3. LIMITS AND CONTINUITY

Algebra reveals much about many functions. However, there are places where the algebra breaks down thanks to division by zero. We have sometimes stated that there is “division by zero”. We do not mean to indicate that we are actually dividing by zero. Instead, our meaning is that we have to avoid that point because the laws of arithmetic fail to be reliable at that point. A natural question to ask is what happens at such ill-defined points? Is there a logically reliable procedure with which we can elicit information about such cantankerous points?

The point of this chapter is to give an answer to what happens when we try to divide by zero. Not that we ever manage to actually divide by zero, instead we find a method to do the next best thing. We describe how to take the *limit* of functions at such points. It turns out division by zero is just one of several other so-called *indeterminant forms*. We will discuss why they are called “indeterminant”. We conclude the chapter by using limits to define *continuous functions.*

Limits are used to make all the basic definitions of calculus. It is thus important for us to gain some familiarity with limits in the interest of better understanding the definition of derivative and integral in the later chapters. I will admit that (at least where limits are concerned) we are not entirely rigorous in this work. There is a more basic method of proof that we will not usually employ. Often the proof is by graph or a table of values or simply a sentence explaining logically how the function behaves close to the limit point will suffice for this course. The heavy lifting for limits typically involves removing the indeterminancy through some algebraic chicanery.

The rigorous definition for the limit is the so-called ϵ-δ definition. As a historical note the ϵ-δ formulation actually came long after Newton and Leibnitz pioneered the subject of calculus. There were contradictions and problems that arose because of the free-wheeling careless way calculus was first discussed (in Europe) in terms of *fluxions* or *infinitesimals*. Only later did Euler, Cauchy, Weirstrauss and other 19th century mathematicians formalize the concept of the limit through the ϵ-δ idea. That said, we will only pay attention to this technical detail in one section. Most of questions we consider in calculus do not cut so finely as to require the ϵ-δ formulation. Typically an *advanced calculus* or *real analysis* course will deal with more serious questions involving the ϵ-δ technique.

3.1. DEFINITIONS OF LEFT AND RIGHT LIMITS

The limit of a function exists only if both the left and right limits of the function exist. Whenever I say "exists" you can replace it with "exists as a real number". For example, $\sqrt{-1} = i$ does not exist as a real number. However, it is true that $\sqrt{-1} = i$ exists as a *complex number*. I digress, let's get back to the limits...

Definition 3.1.1: If $f(x)$ gets closer and closer to a real number L_1 as x approaches a from the left on the number line then we write

$$\lim_{x\to a^-} f(x) = L_1$$

which says that the ***left limit*** of $f(x)$ at a is L_1. Another notation I may use at times for the left limit is $f(x) \to L_1$ as $x \to a^-$.

If $f(x)$ gets closer and closer to a real number L_2 as x approaches a from the right on the number line then we write

$$\lim_{x\to a^+} f(x) = L_2$$

which says that the ***right limit*** of $f(x)$ at a is L_2. Another notation I may use at times for the right limit is $f(x) \to L_2$ as $x \to a^+$.

When the left and right limits of f are both equal, say $L_1 = L$ and $L_2 = L$, then we say that the ***limit*** *of f at a is L* and we write

$$\lim_{x\to a} f(x) = L$$

which can also be written $f(x) \to L$ as $x \to a$.

The following comments apply to all three kinds of limits above. We call a the ***limit point***. If there does not exist a real number which satisfies the limiting condition then we say that the limit does not exist. We can abbreviate that by writing it equals "d.n.e."

$$\lim_{x\to a^+} f(x) = d.n.e. \quad \lim_{x\to a^-} f(x) = d.n.e. \quad \lim_{x\to a} f(x) = d.n.e.$$

Now in the case the limit does not exist there are actually many ways that can happen. Two of which we have a nice notation for:

- if the function outputs arbitrarily large positive values as we approach the limit point then we write ∞ instead of d.n.e.

$$\lim_{x\to a^+} f(x) = \infty \quad \lim_{x\to a^-} f(x) = \infty \quad \lim_{x\to a} f(x) = \infty.$$

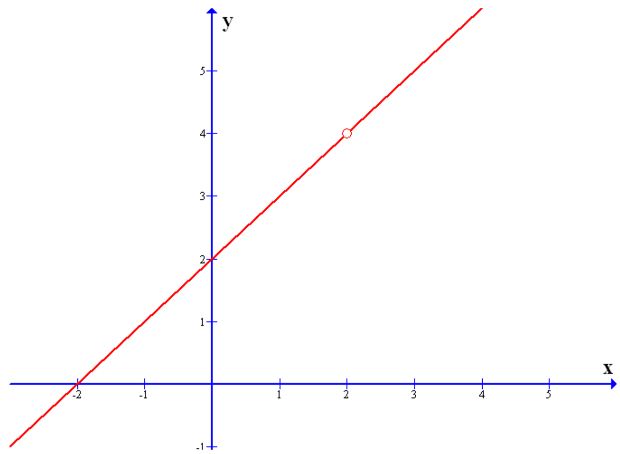
- if the function outputs arbitrarily large negative values as we approach the limit point then we write $-\infty$ instead of d.n.e.

$$\lim_{x\to a^+} f(x) = -\infty \quad \lim_{x\to a^-} f(x) = -\infty \quad \lim_{x\to a} f(x) = -\infty.$$

These correspond to vertical asymptotes in the graph.

