yields many in $P \times_G V$. This makes sense, think about FM.

Assume $\hat{\varphi} : P \to V$ is equivariant with corresponding section φ of the bundle $P \times_G V \to M$. If $\hat{\varphi}(u) = \hat{\varphi}^m(u) V_m$ <u>then</u> for $x \in U$ $\varphi(x) = \varphi^m(x)\tilde{V}_m(x)$. Where $\varphi^m(x) = \hat{\varphi}^m(s(x))$. More over if

$$(\nabla_X \varphi)(x) = (\nabla_X \varphi)^m(x)\tilde{V}_m(x) \quad \text{for } x \in U$$

then in fact $(\nabla_{X_x}\varphi)^m(x) = d_x\varphi^m(X_x) + h_{an}^{\ \ m}(s^*\omega)^a(X_x)\varphi^n(x)$

$$(\nabla_{X} \varphi)^m(x) = d_x\varphi^m(X_x) + h^m_{an}\,(s^*\omega)^a(X)\,\varphi^n(x)$$

for $x \in U$. Take $X = \partial_\mu$ and $A^a_\mu = (s^*\omega)^a(\partial_\mu)$ and we recover the usual physicsy formula,

$$(\nabla_{\partial_\mu}\varphi)^m = \partial_\mu \varphi^m + h^m_{an} A^a_\mu \varphi^n$$

(P, M, π, G) a principal fiber bundle

$G \times V$ linear $\underrightarrow{\text{induces}}$ $\mathfrak{g} \times V \longrightarrow V$

$\{e_a\}$ basis of $\mathfrak{g}$

$\{V_m\}$ basis of V

$$e_a \cdot V_n = h^m_{an} V_m$$

$s : U \longrightarrow P$ a local section

$$\tilde{V}_m(x) = (s(x), V_m)G$$

are local sections of $P \times_G V \longrightarrow M$

Th^m/ $\hat{\varphi} : P \longrightarrow V$ equivariant $\longleftrightarrow \varphi : M \longrightarrow P \times_G V$.

If $\hat{\varphi}(u) = \hat{\varphi}^m(u) V_m$ then $\varphi(x) = \varphi^m(x)\tilde{V}_m(x)$ and more than just this we can relate the components

$$\varphi^m(x) = \hat{\varphi}^m(s(x))$$

Also if $(\nabla_X \varphi)(x) = (\nabla_X \varphi)^m(x)\tilde{V}_m(x)$ then ($X \in \mathfrak{X}(M)$)

$$\boxed{(\nabla_X \varphi)^m(x) = d\varphi^m(X_x) + h^m_{an}(s^*\omega)^a(X_x)\varphi^n(x)}$$

Meaning that for $X = \partial_\mu$

$$(\nabla_{\partial_\mu}\varphi)^m(x) = \partial_\mu \varphi^m(x) + h^m_{an} A^a_\mu(x)\varphi^n(x)$$

All in terms of sections of the vector bundle.

Remark: $D\hat{\varphi} = d\hat{\varphi} + \omega \cdot \hat{\varphi}$. Notice D lives on the bundle & equivariant maps whereas ∇ works on sections of vect. bundles.

$\nabla_X \varphi \longleftrightarrow D\hat{\varphi}$ & $(\nabla_X \varphi)(x) = (u_x, \tilde{X}(\hat{\varphi}(u_x)))G$

Proof: First note that if $\hat{\varphi}(u) = \hat{\varphi}^m(u) V_m$ then

$$\varphi(x) = (s(x), \hat{\varphi}(s(x)))G$$
$$= (s(x), \hat{\varphi}^m(s(x)) V_m)G$$
$$= \hat{\varphi}^m(s(x))\,(s(x), V_m)G$$
$$= \hat{\varphi}^m(s(x))\,\tilde{V}_m(x) =$$
$$= \varphi^m(x)\,\tilde{V}_m(x) \quad \therefore \quad \boxed{\varphi^m(x) = \hat{\varphi}^m(s(x))}$$

So we've fixed the bases so that the components in range of φ transfer to comp. in range of $\hat{\varphi}$.

