Fun with complex variables (5pts.)

A complex number $z = a + ib$ where $i = \sqrt{-1}$. Every complex number can be written in this way. We say $Re(z) = a$ while $Im(z) = b$ thus $z = Re(z) + i\,Im(z)$. In this project we will investigate some basic algebraic identities in the complex setting & then see how certain integrals are greatly simplified with the help of the complex exponential.

1.) Assume that $e^{i\theta} \equiv \cos\theta + i\sin\theta$. Given that assumption show that,

$$\boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

hint: notice $e^{-i\theta} = \cos\theta - i\sin\theta$ then add or subtract with $e^{i\theta}$.

2.) A complex equation is true if the real and imaginary parts of the LHS and RHS match up. Assume that $e^{ia}e^{ib} = e^{i(a+b)}$. Given that assumption, derive formulas for $\sin(a+b)$ & $\cos(a+b)$.

3.) Assume that the chain rule still works, $\frac{d}{d\theta}\left(e^{\pm i\theta}\right) = \pm i e^{\pm i\theta}$. Show that $\frac{d}{d\theta}(\sin\theta) = \cos\theta$ and $\frac{d}{d\theta}(\cos\theta) = -\sin\theta$. You should do these in terms of the imaginary exponential. Hint: $\frac{1}{i} = -i$

4.) Guess the $\int e^{ik\theta}\,d\theta$. Calculate $\int \cos(m\theta)\cos(n\theta)\,d\theta$. Do this by converting the cosines to imaginary exponentials. Treat the cases $m = n$ and $m \neq n$ seperately.

5.) Prove that $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$ and $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$.

6.) Given $e^{i\theta}e^{x} = e^{i\theta + x}$. Calculate $\int e^x \sin(x)\,dx$ without using integration by parts. Instead assume $\int e^{(a+ib)x}dx = \frac{1}{a+ib}e^{(a+ib)x} + C$. (*) Since $\sin(x)$ is a sum of imaginary exponentials and e^x is an ordinary exponential their product is a complex exponential which I told you how to integrate (see *).