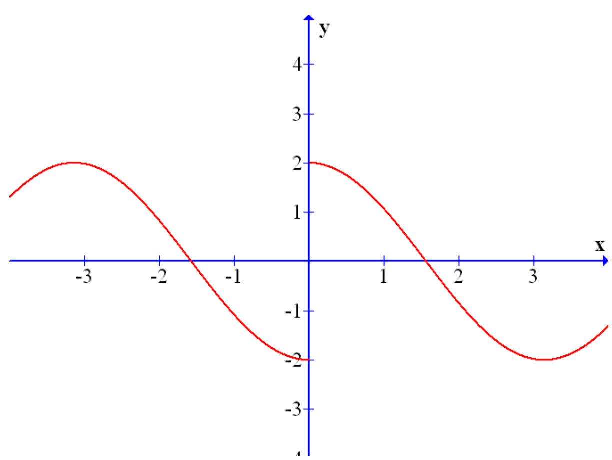
So many words. Let's look at a few pictures.

Let’s begin with a function with a hole in its graph. Suppose that the following is the graph $y = f(x) = (x + 2)(x - 2)/(x - 2)$



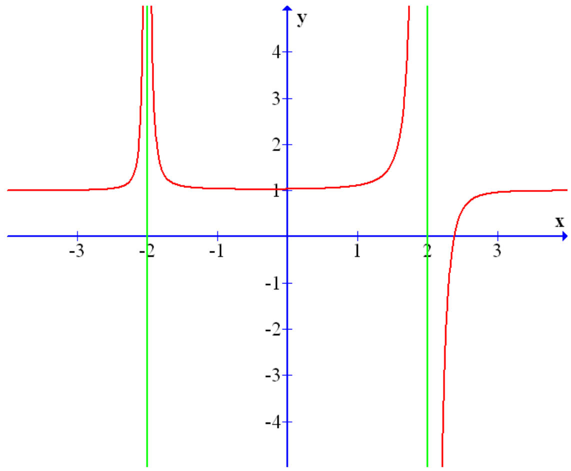
We can easily see that as x approaches 2 from the left or the right we get $f(x)$ closer and closer to 4. So we say that the $\lim_{x\to 2} f(x) = 4$. Notice that the limit point $a = 2$ is not in the domain of the function. That is pretty neat, we can evaluate the limit at $a = 2$ even though $f(2)$ is undefined. For the function pictured above we can see that for limit points other than $a = 2$ we can actually say that $\lim_{x\to a} f(x) = f(a)$.

Next, let’s examine a function which has left and right limits at a particular limit point, but they disagree. I’m tired of f, let’s say the following is the graph $y = g(x)$, let us examine the limit at $a = 0$



We see that as $x \to 0^-$ the function $g(x) \to -2$. On the other hand, as $x \to 0^+$ we observe that $g(x) \to 2$. So the left limit is -2 while the right limit is 2. So the one-sided limits exist but do not agree. Hence we say that the limit of $g(x)$ at zero does not exist. In other words, $\lim_{x\to 0} g(x) = d.n.e.$. *I’ll grant you a bonus point if you can give me an explicit formula for $g(x)$ without breaking it up into cases. (I know there is such a formula cause that’s how I graphed it.)*

Finally let's examine a graph with a few vertical asymptotes. Let us suppose that the following is the graph of $y = h(x)$



I have used green vertical lines to illustrate the vertical asymptotes of the function, these are not part of the graph itself. The function's graph is in red. To begin we discuss the limits at $a = -2$. It is clear that $h(x)$ takes on larger and larger positive values as we get closer and closer to $a = -2$ from the left or right so we can write that $\lim_{x \to -2^-} h(x) = \infty$ and $\lim_{x \to -2^+} h(x) = \infty$ thus

$$\lim_{x \to -2} h(x) = \infty.$$

On the other hand the story at $a = 2$ is a little different. We can see that as se approach two from the left we find the function takes on larger and larger positive values so $\lim_{x \to 2^-} h(x) = \infty$. In contrast, as we approach two from the right side the function takes on larger and larger negative values so $\lim_{x \to 2^+} h(x) = -\infty$. Consequently we find that the two-sided limit at $a = 2$ is not ∞ or $-\infty$, we can only say that

$$\lim_{x \to 2} h(x) = d.n.e.$$

it would not make sense to say it was ∞ since the function does not just get really large and positive at the limit point, likewise it does not make sense to say that the limit was $-\infty$ since the function did not just get really large and negative at the limit point.

Moral of story: limits encapsulate lots of different kinds of behavior both within the domain of the function and also just outside it, like with a hole in the graph or vertical asymptote. The subtle thing to remember is that the limit gets really really really really…. really close to the limit point without actually getting there. This allows us a logical freedom that ordinary algebra will not permit.