$$(\nabla_X \varphi)(x) = (s(x), \tilde{X}(\hat{\varphi})(s(x)))G$$
$$= (s(x), d\hat{\varphi}(\tilde{X}(s(x))))G$$
$$= (s(x), d\hat{\varphi}^m(\tilde{X}(s(x)))V_m)G$$
$$= d\hat{\varphi}^m(\tilde{X}(s(x)))\,(s(x), V_m)G$$
$$= d\hat{\varphi}^m(\tilde{X}(s(x)))\,\tilde{V}_m(x)$$

Expand $(\nabla_X \varphi)(x)$ in basis of vector bundle

$$\boxed{(\nabla_X \varphi)^m(x) = d\hat{\varphi}^m(\tilde{X}(s(x)))}$$

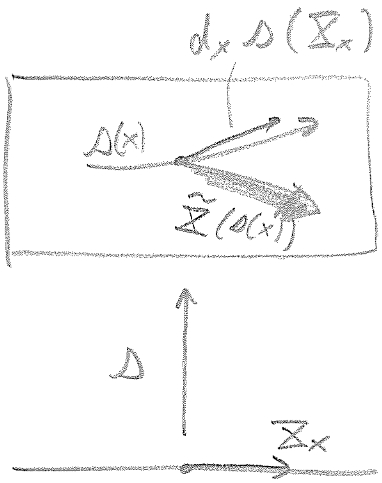
comp. of $\nabla_X \varphi$ relative to $\tilde{V}$ comp. of $d\hat{\varphi}$ relative to V

This will help explain the mystery of the metric confusion last class. We'll see...

Proof Continued (on Dr. Fulp's 48 and we're at 55 now) 9/29/05 82

$$\tilde{X}(s(x)) = d_x s(X_x) - (s^*\omega)(X_x)^{\#}(s(x))$$

A strange result (proved before



the difference is vertical, we proved earlier.

$\star$
$$D\hat{\varphi} = d\hat{\varphi} + \omega\cdot\hat{\varphi}$$
$$d\hat{\varphi}(\text{vert.}) = -\omega_u(\text{vert})\,\hat{\varphi}(u)$$

We have lift on one side & thing being lifted on RHS, anyway

$$d\hat{\varphi}\left(\tilde{X}(s(x))\right) = d\hat{\varphi}\left(d_x s(X_x)\right) - d\hat{\varphi}\left((s^*\omega)(X_x)^{\#}(s(x))\right)$$

$$\overset{\star}{=} d(\hat{\varphi}\circ s)(X_x) + \omega\left[(s^*\omega)(X_x)^{\#}(s(x))\right]\cdot\hat{\varphi}(s(x))$$

$$= d(\hat{\varphi}\circ s)(X_x) + (s^*\omega)(X_x)\cdot\hat{\varphi}(s(x))$$

Remarks to get from 2nd to 3rd equality above

$$\xi = (s^*\omega)(X_x) \in \mathfrak{g} \Rightarrow \xi^{\#}(s(x))$$

Proof Continued

Summary upto now of proof.

$$\boxed{\begin{aligned} (\nabla_X \varphi)(x) &= d\hat{\varphi}^m(\tilde{X}(s(x)))\, \tilde{v}_m(x) \\ d\hat{\varphi}^m(\tilde{X}(s(x))) &= d_x\varphi^m(X_x) + \left[A_x(X_x)\cdot(\hat{\varphi}\circ s)(x)\right]^m \end{aligned}}$$

It'll be easier to work with these formulae as opposed to the actual proof statements. (Should reformulate these things) Continuing we find using linearity of action to pull out scalars,

$$d\hat{\varphi}^m(\tilde{X}(s(x))) = d_x\varphi^m(X_x) + \left[A_x^a(X_x)\hat{\varphi}^n(s(x))(e_a\cdot v_n)\right]^m$$

$$= d_x\varphi^m(X_x) + A_x^a(X_x)\varphi^n(x)\, h_{an}^{\;\;m}$$

$$= \underline{d\varphi^m(X_x) + h_{an}^{\;\;m}(s^*\omega)^a(X_x)\varphi^n(x)} \quad //$$

This shows how to relate equivariant maps to $P\times_G V$

notice $\hat{\varphi}: P \to V$. Is there a diff form on P which is vect-valued which has equivariance property?

