CALCULUS 131:	TEST 4: INTEGRAL CALCULUS.

Show steps clearly for maximum credit. 120pts required, some bonus available.

6pts
Problem 1 State FTC I and FTC II.(good credit may be obtained by writing the central formulas for each theorem below)

$$\text{FTC I} : \quad \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\text{FTC II} : \quad \int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where } F'(x) = f(x).$$

(I ignore the fine print for emphasis)

6pts
Problem 2 Suppose that $\int_0^2 f(x)\,dx = 3$ and suppose g is continuous with $1 \le g(x) \le 4$ for all $x \in [0,2]$. Given this information find the maximum and minumum values possible for $\int_0^2 [g(x) + f(x)]\,dx$.

$$1 \le g(x) \le 4 \implies \int_0^2 1\,dx \le \int_0^2 g(x) \le \int_0^2 4\,dx$$

$$\implies 2 \le \int_0^2 g(x)dx \le 8$$

Consider,

$$I = \int_0^2 [g(x) + f(x)dx] = \int_0^2 g(x)dx + \int_0^2 f(x)dx = \int_0^2 g(x)dx + 3$$

Thus $I - 3 = \int_0^3 g(x)dx$ hence $2 \le I - 3 \le 8$

Therefore, $\underline{5 \le \int_0^2 (g(x)+f(x))dx \le 11}$.

10pts
Problem 3 Suppose that the acceleration of Ron S. is given as a function of time t to be $a(t) = t$. Furthermore, Ron S. undergoes one-dimensional motion along the x-axis where he begins at the origin $x(0) = 0$ with a velocity $v(0) = 1$. **Calculate** the velocity and position of Ron S. as a function of time.

$$a(t) = \frac{dV}{dt} = t \hookrightarrow V(t) = V(0) + \int_0^t \tau\,d\tau = 1 + \frac{t^2}{2}$$

$$V(t) = \frac{dx}{dt} = 1 + t^2/2 \hookrightarrow X(t) = X(0) + \int_0^t \left(1 + \frac{\tau^2}{2}\right)d\tau$$

$$= 0 + \left(\tau + \frac{1}{6}\tau^3\right)\Big|_0^t$$

$$= t + \frac{1}{6}t^3$$

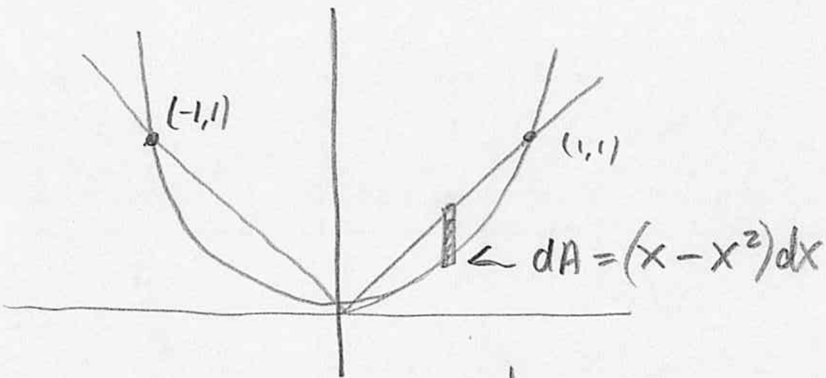
Thus, $\boxed{\underbrace{V(t) = 1 + t^2/2}_{\text{velocity}} \text{ and } \underbrace{X(t) = t + t^3/6}_{\text{position}}}$.

15pts

Problem 4 Calculate area bounded between $y = |x|$ and $y = x^2$. Please show your work including a graph.

$$|x| = x^2 \Rightarrow x^2 = x^4$$
$$\Rightarrow x^4 - x^2 = 0$$
$$\Rightarrow x^2(x+1)(x-1) = 0$$
$$\Rightarrow x = 0, \pm 1$$

intersection pts.

