Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Thanks. Same instructions as previous tests. There are at least 150pts to earn on this exam.

Problem 1 ~~[15pts]~~ 7pts Suppose $f'(x) = x + e^x$ and $f(0) = 0$. Calculate $f(x)$.

$$f(x) = \tfrac{1}{2}x^2 + e^x + C$$

$$f(0) = 1 + C = 0 \Rightarrow \underline{C = -1}.$$

$$\boxed{f(x) = \tfrac{1}{2}x^2 + e^x - 1}$$

Problem 2 ~~[15pts]~~ 7 pts. Calculate $\frac{dg}{dx}$ where g is the function defined by

$$g(x) = \int_{x^2}^{\sin(x)} \sqrt[3]{t^2 + \sqrt{t}}\, dt.$$

$$\frac{dg}{dx} = \left[\sqrt[3]{\sin^2(x) + \sqrt{\sin(x)}}\right]\cos(x) - \left[\sqrt[3]{x^4 + \sqrt{x^2}}\right]2x.$$

Problem 3 ~~[9pts]~~ 7pts. Suppose that $\int_0^1 f(x)\,dx = 3$ and suppose g is continuous with $1 \le g(x) \le 2$ for all $x \in [0,1]$. Given this information find the maximum and minumum values possible for $\int_0^1 [g(x) + f(x)]\,dx$.

$$I \equiv \int_0^1 [g(x) + f(x)]dx = \int_0^1 g(x)\,dx + \int_0^1 f(x)\,dx = \int_0^1 g(x)dx + 3$$

$$\Rightarrow \quad I - 3 = \int_0^1 g(x)\,dx$$

However, $1 \le g(x) \le 2 \quad \forall x \in [0,1] \Rightarrow 1 \le \int_0^1 g(x)dx \le 2$

$$\Rightarrow 1 \le I - 3 \le 2$$

$$\therefore \boxed{4 \le \int_0^1 (g + f)dx \le 5}$$

Problem 4 ~~[11pts]~~ 7pts. Use the definition of the definite integral and the FTC part II to calculate the following infinite sum:

$$\lim_{n\to\infty} \sum_{i=0}^{n} \left[2 + \frac{3i}{n}\right]^2 \frac{3}{n}. = \int_2^5 x^2\,dx = \frac{1}{3}x^3\Big|_2^5 = \frac{1}{3}(125 - 8)$$

$$= \boxed{\frac{117}{3}}$$

$$\Delta x = \frac{3}{n} = \frac{b-a}{n}$$

$$x_i = 2 + i\Delta x$$

$a \Rightarrow b = 5$

Problem 5 ~~[15pts]~~ 10 pts. Integrate. These are all solvable with algebra, basic identities and the basic antiderivatives. (shouldn't need a u-substitution here)

$$\int [\sqrt{x}+2^x]\, dx = \underline{\frac{2}{3}x^{3/2} + \frac{1}{\ln(2)} 2^x + C}.$$

$$\int \tan^2(y)\, dy = \int (\sec^2 y - 1)\, dy = \underline{\tan(y) - y + C}.$$

$\tan^2 y + 1 = \sec^2 y$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \underline{\sin^{-1}(x) + C}.$$

$$\int \sec(x)\tan(x)\, dx = \underline{\sec(x) + C}.$$

omit $$\left\{ \int \frac{z^2+\sqrt{z}}{\sqrt[3]{27z}}\, dz = \int \frac{z^2+\sqrt{z}}{3\sqrt[3]{z}}\, dz = \frac{1}{3}\int z^{5/3}\, dz + \frac{1}{3}\int z^{1/6}\, dz \right.$$

$$= \underline{8z^{8/3} + \frac{2}{7}z^{7/6} + C}.$$

$$\int_{-3}^{-2} \frac{1}{x}\, dx = \ln|x|\Big|_{-3}^{-2}$$

$$= \ln(|-2|) - \ln(|-3|) = \boxed{\ln(2/3)}$$

Problem 6 ~~[15pts]~~ 5 pts Integrate.

$$\int_0^{2\pi} |\sin(x)|\, dx = \int_0^{\pi} \sin(x)\, dx + \int_{\pi}^{2\pi} -\sin(x)\, dx = -\cos(x)\Big|_0^{\pi} + \cos(x)\Big|_{\pi}^{2\pi}$$

$$= -\cos(\pi) + \cos(0) + \cos(2\pi) + 1$$

$$= \boxed{4.}$$

$$|\sin(x)| = \begin{cases} \sin(x) & : 0 \le x \le \pi \\ -\sin(x) & : \pi \le x \le 2\pi \\ \vdots & \end{cases}$$

Problem 7 ~~[15pts]~~ 5 pts Suppose $\frac{dg}{dt}$ gives the rate at which cats jump out a window. What does $\underbrace{\int_1^4 \frac{dg}{dt} dt}$ represent given that $t = 0$ corresponds to noon and t is in hours.