Sazbo for example, "Equivariant Cohomology"

Need differential forms which are equivariant...

Samir did some work on this for local functionals...

Metric & Equivariance revisted

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$g = g_{ij}\; e^i \otimes e^j$$

If $e_1, e_2, \dots, e_n$ are linearly indep. vector fields in the nbhd of a point then $\{e^i(p)\}$ is dual to $\{e_i(p)\}$.

$$\hat{g} : \mathcal{F}M \longrightarrow T_2^0 \mathbb{R}^m$$

$$\hat{g}((p, \{e_i\})) = g^e_{ij}(p)(r^i \otimes r^j)$$

This encodes coordinate changes automatically.

$$s(p) = (p, e_i(p))$$

$$s : U \longrightarrow \mathcal{F}M$$

Consider then

$$D_x \hat{g}(Y_x) \equiv d\hat{g}(hY_x)$$

And we also know $D\hat{g} = d\hat{g} + \omega \cdot \hat{g}$ then according to proof summary we read that

$$(\nabla_{\partial_\mu} g)_{ij}(x) = d_x\, g_{ij}(\partial_\mu) + \left[(A_\mu(x)) \cdot (\hat{g} \circ s)(x)\right]_{ij}$$

$$= (\nabla_{\partial_\mu} g_{ij})(x) + [\quad ? \quad]$$

What is

$$\left[A_\mu(x)\cdot(\hat{g}\circ s)(x)\right]_{ij} \quad \text{precisely?}$$

Well $GL(n)$ acts on $T_2^0\mathbb{R}^n$ by

$$a\cdot(\lambda^i\otimes\lambda^j) = (a\cdot\lambda^i)\otimes\lambda^j + \lambda^i\otimes(a\cdot\lambda^j)$$
$$= a^{-1}{}^i_k(\lambda^k\otimes\lambda^j) + a^{-1}{}^j_\ell(\lambda^i\otimes\lambda^\ell)$$

Need the Lie algebra action though. Let us differentiate essentially let $\xi\in gl(n)$

$$\boxed{\xi\cdot(\lambda^i\otimes\lambda^j) = -\xi^i_k(\lambda^k\otimes\lambda^j) - \xi^j_\ell(\lambda^i\otimes\lambda^\ell)}$$

Continuing, $\hat{g}\circ s = (\hat{g}\circ s)_{ij}(\lambda^i\otimes\lambda^j)$

$$(\nabla_{\partial_\mu}g)_{ij} = \nabla_{\partial_\mu}g_{ij} + g_{IJ}(A_\mu(x)\cdot\lambda^I\otimes\lambda^J)$$
$$= \nabla_{\partial_\mu}g_{ij} + g_{IJ}\left[-[\Gamma_\mu]^I_k\lambda^k\otimes\lambda^J - (\Gamma_\mu)^J_L\lambda^I\otimes\lambda^k\right]_{ij}$$
$$= \nabla_{\partial_\mu}g_{ij} + \left[-g_{jk}\Gamma^k_{\mu i}(\lambda^i\otimes\lambda^j) - g_{i\ell}\Gamma^\ell_{\mu j}\lambda^i\otimes\lambda^j\right]_{ij}$$
$$= \boxed{\nabla_{\partial_\mu}g_{ij} - g_{jk}\Gamma^k_{\mu i} - g_{i\ell}\Gamma^\ell_{\mu j} = (\nabla_{\partial_\mu}g)_{ij}}$$

Curvature

Def^{n}/ If ω is a principal connection on a PFB (P, M, π, G) then its <u>curvature</u> is the two-form defined by the covariant derivative D_u on ω

$$\Omega_u(X,Y) = d_u\omega(hX, hY) \equiv D_u\omega(X,Y)$$

<u>Proposition</u>: If ω is a principal connection on (P, M, π, G) then $\Omega_u(X,Y) = D_u\omega(X,Y) = d_u\omega(X,Y) + [\omega(X), \omega(Y)]$

<u>Remarks</u>:

$\varphi : P \longrightarrow V$

$\varphi(u \cdot a) = a^{-1} \cdot \varphi(v)$

ω

$R_g^*\omega = Ad(g^{-1})\omega$

$i_g(x) = gxg^{-1}$

$Ad(g) = d_e i_g \quad g \in G, x \in G$

$ad = d(Ad)$

(had exercise in fiber bundles)

<u>Remark</u>: You need two facts, proved in fiber bundles,

(1.) If $\xi, \eta \in \mathfrak{g}$ then $[\xi^\#, \eta^\#]_{\mathfrak{X}(VP)} = [\xi, \eta]^\#$

(2.) If X is a vector field on M and $\xi \in \mathfrak{g}$ then $[\underset{\uparrow}{\tilde{X}}, \xi^\#] = 0$

horizontal lift of X

Locally W horizontal lift looks like $W = f^\mu \tilde{\partial}_\mu$ whereas V vertical looks like $V = g^a e_a^\#$ $\quad \langle e_a \rangle = \mathfrak{g}$ where f^μ & g^a are functions, $[V, W] = [f^\mu \tilde{\partial}_\mu, g^a e_a^\#]$

Remark Continued

$$[W, V] = [f^\mu \tilde{\partial}_\mu, g^a e_a^\#]$$

$$= f^\mu(\tilde{\partial}_\mu g^a) e_a^\# \pm g^a e_a^\#(f^\mu) \partial_\mu + f^\mu g^\mu \underbrace{[\tilde{\partial}_\mu, e_a^\#]}_{0}$$

So its special that $[\tilde{X}, \xi^\#] = 0$.
Have to take $X, Y \in T_u P$, but need vector fields and then % up into cases vv, vh, hv, hh. Need to extend vector $\longrightarrow$ vector field via bump-fnct...

Notational Juggling

$\{e_a\}$ a basis of $\mathfrak{g}$

$\omega = \omega^a e_a$

$$D\omega(X,Y) = d\omega(X,Y) + \underbrace{[\omega(X), \omega(Y)]}_{\text{Lie Algebra bracket}}$$

$$= d\omega^a(X,Y) e_a + [\omega^b(X) e_b, \omega^c(Y) e_c]$$

$$= d\omega^a(X,Y) e_a + \omega^b(X)\omega^c(Y)[e_b, e_c]$$

$$= d\omega^a(X,Y) e_a + \omega^b(X)\omega^c(Y) f^a_{bc} e_a$$

$$= [d\omega^a(X,Y) + \omega^b(X)\omega^c(Y) f^a_{bc}] e_a$$

$$= [d\omega^a(X,Y) + \tfrac{1}{2} f^a_{bc}(\omega^b \wedge \omega^c)(X,Y)] e_a$$

$$\therefore \boxed{D\omega^a = d\omega^a + \tfrac{1}{2} f^a_{bc}\, \omega^b \wedge \omega^c}$$

Notation changing on $D\omega$ continued

If ξ, ζ are Lie-algebra valued 1-forms on a manifold Q and $\{e_a\}$ is a basis of the Lie algebra

$$\xi = \xi^a \otimes e_a$$
$$\zeta = \zeta^b \otimes e_b$$

Can then define brackets of these (standard algebraic way)

$$[\xi, \zeta] = (\xi^a \wedge \zeta^b) \otimes [e_a, e_b]$$

This has some funny consequences & explains the funny $\frac{1}{2}$'s...

$$[\omega, \omega] = (\omega^a \wedge \omega^b) \otimes (f_{ab}^c e_c)$$
$$= f_{ab}^c (\omega^a \wedge \omega^b) \otimes e_c$$

We would just drop the tensor, but our formula has a $\frac{1}{2}$ in it. How can this be? Well there's another hidden convention here,

$$\boxed{\begin{aligned} D\omega(X,Y) &= d\omega(X,Y) + \tfrac{1}{2}[\omega,\omega](X,Y) \\ &\text{a.k.a.} \\ D\omega(X,Y) &= d\omega(X,Y) + [\omega(X), \omega(Y)] \end{aligned}}$$