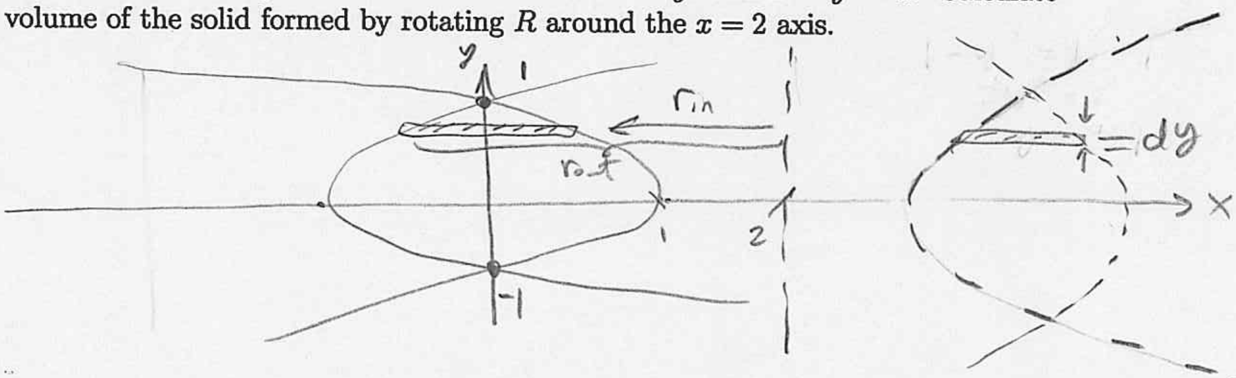


$\leftarrow dA = (x - x^2)dx$

$$\text{Symmetry} \Rightarrow A = 2\int_0^1 (x - x^2)\,dx = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{6} = \boxed{\frac{1}{3}}$$

15pts

Problem 5 Let R be the region bounded between $x = 1 - y^2$ and $x = y^2 - 1$. Calculate the volume of the solid formed by rotating R around the $x = 2$ axis.



$$r_{in} = 2 - x_R = 2 - (1 - y^2) = 1 + y^2$$
$$r_{out} = 2 - x_L = 2 - (y^2 - 1) = 3 - y^2$$
$$dV = \pi\left((3 - y^2)^2 - (1 + y^2)^2\right)dy$$
$$= \pi\left(9 - 6y^2 + \cancel{y^4} - [1 + 2y^2 + \cancel{y^4}]\right)dy$$
$$= \pi\left(8 - 8y^2\right)dy$$
$$= 8\pi\left(1 - y^2\right)dy$$

$$V = \int_{-1}^{1} 8\pi(1 - y^2)\,dy = 16\pi\int_0^1 (1 - y^2)\,dy$$
$$= 16\pi\left(1 - \frac{1}{3}\right)$$
$$= 16\pi\left(\frac{2}{3}\right) = \boxed{\frac{32\pi}{3}}$$

60pts

6. Find the following indefinite integrals. (10 pts each)

a. $\int \frac{2\sqrt[5]{x^6}+7x}{x^2}\,dx =$

$= \int \left(2x^{\frac{6}{5}-2} + 7x^{-1}\right)dx$

$= \int \left(2x^{-\frac{4}{5}} + 7x^{-1}\right)dx$

$= \boxed{10x^{1/5} + 7\ln|x| + C}$

$u = \cos^2(x) + 2$

$du = 2\cos(x)\sin(x)\,dx$

$= 2\sin(2x)\,dx$

b. $\int \left[\sin(2x)\, e^{\cos^2 x + 2}\right]dx =$

$= \int e^u\,du$

$= e^u + C$

$= \boxed{e^{\cos^2(x)+2} + C}$

c. $\int \frac{3}{x^2}\cot\left(\frac{1}{x}\right)dx =$

$= \int -3\cot(u)\,du$

$\boxed{u = 1/x \quad du = -dx/x^2}$

$= -3\int \frac{\cos(u)\,du}{\sin(u)}$

$= -3\int \frac{dw}{w}$

$w = \sin(u)$
$dw = \cos(u)\,du$

$= -3\ln|w| + C$

$= \boxed{-3\ln\left|\sin(1/x)\right| + C}$

$u = x^2+3$
$du = 2x\,dx$

d. $\int \frac{7x}{\sin^2(x^2+3)}\,dx =$

$= \int \frac{\frac{7}{2}du}{\sin^2(u)}$

$= \frac{7}{2}\int \csc^2(u)\,du$

$= -\frac{7}{2}\cot(u) + C$

$= \boxed{-\frac{7}{2}\cot(x^2+3) + C}$

e. $\int \frac{3\cos\left(\sqrt[3]{x^2}\right)}{\sqrt[3]{x}}\,dx =$

$= \int 3\cos(u)\,\frac{3du}{2}$

$u = x^{2/3}$
$du = \frac{2}{3}x^{-1/3}dx$
$\frac{dx}{\sqrt[3]{x}} = \frac{3du}{2}$

$= \frac{9}{2}\sin(u) + C$

$= \boxed{\frac{9}{2}\sin(x^{2/3}) + C}$

$u = \cos^{-1}(x)$
$du = \frac{-dx}{\sqrt{1-x^2}}$

f. $\int \frac{4^{\cos^{-1}x}}{\sqrt{1-x^2}}\,dx =$

$= -\int 4^u\,du$

$= \boxed{\frac{-1}{\ln(4)}\,4^{\cos^{-1}(x)} + C}$

g. $\int e^x \tan(2e^x - 5)\,dx =$ $u = 2e^x - 5$, $du = e^x dx$

$= \int \tan(u)\,du$

$= \int \frac{\sin(u)\,du}{\cos(u)}$

$= -\int \frac{dw}{w}$

$= -\ln|\cos(u)| + C$

$= \boxed{-\ln|\cos(2e^x - 5)| + C}$

h. $\int \frac{2x+5}{x^2+3}\,dx =$

$= \int \frac{2x\,dx}{x^2+3} + \int \frac{(5/3)\,dx}{1+(x/\sqrt{3})^2}$

$= \int \frac{du}{u} + \frac{5\sqrt{3}}{3}\int \frac{dw}{1+w^2}$ $w = x/\sqrt{3}$, $\sqrt{3}\,dw = dx$, $u = x^2+3.$