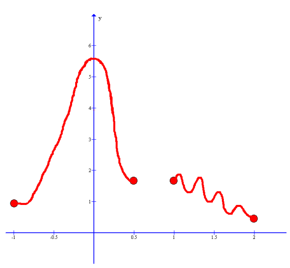
There is something called nonstandard analysis where infinitesimals and infinity are actually "numbers" and in that context limits are traded for formal algebraic ideas, but you have many many math courses before that is something I should tell you, oh oops.

3.2. CONTINUOUS FUNCTIONS

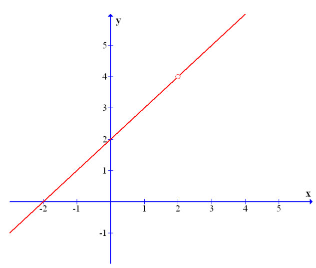
Almost all the functions that arise in basic applications are continuous or piecewise continuous (will discuss later). Without further ado,

> ***Definition 3.2.1:*** A function f is ***continuous at a*** if
>
> $$\lim_{x \to a} f(x) = f(a).$$
>
> When f is continuous for each point inside $I \subseteq dom(f)$ then we say the function f is ***continuous on I***. For endpoints in $I \subseteq dom(f)$ we relax the double-sided limit to the appropriate single sided limit to be fair. Now if $dom(f)$ is a connected subset of the real numbers then we say that f is ***continuous*** if f is continuous on $dom(f)$.

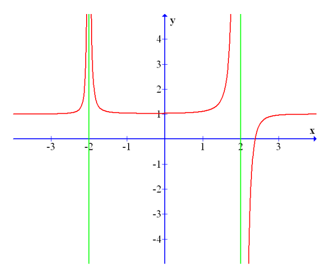
Graphically you might recall that a function is continuous if you can draw its graph without lifting your pencil (or crayon etc…).



i.) not connected



ii.) hole in graph



iii.) vertical asymptotes

Case i. is not continuous because the domain is not connected. However, the function in case i. is continuous on the two separate intervals pictured. Case ii. has a hole in the graph at $x = 2$ so the function is not continuous at 2 hence the function is discontinuous. Case iii. has vertical asymptotes at $x = \pm 2$ so the function is clearly discontinuous at those points. However, at all other points the function in iii. is continuous. We could say that in all the cases above if a point is on the *interior* of the domain then the function is continuous at that point. By *interior* we simply mean that the point is not quite to the boundary of the domain, so an *interior point* has some distance between itself and say the hole in the graph or a vertical asymptote, or maybe just the endpoint of the domain as in case i.

3.3. EVALUATING BASIC LIMITS

Customarily the theorem which I am about to give is motivated by a longer discussion of limits. My philosophy is that it is better to just state this theorem so we can use it. The proof of the theorem actually follows from the properties of limits which I give in the section after this. Anyway, this theorem is very important, perhaps the most important theorem I will give you concerning continuity. It gives the basic building blocks we have to use.

Theorem 3.3.1: The elementary functions given in section 2.4 are all continuous at each point in the interior of their domains.

In other words polynomial, rational, algebraic, trigonometric, exponential, logarithmic, hyperbolic trigonometric, etc… discussed in § 2.4 are continuous where their formulas make sense. If we are not at a vertical asymptote or hole in the graph then elementary functions are continuous. I should mention that there exist non-elementary functions which are discontinuous everywhere. Those sort of functions arise in the study of *fractals*.

Example 3.3.1: In each of the limits below the limit point is on the interior of the domain of the elementary function so we can just evaluate to calculate the limit.

$i.)\ \lim_{x\to 3}(\sin(x)) = \sin(3)$

$ii.)\ \lim_{x\to -2}\left(\frac{\sqrt{x^2-3}}{x+5}\right) = \frac{\sqrt{4-3}}{-2+5} = \frac{1}{3}$

$iii.)\ \lim_{h\to 0}(\sin^{-1}(h)) = \sin^{-1}(0) = 0$

$iv.)\ \lim_{x\to a}(x^3+3x^2-x+3) = a^3+3a^2-a+3.$

We did not even need to look at a graph to calculate these limits. Of course it is also possible to evaluate most limits via a graph or a table of values, but those methods are less reliable..

I may ask you to calculate a particular limit a particular way. However, if I don't say one way or the other you are free to think for yourself. Sometimes a graph is a good solution, sometimes a table of values is convenient, sometimes we can use Theorem 3.3.1 or properties I'll discuss in the next section. The example below illustrates the table of values idea.

Example 3.3.2: Using a table of values to see $\lim_{x\to 0}(\frac{\sin(x)}{x}) = 1$

x	sin(x)/x
0.5	0.958851
0.2	0.993347
0.1	0.998334
0.01	0.999983
0.001	0.999999

Now the limit consider in Example 3.3.2 is not nearly as obvious as the limits in Example 3.3.1. I should mention that the limit has *indeterminant form of type 0/0* since both $\sin(x)$ and x tend to zero as x goes to zero. One of main goals in this chapter is to learn how to analyze indeterminant forms. I do not recommend the table of values method for most problems. It will work, but it's kind of like painting your car with a paint brush. I do it once, but probably not when I was trying to impress anybody. It is a good way to gain intuition about a limit, but I would like to see us use more solid arguments for the final argument.