$g(4) - g(1)$ = total # of cats jumped out window from 1pm to 4pm.

50pts

Problem 8 ~~[15pts]~~ Integrate. Show work where necessary. Indicate all u-substitutions either implicitly or explicitly.

$$\int_{-3}^{-2} (x+3)^8\, dx = \int_0^1 u^8\, du$$

$$u = x+3, \quad du = dx$$

$$u(-3) = 0, \quad u(-2) = 1.$$

$$= \frac{u^9}{9}\Big|_0^1$$

$$= \boxed{\frac{1}{9}}$$

$$\int \frac{\cos(\ln(x))}{x}\, dx = \int \cos(\ln(x))\, d(\ln(x))$$

$$= \boxed{\sin(\ln(x)) + C}$$

assume $0 < a < b$,

$$\int_{a^2}^{b^2} \frac{\cos(\sqrt{t})}{\sqrt{t}}\, dt = \int_a^b 2\cos(u)\, du$$

$$u = \sqrt{t}, \quad du = \frac{dt}{2\sqrt{t}}$$

$$u(b^2) = \sqrt{b^2} = b$$

$$u(a^2) = \sqrt{a^2} = a$$

$$= 2\sin(u)\Big|_a^b$$

$$= \boxed{2\sin(b) - 2\sin(a)}$$

$$\int \frac{x+1}{2x+3}\,dx = \int \left[\frac{\frac{1}{2}u - \frac{3}{2} + 1}{u}\right]\frac{du}{2}$$

Let $u = 2x+3$
$du = 2dx$ and
$x = \frac{1}{2}(u-3)$.

$$= \frac{1}{4}\int\left(\frac{u-1}{u}\right)du$$

$$= \frac{1}{4}\int\left(1 - \frac{1}{u}\right)du = \frac{1}{4}u - \frac{1}{4}\ln|u| + C$$

$$= \boxed{\frac{1}{4}(2x+3) - \frac{1}{4}\ln|2x+3| + C}$$

$$\boxed{= \frac{1}{2}x + \frac{3}{4} - \frac{1}{4}\ln|2x+3| + C.}$$

$$\int \cos^3(x)\,dx$$

$$= \int \cos^2(x)\cos(x)\,dx$$

$$= \int \left(1-\sin^2(x)\right)d(\sin(x))$$

$$= \boxed{\sin(x) - \frac{1}{3}\sin^3(x) + C}$$

(move) $\Big\{ \displaystyle\int \sin^2(\theta)\,d\theta = \int\left[\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right]d\theta$

$$= \boxed{\frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + C}$$

Problem 9 [~~15pts~~ 10pts.] Suppose that the acceleration of Herbert is given as a function of time t to be $a(t) = t$. Furthermore, Herbert undergoes one-dimensional motion along the x-axis where he begins at the origin with a velocity $v(0) = 1$. **Calculate** the velocity and position of Herbert as a function of time.

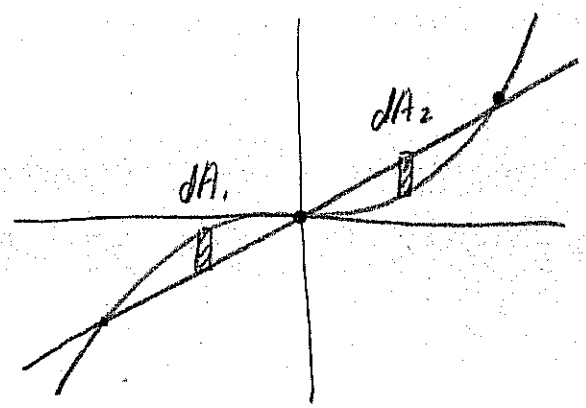
Remember, $a = \frac{dv}{dt}$ and $v = \frac{dx}{dt}$ hence,

$$V(t) = V(0) + \int_0^t \tau\,d\tau = \boxed{1 + \frac{1}{2}t^2 = V(t)}$$

$$X(t) = X(0) + \int_0^t\left(1 + \frac{1}{2}\tau^2\right)d\tau = \boxed{t + \frac{1}{6}t^3 = X(t)}$$

(could also adopt approach like Problem 1.)

Problem 10 [15pts] Calculate area bounded between $y = x$ and $y = x^3$. Include a graph which indicates the typical infinitesimal region and include algebra which justifies the bounds of the integration you did to calculate the area. (in short, show your work please)



$-1 \le x \le 0: \; dA_1 = (x^3 - x)\,dx$

$0 \le x \le 1: \; dA_2 = (x - x^3)\,dx$

$x = x^3$

$x^3 - x = x(x+1)(x-1) = 0$

$x = 0, \; x = \pm 1$ intersection pts.