There are yet other methods of expressing this

$$[\xi, \zeta] \propto [\xi_\mu, \zeta_\nu]\, dx^\mu \wedge dx^\nu$$

ω is on a PFB

Let $s : U \longrightarrow P$ be a local section

(1.) $\Omega(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)]$

(2.) $\Omega^c = d\omega^c + \frac{1}{2} f_{ab}^{\ c}(\omega^a \wedge \omega^c)$

(3.) $\Omega^c = d\omega^c + \frac{1}{2}[\omega, \omega]^c$

Pull-back under s and let $X = \partial_\mu$ and $Y = \partial_\nu$ and we'll recover some familiar things, $s^* d\omega = d(s^*\omega)$

(1.) $F_{\mu\nu} = dA(\partial_\mu, \partial_\nu) + [A_\mu, A_\nu]$

$= (\partial_\rho A_\lambda) dx^\rho \wedge dx^\lambda (\partial_\mu, \partial_\nu) + [A_\mu, A_\nu]$

$= (\partial_\rho A_\lambda)[\delta_\mu^\rho \delta_\nu^\lambda - \delta_\nu^\rho \delta_\mu^\lambda] + [A_\mu, A_\nu]$

$= \boxed{\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F_{\mu\nu}}$

(2.) $\boxed{F^c = dA^c + \frac{1}{2} f_{ab}^{\ c}(A^a \wedge A^b)}$

real-valued one-forms.

$A^a : TU \longrightarrow \mathbb{R}$, $A^a = A_\mu^a dx^\mu$

$A = A_\mu^a dx^\mu e_a = A_\mu^a (dx^\mu \otimes e_a)$

(3.) $F^c = dA^c + \frac{1}{2}[A, A]^c$

a.k.a $\boxed{F = dA + \frac{1}{2}[A, A]}$ eg. Bertlmann

If $\mathfrak{g}$ is a Lie Algebra of matrices with basis $\{T_a\}$ one defines (ala Bertlmann) a wedge product of $\xi, \zeta \in \Lambda^p(Q, \mathfrak{g})$ by

$$\underset{\text{manifold}}{Q}, \quad \underset{\text{Lie Algebra}}{\mathfrak{g}}$$

$$\zeta \wedge \xi = (\zeta^a \otimes T_a) \wedge (\xi^b \otimes T_b)$$

$$= (\zeta^a \wedge \xi^b) \otimes (T_a T_b)$$

need matrix multiplication it's not even Lie Algebra valued, its matrix valued.

<u>But</u> if we exploit the skew-symmetry of wedge

$$(\zeta \wedge \xi) - (-1)^{pq}(\xi \wedge \zeta)$$

$$= (\zeta^a \wedge \xi^b) \otimes (T_a T_b) - (-1)^{pq}(\xi^b \wedge \zeta^a)(T_a T_b)$$

$$= (\zeta^a \wedge \xi^b) \otimes (T_a T_b) - (\zeta^a \wedge \xi^b) \otimes (T_a T_b)$$

$$= (\zeta^a \wedge \xi^b) \otimes (T_a T_b - T_b T_a)$$

$$= f_{ab}^{\ c} (\zeta^a \wedge \xi^b) \otimes T_c$$

Then consider case where $\zeta = \omega = \xi$ and get

$$[\omega, \omega] = \underbrace{2\omega \wedge \omega}_{\text{wedge on Lie Algebra valued one forms.}} = 2 f_{ab}^{\ c} (\omega^a \wedge \omega^c) T_c$$

$$\therefore \quad \Omega = d\omega + \tfrac{1}{2}[\omega, \omega] = \boxed{d\omega + \omega \wedge \omega = \Omega}$$

$$\Omega = d\omega + \omega \wedge \omega$$

Pull it back and you'll see,

$$F = dA + A \wedge A = \text{"} dA + A^2 \text{"}$$

Th^m/ Let ω be a principal connection on a PFB (P, M, π, G) then if $g \in G$ then the pull-back of curvature under Right translation, gives,

$$R_g^* \Omega = Ad(g^{-1}) \Omega$$

$$\omega(\xi^\#) = \xi$$

$$\Omega(\xi^\#, \eta^\#) = 0$$