Indeterminant forms:

The first three of these we encounter most often. We will need to wait a little bit before we tackle some of the trickier cases. But, just to give you an idea of all the different ways a limit can be *undetermined*, here they are.

- We say that $\lim(\frac{f}{g})$ is of "type $\frac{0}{0}$" if $\lim f = 0$ and $\lim g = 0$.
- We say that $\lim(fg)$ is of "type 0∞" if $\lim f = 0$ and $\lim g = \infty$.
- We say that $\lim(\frac{f}{g})$ is of "type $\frac{\infty}{\infty}$" if $\lim f = \infty$ and $\lim g = \infty$.
- We say that $\lim(f - g)$ is of "type $\infty - \infty$" if $\lim f = \infty$ and $\lim g = \infty$.
- We say that $\lim(f^g)$ is of "type 0^0" if $\lim f = 0$ and $\lim g = 0$.
- We say that $\lim(f^g)$ is of "type ∞^0" if $\lim f = \infty$ and $\lim g = 0$.
- We say that $\lim(f^g)$ is of "type 1^∞" if $\lim f = 1$ and $\lim g = \infty$.

When we encounter such limits we have to do some ***thinking*** and/or ***work*** to unravel the *indeterminancy*. We saw the table of values revealed the mystery of $\sin(x)/x$ in Example 3.3.2. We will learn better methods in future sections.

3.4. PROPERTIES OF LIMITS

The notation $\lim$ is meant to include left, right and double-sided limits.

Proposition 3.4.1: Let $c \in \mathbb{R}$ and suppose that the limits of the functions f and g exist, that means $\lim(f) \in \mathbb{R}$ and $\lim(g) \in \mathbb{R}$, then

$i.)\ \lim(f \pm g) = \lim(f) \pm \lim(g))$

$ii.)\ \lim(cf) = c\lim(f)$

$iii.)\ \lim(fg) = \lim(f)\lim(g)$

$iv.)\ \lim(f/g) = \lim(f)/\lim(g)$ given $\lim(g) \neq 0$

$v.)\ \lim(f(g(x))) = f(\lim(g(x)))$ given f continuous at $\lim(g)$.

It should be emphasized that we need to know that the limits of both functions exist for this proposition to work. You **cannot** just glibly say $\lim g = 0$ and $\lim f = \infty$ so $\lim(fg) = \lim f \lim g = 0 \cdot \infty = 0$. This kind of reasoning is not allowed because there are cases where it fails. It could be that $0 \cdot \infty = 1$ or 2 or 3 or -75 or 42 etc... it is ***undetermined.*** We need to know that $\lim(f) \in \mathbb{R}$ and $\lim(g) \in \mathbb{R}$ or else we cannot break up limits as described in Proposition 3.4.1. Ok, enough about what not to do, let's see what we can do.

Example 3.4.1: I am going to comment out to the side as we apply the properties listed in Proposition 3.4.1,

$$\begin{aligned}
\lim_{x\to 0}(\sin(x) + \cos(e^x)) &= \lim_{x\to 0}(\sin(x)) + \lim_{x\to 0}(\cos(e^x)), \quad \text{(used i.)} \\
&= \sin(0) + \cos(\lim_{x\to 0} e^x), \quad \text{(used v.)} \\
&= \sin(0) + \cos(e^0) \\
&= \cos(1).
\end{aligned}$$

notice that the first step was not really justified until we learned that both $\lim_{x\to 0}(\sin(x))$ and $\lim_{x\to 0}(\cos(e^x))$ exist. Also I should mention that we have just used the continuity of sine, cosine and the exponential function.

3.5. ALGEBRAICALLY DETERMINING LIMITS

We have established all the basics. Now it is time for us to do some real thinking. The examples given in this section illustrate all the basic algebra tricks to unravel undetermined limits. I like to say we do algebra to determine the limit. The limits are not just decoration, many times an expression with the limit is correct while the same expression without the limit is incorrect. On the other hand we should not write the limit if we do not need it in the end. How do we know when and when not? We practice.

Example 3.5.1:

$$\lim_{x \to -2} \left(\frac{x+2}{x^2+3x+2} \right).$$

Notice that this limit is of type 0/0 since the numerator and denominator are both zero when take the limit at -2.

$$\begin{aligned} \lim_{x \to -2} \left(\frac{x+2}{x^2+3x+2} \right) &= \lim_{x \to -2} \left(\frac{x+2}{(x+2)(x+1)} \right) \\ &= \lim_{x \to -2} \left(\frac{1}{x+1} \right) \\ &= \frac{1}{-2+1} \\ &= -1. \end{aligned}$$

The second step where we cancelled $x+2$ with $x+2$ is valid *inside the limit* because we do not have $x = -2$ in the limit. We get very close, but that is the difference, this is not division by zero.

