MATH 131 TEST 3

Show your work. No graphing calculators. This is a timed test, use your time wisely, some problems are worth more than others. All arguments should be given in notation used in lecture. Thanks!

Problem 1 [4pts] State the Mean Value Theorem.

If f is a continuous function on $[a, b]$ and differentiable on (a, b) then $\exists\, c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Problem 2 [16pts] Calculate the following basic indefinite integrals:

(a.) $\int \frac{x + x^2}{\sqrt{x}}\, dx = \int \left(x^{1/2} + x^{3/2}\right) dx = \underline{\frac{2}{3} x^{3/2} + \frac{2}{5} x^{5/2} + C}$.

(b.) $\int \frac{dx}{4 + x^2} = \frac{1}{4} \int \frac{dx}{1 + (x/2)^2} = \frac{1}{4} \cdot \left(\frac{1}{\frac{1}{2}} \tan^{-1}\left(\frac{x}{2}\right) \right) + C = \boxed{\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + C}$

(c.) $\int 2^x\, dx = \boxed{\frac{1}{\ln(2)} 2^x + C}$

(d.) $\int \sec(x) \tan(x)\, dx = \boxed{\sec(x) + C}$

Problem 3 [10pts] Find extreme values of $f(x) = x^3 - 3x^2 + 7$ on $[-1, 4]$.

$f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ if $\underbrace{x = 0,\ x = 2}_{\text{critical \#s}}$

$f(-1) = -1 - 3 + 7 = 3$

$f(0) = 7$

$f(2) = 8 - 3(4) + 7 = \boxed{3 \leftarrow \text{minimum value}}$

$f(4) = 64 - 3(16) + 7 = \boxed{23 \leftarrow \text{max. value}}$

By closed interval method.

Problem 4 [10pts] Let $f(x) = \ln(x^2 + 4x + 5)$. Locate all critical numbers for f and use the second derivative test to find the local extrema for the function.

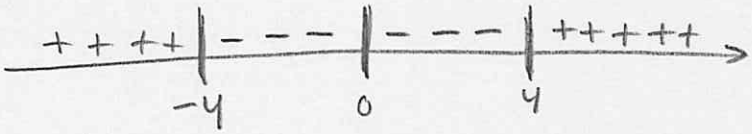
$$\frac{df}{dx} = \frac{2x+4}{x^2+4x+5} = \frac{2(x+2)}{(x+2)^2+1} \Rightarrow \boxed{x = -2 \text{ only critical \#}}$$

$$\frac{d^2f}{dx^2} = \frac{2(x^2+4x+5)-(2x+4)(2x+4)}{(x^2+4x+5)^2}$$

$$f''(-2) = \frac{2(1)-(0)(0)}{1^2} = 2 > 0 \Rightarrow \boxed{f(-2) \text{ is local minimum.}}$$

Problem 5 [10pts] Find the intervals of increase and decrease for the function $f(x) = x + \frac{16}{x}$. Also, classify any local extreme values by applying the first derivative test.

$$f'(x) = 1 - \frac{16}{x^2} = 0 \Rightarrow x = 16 \Rightarrow \boxed{\begin{array}{l} x = \pm 4 \text{ critical \#'s} \\ \bullet \text{ also } x=0 \text{ is critical} \end{array}}$$



1st derivative test ⇒ $f(-4)$ is local maximum.
$f(4)$ is local minimum.

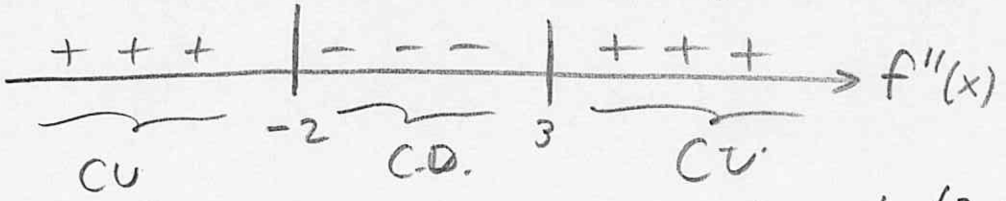
Observe f is inc. on $(-\infty,-4)$ and $(4,\infty)$
f is dec on $(-4,0)$ and $(0,4)$.

Problem 6 [10pts] Determine where $f(x) = x^4 - 2x^3 - 36x^2$ is concave up (CU) or concave down (CD). Also, find any inflection points.

$$f'(x) = 4x^3 - 6x^2 - 72x$$

$$f''(x) = 12x^2 - 12x - 72 = 12(x^2 - x - 6) = 12(x-3)(x+2)$$

Observe $f''(3) = 0$ and $f''(-2) = 0$ these give points at which the concavity might change.