$= \ln|u| + \frac{5}{\sqrt{3}} \tan^{-1}(w) + C$

$= \boxed{\ln(x^2+3) + \frac{5}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C}$

i. $\int 5x^2 \csc(x^3+2)\cot(x^3+2)\,dx =$

$= \int \frac{5}{3} \csc(u)\cot(u)\,du$: $u = x^3+2$, $du = 3x^2dx$, $x^2dx = \frac{1}{3}du$

$= -\frac{5}{3}\csc(u) + C$

$= \boxed{-\frac{5}{3}\csc(x^3+2) + C}$

j. $\int_3^{18} x\sqrt{x-2}\,dx =$ $u = x-2$, $x = u+2$, $u(18) = 16$, $u(3) = 1$

$= \int_1^{16} (u+2)\sqrt{u}\,du$

$= \int_1^{16} \left(u^{3/2} + 2u^{1/2}\right) du$

$= \left(\frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2}\right)\Big|_1^{16}$

$= \boxed{\frac{2}{5}(16^{5/2} - 1) + \frac{4}{3}(16^{3/2} - 1)}$

$= \frac{2}{5}(2^{10} - 1) + \frac{4}{3}(2^6 - 1)$

6pts

Problem 9 Use the definition of the definite integral and the FTC part II to calculate the following infinite sum:

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left[2 + \frac{3i}{n}\right]^2 \frac{3}{n} = \lim_{n\to\infty} \sum_{i=0}^{n} x_i^2 \Delta x$$

$a = 2$

$b = a + n\Delta x$

$\Delta x = \frac{3}{n}$

$b = 2 + n\left(\frac{3}{n}\right) = 5.$

$x_i = a + i\Delta x$

$= 2 + i\left(\frac{3}{n}\right)$

$= 2 + \frac{3i}{n}$

$$= \int_2^5 x^2\,dx$$

$$= \frac{x^3}{3}\Big|_2^5$$

$$= \frac{1}{3}(5^3 - 2^3)$$

$$= \frac{117}{3} = \boxed{39}$$

6pts

Problem 7 Calculate $\frac{dg}{dx}$ where g is the function defined by

$$g(x) = \int_{x^2}^{\sin(x)} e^{-t^2}\,dt. = \boxed{e^{-\sin^2(x)}\cos(x) - e^{-x^4}(2x)} \quad \text{by FTC III.}$$

$$\frac{d}{dx}[\sin(x)] \qquad \frac{d}{dx}[x^2]$$

6pts

Problem 8 Suppose $A(x) = \int_0^x f(t)\,dt$ for $x \in \mathbb{R}$. Furthermore, suppose that f is a continuous function on $\mathbb{R}$ which is positive on $(1, \infty)$ and negative on $(-\infty, 1)$. Answer the questions below, if the data is inconclusive then explain why.

(a.) if possible, given this data, find where A has a critical point. Is this value a maximum, a minimum or neither.

$$\frac{dA}{dx} = f(x) \quad \text{by FTC I.}$$

only critical # is $x=1$.

$$- - - - \mid + + + + \quad \frac{dA}{dx} = f(x)$$

$A(1)$ is minimum value by 1^{st} der. test at critical # $x=1$.

(b.) if possible, given this data, find where f has a critical point. Is this value a maximum, a minimum or neither.

No, I cannot find critical pts for $f(x)$ or, even if any such points exist. If $k > 0$,

$$f(x) = (x-1)(x^2+k) \quad \text{has} \quad f(1) = 0$$

and the $f(x) < 0$ for $x < 1$ and $f(x) > 0$ for $x > 1$.

However, notice

$$\frac{df}{dx} = (x^2+k) + (x-1)(2x)$$
$$= k + x^2 + 2x^2 - 2x$$
$$= k + 3x^2 - 2x$$

Set $k = 5$ then $f'(x) = 5 + 3x^2 - 2x$ and $f'(5) = 0$.

Set $k = 10$ then
$$f'(x) = 3x^2 - 2x + 10$$
$$= 3(x^2 - \tfrac{2}{3}) + 10$$
$$= 3\left[\left(x - \tfrac{1}{3}\right)^2 - \tfrac{1}{9}\right] + 10$$
$$= 3\left(x - \tfrac{1}{3}\right)^2 + \tfrac{29}{30} \neq 0 \quad \forall x \in \mathbb{R}.$$

Thus, the given data is inconclusive on the question of b.//