Example 3.5.2: The limit below is also type 0/0 to begin with,

$$\begin{aligned} \lim_{x \to 0} \left(\frac{3x+x^2}{x^3+2x^2+x} \right) &= \lim_{x \to 0} \left(\frac{x(x+3)}{x(x^2+2x+1)} \right) \\ &= \lim_{x \to 0} \left(\frac{(x+3)}{(x^2+2x+1)} \right) \\ &= \frac{0+3}{0+0+1} \\ &= 3. \end{aligned}$$

I reiterate, we can cancel the x inside the limit because $x \neq 0$ within the limit. Again we see that factoring and cancellation has allowed us to modify the limit so that we could reasonably plug in the limit point in the simplified limit. This is often the goal.

We see that sometimes algebraic manipulations will change an undetermined form to a determined form, by which I simply mean an expression which does not violate the laws of real arithmetic when you plug in the limit point.

Example 3.5.3:

$$\begin{aligned} \lim_{\theta \to 0}\left(\frac{\tan(\theta)}{\sin(\theta)}\right) &= \lim_{\theta \to 0}\left(\frac{\sin(\theta)}{\cos(\theta)}\frac{1}{\sin(\theta)}\right) \\ &= \lim_{\theta \to 0}\left(\frac{1}{\cos(\theta)}\right) \\ &= \frac{1}{\cos(0)} \\ &= 1. \end{aligned}$$

Example 3.5.4: the first step is a time-honored trick, it is nothing more than multiplication by 1. So if you encounter a similar problem try a similar trick.

$$\begin{aligned} \lim_{h \to 0}\left(\frac{\sqrt{4+h}-2}{h}\right) &= \lim_{h \to 0}\left(\frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)}\right) \\ &= \lim_{h \to 0}\left(\frac{4+h+2\sqrt{4+h}-2\sqrt{4+h}-4}{h(\sqrt{4+h}+2)}\right) \\ &= \lim_{h \to 0}\left(\frac{h}{h(\sqrt{4+h}+2)}\right) \\ &= \lim_{h \to 0}\left(\frac{1}{\sqrt{4+h}+2}\right) \\ &= \frac{1}{\sqrt{4}+2} \\ &= \frac{1}{4}. \end{aligned}$$

Example 3.5.5: Here the trick is to combine the fractions in the numerator by finding the common denominator of $4x$.

$$\begin{aligned} \lim_{x \to -4}\left(\frac{\frac{1}{4}+\frac{1}{x}}{4+x}\right) &= \lim_{x \to -4}\left(\frac{\frac{x+4}{4x}}{4+x}\right) \\ &= \lim_{x \to -4}\left(\frac{x+4}{4x}\cdot\frac{1}{4+x}\right) \\ &= \lim_{x \to -4}\left(\frac{1}{4x}\right) \\ &= \frac{-1}{16}. \end{aligned}$$

Example 3.5.6:

$$\lim_{x\to 3}\left(\frac{(x-3)\cos(x-3)}{x(x^2-5x+6)}\right) = \lim_{x\to 3}\left(\frac{(x-3)\cos(x-3)}{x(x-3)(x-2)}\right)$$
$$= \lim_{x\to 3}\left(\frac{\cos(x-3)}{x(x-2)}\right)$$
$$= \frac{\cos(3-3)}{3(3-2)}$$
$$= \frac{1}{3}.$$

Example 3.5.7: Piecewise defined functions can require a bit more care. Sometimes we need to look at one-sided limits.

$$\lim_{x\to 0}\left[\frac{|x|}{x}\right] = ?$$

recall that the notation $|x|$ is the absolute value of x, it is the distance from zero to x on the number line.

$$|x| = \begin{cases} -x & : x < 0. \\ x & : x \geq 0. \end{cases}$$

In the left limit $x \to 0^-$ we have $x < 0$ so $|x| = -x$ thus,

$$\lim_{x\to 0^-}\left[\frac{|x|}{x}\right] = \lim_{x\to 0^-}\left[\frac{-x}{x}\right] = \lim_{x\to 0^-}\left[\frac{-1}{1}\right] = -1.$$

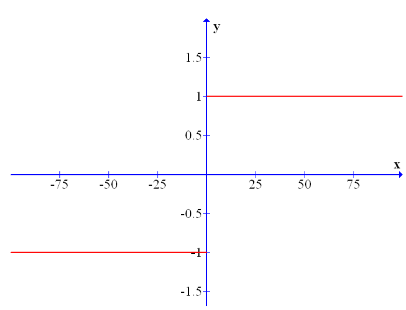
In the right limit $x \to 0^+$ we have $x > 0$ so $|x| = x$ thus,

$$\lim_{x\to 0^-}\left[\frac{|x|}{x}\right] = \lim_{x\to 0^-}\left[\frac{x}{x}\right] = \lim_{x\to 0^-}\left[\frac{1}{1}\right] = 1.$$

Consequently we find that the limit in question does not exist since the left and right limits disagree.

$$\lim_{x\to 0}\left[\frac{|x|}{x}\right] = d.n.e.$$

The function we just looked at in Example 3.5.7. is an example of a *step function*. They are very important to engineering since they model switching. The graph of $y = |x|/x$ looks like a single stair step,



3.6. SQUEEZE THEOREM

There are limits not easily solved through algebraic trickery. Sometimes the "Squeeze" or "Sandwich" Theorem allows us to calculate the limit.