Observe f is C.U. on $(-\infty,-2)$ and $(3,\infty)$.
f is C.D on $(-2,3)$.

Thus $(-2, f(-2))$, $(3, f(3))$ are points of inflection.

$$* \left(x<0 \Rightarrow \frac{1}{\sqrt{x^2}} = \frac{1}{-x} . \right)$$

Problem 7 [10pts] Calculate the following limits by algebra and common sense.

(*a.*) $$\lim_{x\to\infty}\left[\frac{e^x+e^{2x}}{2e^{3x}+e^x}\right] = \lim_{x\to\infty}\left[\frac{e^{-2x}+e^{-x}}{2+e^{-2x}}\right] = \frac{0+0}{2+0} = \boxed{0.}$$

(*b.*) $$\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{x+5} = \lim_{x\to-\infty}\frac{\frac{-1}{x}\sqrt{x^2+2}}{\frac{-1}{x}(x+5)} \; *$$

$$= \lim_{x\to-\infty}\left(\frac{\sqrt{1+2/x^2}}{-1-5/x}\right) = \frac{\sqrt{1+0}}{-1+0} = \boxed{-1.}$$

Problem 8 [15pts] Calculate the following limits using L'Hospital's rule where appropriate. Indicate usage of the rule either with the notation we used in lecture or in words to the side.

(*a.*) $$\lim_{x\to1}\frac{\ln(x)}{4x-x^2-3} \underset{(\frac{0}{0})}{\not=} \lim_{x\to1}\left(\frac{1/x}{4-2x}\right) = \lim_{x\to1}\left(\frac{1}{4x-2x^2}\right) = \frac{1}{4-2} = \boxed{\frac{1}{2.}}$$

(*b.*) $$\lim_{x\to0^+}x^{4x} = \exp\Big(\underbrace{\lim_{x\to0^+}\left(\ln\left(x^{4x}\right)\right)}_{*}\Big) = \exp(0) = \boxed{1.}$$

$$* = \lim_{x\to0^+}\left[4x\ln(x)\right] = \lim_{x\to0^+}\left[\frac{4\ln x}{1/x}\right] \underset{(\frac{\infty}{\infty})}{\not=} \lim_{x\to0^+}\left[\frac{4/x}{-1/x^2}\right] = \lim_{x\to0^+}\left[-4x\right] = 0$$

(*c.*) $$\lim_{x\to0}[\csc(ax)\sin(bx)] = \lim_{x\to0}\left(\frac{\sin bx}{\sin ax}\right)$$ (Let a, b be nonzero constants.)

$$\underset{(\frac{0}{0})}{\not=} \lim_{x\to0}\left(\frac{b\cos bx}{a\cos ax}\right) = \frac{b\cos(0)}{a\cos(0)} = \boxed{\frac{b}{a.}}$$

Problem 9 [5pts] Find the second-order Taylor polynomial for $f(x) = e^{-x^2}$ centered at $x = 0$.

$$f'(x) = -2xe^{-x^2} \Rightarrow f'(0) = 0$$

$$f''(x) = -2\left[e^{-x^2} + x(-2x)e^{-x^2}\right] \Rightarrow f''(0) = -2$$

$$T_2(x) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 = 1 + \tfrac{1}{2}(-2)x^2 = \boxed{1 - x^2}$$

Problem 10 [10pts] Suppose $x \lesseqgtr 1$ and $y \geq 1$. Furthermore, suppose that the product of x and y is 100. Find x, y for which the sum $x + y$ is smallest. Also, x, y for which the sum $x + y$ is largest.

$100 = xy$ consider $S = x + y = x + \frac{100}{x}$

$$\frac{dS}{dx} = 1 - \frac{100}{x^2} = 0 \Rightarrow 100 = x^2 \Rightarrow \underline{x = \pm 10}$$

Thus, for $x \geq 1$ we find critical # $\underline{x = 10}$.

Observe the values for x fall into a closed interval!

$x = \frac{100}{y}$ but $y \geq 1$ $\Rightarrow$ $x \leq 100$. Thus $x \in [1, 100]$.