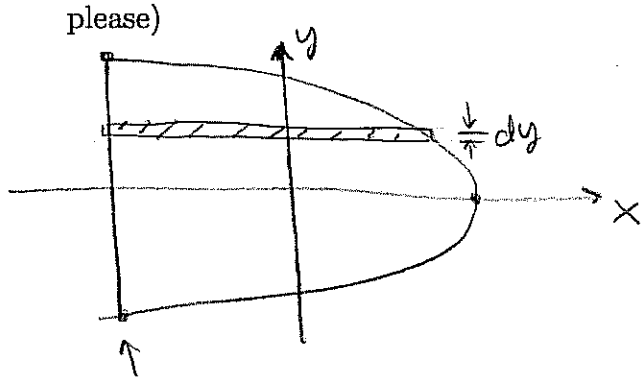
$$A = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 (x - x^3)\,dx$$

$$= 2\int_0^1 (x - x^3)\,dx$$

$$= 2\left(\tfrac{1}{2} - \tfrac{1}{4}\right)$$

$$= \boxed{1/2}$$

Problem 11 [15pts] Calculate area bounded between $x = 2 - y^2$ and $x = -2$. Include a graph which indicates the typical infinitesimal region and include algebra which justifies the bounds of the integration you did to calculate the area. (in short, show your work please)



$x_L = x_R$

$-2 = 2 - y^2$

$y^2 = 4$

$y = \pm 2$

$dA = (x_R - x_L)\,dy$

$= [2 - y^2 - (-2)]\,dy$

$= (4 - y^2)\,dy$

$$A = \int_{-2}^{2} (4 - y^2)\,dy$$

$$= \left(4y - \tfrac{1}{3}y^3\right)\Big|_{-2}^{2}$$

$$= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right)$$

$$= 16 - 16/3$$

$$= \frac{48 - 16}{3}$$

$$= \boxed{32/3}$$

Problem 12 ~~[9pts]~~ 7 pts. Suppose $A(x) = \int_1^x f(t)\,dt$ for $x \in \mathbb{R}$. Furthermore, suppose that f is a continuous function on $\mathbb{R}$. Answer the questions below in view of the information just given: (Feel free to use any important theorems which I presented in lecture to answer the questions above. (I do mean for $k \in \mathbb{N}$ in part (3.))).

1. if f is differentiable on $\mathbb{R}$ then is A also differentiable on $\mathbb{R}$?
2. if f is not differentiable then is A also a differentiable function ?
3. if f is k-times continuously differentiable then A is $(k+1)$-times continuously differentiable ?

1.) yes, in fact $\frac{dA}{dx} = f(x)$ by FTC I thus A is differentiable and hence A is continuous.

2.) yes, note FTC I only requires f continuous. The integration process smooths out kinks.

3.) yes. Since $\frac{dA}{dx} = f(x) \Rightarrow \frac{d^{k+1}A}{dx^{k+1}} = \underbrace{f^{(k)}(x)}_{\text{continuous by assumption}}$

$\Rightarrow A^{(k+1)}$ continuous.

Problem 13 ~~[9pts]~~ 15 pts. Integrate.

$$\int e^x \cosh(x)\,dx = \int \frac{1}{2} e^x(e^x + e^{-x})\,dx$$

$$= \int \left(\frac{1}{2} e^{2x} + \frac{1}{2}\right) dx = \boxed{\frac{1}{4} e^{2x} + \frac{x}{2} + C}$$

$$\int [\cosh^2(x) - \sinh^2(x)]\,dx = \int dx = \boxed{x + C}$$

$$\int [\sin(\theta)\cos(3\theta) + \cos(\theta)\sin(3\theta)]\,d\theta = \int \sin(4\theta)\,d\theta$$

$$= \boxed{\frac{-1}{4}\cos(4\theta) + C}$$

$$\int \sin(ax)\cos(bx)\,dx = \int \left[\frac{1}{2}\sin(a+b)x - \frac{1}{2}\sin(a-b)x\right]dx$$

$$= \boxed{\frac{-1}{2(a+b)}\cos[(a+b)x] + \frac{1}{2(a-b)}\cos[(a-b)x] + C}$$

$$\int 2^x e^x\,dx = \int e^{x\ln(2)} e^x\,dx$$

$2^x = e^{\ln(2^x)} = e^{x\ln(2)}$

$$= \int e^{[\ln(2)+1]x}\,dx = \boxed{\frac{e^x 2^x}{\ln(2)+1} + C}$$