> *Theorem 3.6.1:(Squeeze Theorem)* Let $g(x) \leq f(x) \leq h(x)$ for all x near a then we find that the limits at a follow the same ordering,
>
> $$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x).$$
>
> Moreover, if $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} f(x) = L$.

We can think of $h(x)$ as the top slice of the sandwich and $g(x)$ as the bottom slice. The function $f(x)$ provides the BBQ or peanut butter or whatever you want to put in there.

Example 3.6.1: Use the Squeeze Theorem to calculate $\lim_{x \to 0}(\ x^2 \sin(\frac{1}{x}) \ )$. Notice that the following inequality is suggested by the definition or graph of sine

$$-1 \leq \sin(x) \leq 1$$

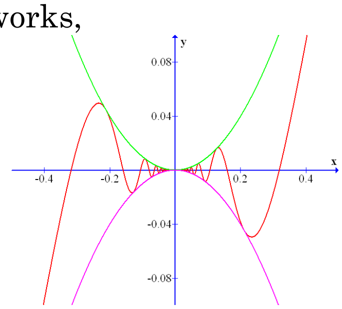
Now multiply by x^2 which is positive if $x \neq 0$ so the inequality is maintained,

$$-x^2 \leq x^2 \sin(x) \leq x^2$$

We identify that $g(x) = -x^2$ and $h(x) = x^2$ sandwich the function $f(x) = x^2 \sin(\frac{1}{x})$ near $x = 0$. Moreover, it is clear that

$$\lim_{x \to 0}(\ x^2 \ ) = 0 \qquad \lim_{x \to 0}(\ -x^2 \ ) = 0.$$

Therefore, by the Squeeze Theorem, $\lim_{x \to 0}(\ x^2 \sin(\frac{1}{x}) \ )$. Graphically we can see why this works,



$g(x) = \text{purple}$
$f(x) = \text{red}$
$h(x) = \text{green}$

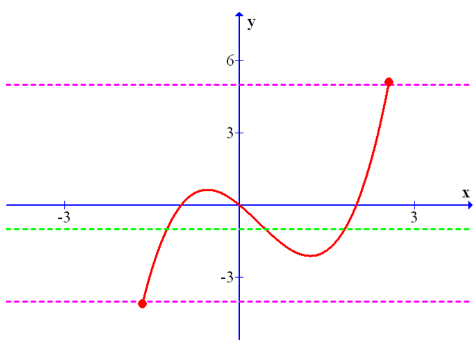
Incidentally, you might be wondering why we could not just use Proposition 3.4.1 part iii. The problem is that since the limit of $\sin(\frac{1}{x})$ at zero does not exist (*if you look at the graph of the function* $\sin(\frac{1}{x})$ *you'll see that it oscillates wildly near zero*) we have no right to apply the proposition.

> *Example 3.6.2:* Suppose that all we know about the function $f(x)$ is that it is sandwiched by $1 \leq f(x) \leq x^2 + 2x + 2$ for all x. Can we calculate the limit of $f(x)$ as $x \to -1$? Well, notice that
>
> $$\lim_{x \to -1}(1) = 1 \qquad \lim_{x \to -1}(x^2 + 2x + 2) = 1.$$
>
> Therefore, by the Squeeze Theorem, $\lim_{x \to -1} f(x) = 1$.

3.7. INTERMEDIATE VALUE THEOREM

> ***Theorem 3.7.1 (I.V.T.):*** Suppose that f is continuous on an interval $[a, b]$ with $f(a) \neq f(b)$ and let N be a number such that N is between $f(a)$ and $f(b)$ then there exists $c \in (a, b)$ such that $f(c) = N$.

Notice that this theorem only tells us that there exists a number c, it does not actually tell us how to find that number. This theorem is quite believable if you think about it graphically. Essentially it says that if you draw a horizontal line $y = N$ between the lines $y = f(a)$ and $y = f(b)$ then since the function is continuous we must cross the line $y = N$ at some point. Remember that the graph of a continuous function has no jumps in it so we can't possibly avoid the line $y = N$. Let me draw the situation for the case $f(a) < f(b)$,



> Green line is $y = N$. Purple lines are $y = f(a)$ and $y = f(b)$. In this example there is more than one point c such that $f(c) = N$. There must be ***at least*** one such point provide that the function is continuous.

The IVT can be used for an indirect manner to locate the zeros of continuous functions. The theorem motivates an iterative process of divide and conquer to find a zero of the function. Essentially the point is this, if a continuous function changes from positive to negative or vice-versa on some interval then it must be zero at least one place on that interval. This observation suggests we should guess where the function is zero and then look for smaller and smaller intervals where the function has a sign change. We can just keep zooming in further and further and getting closer and closer to the zero. Perhaps you have already used the IVT without realizing it when you looked for an intersection point on your graphing calculator.

Example 3.7.1: Show that there exists a zero of the polynomial $P(x) = 4x^3 - 6x^2 + 3x - 2$ on the interval $[1, 2]$. Observe that,

$$P(1) = 4 - 6 + 3 - 2 = -1 < 0$$
$$P(2) = 32 - 24 + 6 - 2 = 12 > 0$$

We know that P is continuous everywhere and clearly $P(1) < 0 < P(2)$ so by the IVT we find there exists some point $c \in (1, 2)$ such that $P(c) = 0$. To find what c is precisely would require more work.