Check endpoints; $S(1) = 1 + \frac{100}{1} = 101$, $S(100) = 100 + \frac{100}{100} = 101$.

and $S(10) = 10 + \frac{100}{10} = 20$ is the minimum sum.

$$\boxed{\begin{array}{llll} x, y = 1, 100 & \hookrightarrow & S = 101 & \text{max.} \\ \hline x, y = 10, 10 & \hookrightarrow & S = 10 + 10 & \text{min.} \end{array}}$$

Problem 11 [10pts] Suppose the acceleration as a function of time is given by $a(t) = 18t$. Furthermore, suppose $v(0) = 2$ and $s(0) = 1$. Find the velocity and position as functions of time t; that is find $v(t)$ and $s(t)$.

$$a = \frac{dv}{dt} = 18t \Rightarrow v(t) = 9t^2 + C_1, \quad v(0) = C_1 = 2.$$

$$\therefore \boxed{v(t) = 9t^2 + 2}$$

$$v = \frac{ds}{dt} = 9t^2 + 2 \Rightarrow s(t) = 3t^3 + 2t + C_2$$

$$s(0) = C_2 = 1.$$

$$\therefore \boxed{s(t) = 3t^3 + 2t + 1}$$

Problem 12 [5pts] Suppose a smooth function f has critical points at $x = -1$ and $x = 2$. Furthermore, suppose that $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.

1. what is $f''(1)$ and how do you know that to be true from the given information?
2. find the intervals of increase and decrease for f

1.) $\underset{1}{+ + + + | - - - -} \rightarrow f''(x)$

Consider $(0,2)$. Notice f smooth $\Rightarrow f''$ continuous

thus $f''(0) < 0 < f''(2) \Rightarrow \exists c \in (0,2)$ (IVT)

for which $f''(c) = 0$, but by given data

we need $\underline{f''(1) = 0}$ as $f''(c) \neq 0 \ \forall c \neq 1$.

2.) Observe, $f''(-1) > 0 \Rightarrow f(-1)$ local min. Likewise $f''(2) < 0$

hence $f(2)$ is local max. (By 2^{nd} der. test.) it follows

from 1^{st} derivative test $- - - \underset{-1}{|} + + + \underset{2}{|} - - - - \rightarrow f'$ thus f inc. on $(-1,2)$

f dec. on $(-\infty,-1)$ and $(2,\infty)$.

Problem 13 [2+3pts] Suppose F is an antiderivative of f on all of $\mathbb{R}$. Is F continuous? Circle either [✓ **YES**] or **NO**. Explain your choice.

By $\underline{\text{def}^{\underline{n}}}$ $\frac{dF}{dx} = f$ hence F is differentiable

$\Rightarrow F$ is continuous.

Problem 14 [5pts] Suppose $1 \le f'(x) \le 5$ for all $x \in \mathbb{R}$. If $f(3) = 2$ then what is the minimum and maximum values possible for $f(7)$? Assume that f is smooth on $\mathbb{R}$ and justify your work with an appropriate calculus theorem.

Apply the MVT to $[3,7]$. We find $\exists c \in [3,7]$

for which $f'(c) = \frac{f(7) - f(3)}{7-3} = \frac{f(7) - 2}{4}$

$$\Rightarrow \quad f(7) = 4f'(c) + 2$$

But, $1 \le f'(c) \le 5 \Rightarrow 4 \le 4f'(c) \le 20 \Rightarrow \underset{\text{min}}{\underline{6}} \le \overbrace{4f'(c)+2}^{f(7)} \le \underset{\text{max.}}{\underline{22}}$

Problem 15 [5pts] You are given that $f(x) = 120x^2 + 30x^3 - 25x^4 - 9x^5 + 2x^6$. Find the intervals of concavity (CU and CD). It is useful to note that $f''(-1) = 0$ but $(-1, f(-1))$ is not a point of inflection. (work on back page of previous sheet and put answer here thanks!)

$$f'(x) = 240x + 90x^2 - 100x^3 - 45x^4 + 12x^5$$
$$f''(x) = 240 + 180x - 300x^2 - 180x^3 + 60x^4$$
$$= 60(4 + 3x - 5x^2 - 3x^3 + x^4)$$
$$= \underbrace{60(x+1)^2(x-1)(x-4)}_{\text{By next page}}$$

$+ + + \underset{-1}{|} + + + \underset{1}{|} - - - \underset{4}{|} + + + \rightarrow f''$

f is C.U on $(-\infty, 1)$ and $(4,\infty)$

f is C.D on $(1,4)$.

Long Division for Problem 15:

$$\begin{array}{r|l} & x^2 - 5x + 4 \\ \hline x^2+2x+1 & x^4 - 3x^3 - 5x^2 + 3x + 4 \\ & x^4 + 2x^3 + x^2 \\ \hline & -5x^3 - 6x^2 + 3x + 4 \\ & -5x^3 - 10x^2 - 5x \\ \hline & 4x^2 + 8x + 4 \\ & 4x^2 + 8x + 4 \\ \hline & \boxed{0} \end{array}$$

Thus $f''(x) = 60(x+1)^2(x^2-5x+4)$

$= 60(x+1)^2(x-1)(x-4)$