Example 3.7.2: Does $\tan^{-1}(x) = -\cos(x)$ for some $x \in (-2, 2)$?
Lets rephrase the question. Does $f(x) = \tan^{-1}(x) + \cos(x) = 0$ for some $x \in (-2, 2)$? This is the same question, but now we can use the IVT plus the sign change idea. Observe,

$$f(-2) = \tan^{-1}(-2) + \cos(-2) = -1.52$$
$$f(2) = \tan^{-1}(2) + \cos(2) = 0.691$$

Obviously $f(-2) < 0 < f(2)$ and both $\tan^{-1}(x)$ and $\cos(x)$ are continuous everywhere so by the IVT there is some $c \in (-2, 2)$ such that $f(c) = 0$. Clearly that point has

$$\tan^{-1}(c) = -\cos(c).$$

If you examine the graphs of $y = \tan^{-1}(x)$ and $y = -\cos(x)$ you will find that they intersect at $c = -0.82$ (approximately).

Let me take a moment to write an algorithm to find roots. Suppose we are given a continuous function f, we wish to find c such that $f(c) = 0$.

1.) Guess that f is zero on some interval (a_0, b_0).
2.) Calculate $f(a_0)$ and $f(b_0)$ if they have opposite signs go on to 3.) otherwise return to 1.) and guess differently.
3.) Pick $c_1 \in (a_0, b_0)$ and calculate $f(c_1)$.
4.) If the sign of $f(c_1)$ matches $f(a_0)$ then say $a_1 = c_1$, and let $b_1 = b_0$
If the sign of $f(c_1)$ matches $f(b_0)$ then say $b_1 = c_1$, and let $a_1 = a_0$.
5.) Pick $c_2 \in (a_1, b_1)$ and calculate $f(c_2)$.
6.) If the sign of $f(c_2)$ matches $f(a_1)$ then say $a_2 = c_2$, and let $b_2 = b_1$
If the sign of $f(c_2)$ matches $f(b_1)$ then say $b_2 = c_2$, and let $a_2 = a_1$.

And so on… If we ever found $f(c_k) = 0$ then we stop there. Otherwise, we can repeat this process until the subinterval (a_k, b_k) is so small we know the zero to some desired accuracy. Say you wanted to know 2 decimals with certainty, if you did the iteration until the length of the interval (a_k, b_k) was 0.001 then you would be more than certain.

3.8. PRECISE DEFINITION OF LIMIT

You might read the article by Dr. Monty C. Kester posted on Blackboard. It helps motivate the definition I give now.

Definition 3.8.1: We say that the limit of a function f at $a \in \mathbb{R}$ exists and is equal to $L \in \mathbb{R}$ iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

Notice we do not require that the limit point be in the domain of the function. The zero in $0 < |x - a| < \delta$ is precise way of saying that we do not consider the limit point in the limit. All other x that are within δ units of the limit point a are included in the analysis (recall that $|a - b|$ gives the distance from a to b on the number line). If the limit exists then we can choose the δ such that the values $f(x)$ are within ϵ units of the limiting value L.

Example 3.8.1: Prove that

$$\lim_{x \to 3}(2x) = 6.$$

Let us examine what we need to produce. We need to find a δ such that

$$|x - 3| < \delta \implies |2x - 6| < \epsilon$$

The way this works is that ϵ is chosen to start the proof so we cannot adjust ϵ, however the value for δ we are free to choose. But, whatever we choose it must do the needed job, it must make the implication hold true. I usually look at what I want to get in the end and work backwards. We want, $|2x - 6| < \epsilon$. Notice

$$|2x - 6| = |2(x - 3)| = |2||x - 3| = 2|x - 3| < 2\delta$$

If we choose δ such that $2\delta = \epsilon$ then it should work. So we will want to use $\delta = \epsilon/2$ in our proof. Let's begin the proof:

Let $\epsilon > 0$ *choose* $\delta = \epsilon/2$. *If* $x \in \mathbb{R}$ *such that* $0 < |x - 3| < \delta$ *then*

$$|2x - 6| = 2|x - 3| < 2\delta = 2(\epsilon/2) = \epsilon.$$

Therefore, $\lim_{x \to 3}(2x) = 6$.

I put the proof in italics to alert you to the fact that the rest of this jibber-jabber was just to prepare for the proof. Often a textbook will just give the proof and leave it to the reader to figure out how the proof was concocted.

I'll now give a formal proof that the limit is linear. This proof I include to show you how these things are argued, you are responsible for problems more like the easy example unless I specifically say otherwise. If I were to put this on a test I'd warn you it was coming (or it would be a bonus question)

Proposition 3.8.2: If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then $\lim_{x\to a}[f(x)+g(x)] = L_1 + L_2$. In other words,

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

Proof: Let $\epsilon > 0$ *and assume that* $\lim_{x\to a} f(x) = L_1$ *and* $\lim_{x\to a} g(x) = L_2$.

Clearly $\epsilon/2 > 0$ *thus as* $\lim_{x\to a} f(x) = L_1$ *there exists* $\delta_1 > 0$ *such that* $0 < |x-a| < \delta_1$ *implies* $|f(x) - f(a)| < \epsilon/2$*. Likewise, as* $\lim_{x\to a} g(x) = L_2$ *there exists* $\delta_2 > 0$ *such that* $0 < |x-a| < \delta_2$ *implies* $|g(x) - g(a)| < \epsilon/2$*.*

Define $\delta_3 = min(\delta_1, \delta_2)$*. Suppose* $x \in \mathbb{R}$ *such that* $0 < |x-a| < \delta_3$ *then* $0 < |x-a| < \delta_1$ *and* $0 < |x-a| < \delta_2$ *because* $\delta_3 \leq \delta_1, \delta_2$*. Consider then,*

$$\begin{aligned} |(f+g)(x) - (L_1+L_2)| &= |f(x) - L_1 + g(x) - L_2| \\ &\leq |f(x) - L_1| + |g(x) - L_2| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

Hence, for each $\epsilon > 0$ *there exists* $\delta_3 > 0$ *such that* $|(f+g)(x) - (L_1+L_2)| < \epsilon$ *whenever* $0 < |x-a| < \delta_3$*. Thus,* $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$*.*

The proof I just gave may leave you with some questions. Such as:

- Where did the $\epsilon/2$ come from ?

- Where did the $\delta_3 = min(\delta_1, \delta_2)$ come from?

Short answer, imagination. Longer answer, we typically work these sort of proofs backwards as in Example 3.8.1.

As I said before, you start with what you want to show then determine how you should use the given data to prove the conclusion. There are a few facts which are helpful in these sorts of arguments. Let's make a collection:

- If $a < b$ and $b < c$ then $a < c$.
- Let $\delta > 0$. If $a < b$ then $a\delta < b\delta$. (preserved inequality)
- Let $\gamma < 0$. If $a < b$ then $\gamma a > \gamma b$. (reversed inequality)
- $|-a| = |a|$
- $|ab| = |a||b|$
- $|a| \geq 0$
- $0 \leq |x-a|$
- Let $\delta > 0$ then $|x-a| < \delta$ is equivalent to $-\delta < x-a < \delta$
- The triangle inequality; $|a+b| \leq |a| + |b|$
- $|a-b| \leq |a| + |b|$
- Let $\delta > 0$ then if we add to the denominator of some fraction it makes the fraction smaller: (assuming $b > 0$)

$$\frac{a}{b+\delta} < \frac{a}{b}$$

- Let $\delta > 0$ then if we subtract from the denominator of some fraction it makes the fraction larger: (assuming $b > 0$)

$$\frac{a}{b-\delta} > \frac{a}{b}$$

Now, I doubt we will use all these tricks. In include them here because if you do take a course in real analysis you'll need to know these things. Sadly, not all real analysis books make any attempt to organize or be clear about these basic tools. (I speak from bad experience) Enough about all that let's try some more examples.

Example 3.8.2: (this is the bow-tie proof) Prove that

$$\lim_{x\to 3}(4x-5)=7.$$

For each $\epsilon > 0$ we need to find a δ such that

$$|x-3|<\delta \implies |(4x-5)-7|<\epsilon$$

Observe, given that $|x-3|<\delta$ we have

$$|4x-12| = 4|x-3| < 4\delta$$

If we choose δ such that $4\delta = \epsilon$ then it should work. So we will want to use $\delta = \frac{\epsilon}{4}$ in our proof. Let's begin the proof:

Let $\epsilon > 0$ *choose* $\delta = \epsilon/4$. *If* $x \in \mathbb{R}$ *such that* $0 < |x-3| < \delta$ *then*

$$|f(x)-L| = |(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = \epsilon.$$

Therefore, $\lim_{x\to 3}(4x-5)=7$.

I put the proof in italics to alert you to the fact that the rest of this jibber-jabber was just to prepare for the proof. Often a textbook will just give the proof and leave it to the reader to figure out how the proof was concocted.

The text also discusses a technical definition for what is meant by limits that go to infinity or negative infinity. We will not cover those this semester. You will have a problem like Example 3.8.1 or 3.8.2 on the first test. It will be worth 10 points.

Example 3.8.3: Prove that

$$\lim_{x \to 0}(x^2) = 0.$$

For each $\epsilon > 0$ we need to find a δ such that

$$0 < |x - 0| < \delta \implies |x^2| < \epsilon$$

Observe, given that $|x| < \delta$ we have

$$|x^2| = |x||x| < \delta\delta = \delta^2$$

If we choose δ such that $\delta^2 < \epsilon$ then it should work. So we will want to use $\delta = \sqrt{\epsilon}$ in our proof. Let's begin the proof:

Let $\epsilon > 0$ *choose* $\delta = \sqrt{\epsilon}$*. If* $x \in \mathbb{R}$ *such that* $0 < |x| < \delta$ *then*

$$|f(x) - L| = |x^2| = |x|^2 < \delta^2 = (\sqrt{\epsilon})^2 = \epsilon.$$

Therefore, $\lim_{x \to 0}(x^2) = 0$.

I put the proof in italics again as to emphasize the distinction between preparing for the proof and stating the proof.