4. DERIVATIVES

We will define the derivative of a function in this chapter. The need for a derivative arises naturally within the study of the motion of physical bodies.

You are probably already familiar with the *average velocity* of a body. For example, if a car travels 100 miles in two hours then it has an average velocity of 50 mph. That same care may not have traveled the same velocity the whole time though, sometimes it might have gone 70mph at the bottom of a hill, or perhaps 0mph at a stoplight. Well, this concept I just employed used the idea of *instantaneous velocity*. It is the velocity measured with respect to an instant of time.

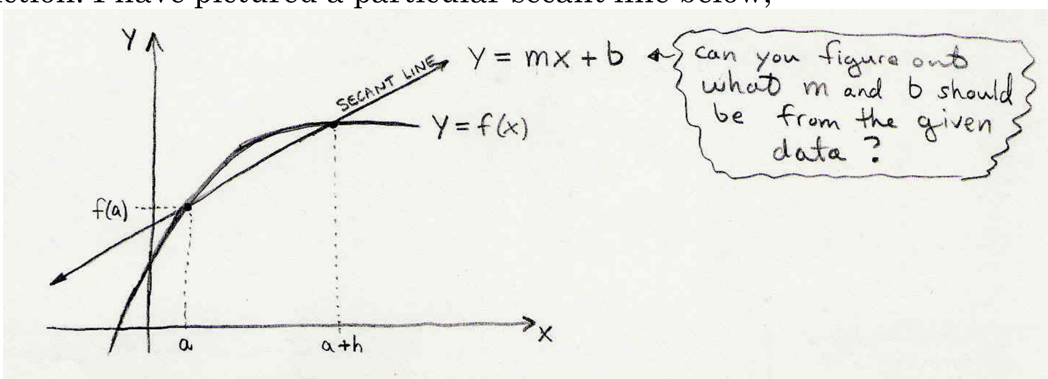
How small is an "instant"? Well, it's pretty small. You might imagine that this "instant" is some agreed small unit of time. That is not the case, there is no natural standard for all processes. I suppose you could argue with the policeman that your average rate of speed to school was 30mph (taking the "instant" to be 10 minutes for me) but I bet all he'll care about is the 40mph you did through the 20mph school zone. The "velocity" of a car as measured by radar is essentially the instantaneous velocity. It is the time rate change in distance for an arbitrarily small increment of time. It seems intuitive to want such a description of motion, I have a hard time thinking about how we would describe motion without instantaneous velocity. But, then I have (we all have) grown up under the influence of Isaac Newton's ideas about motion. Certainly he was not alone in the development of these ideas, Galileo, Kepler and a host of others also pioneered these concepts which we take for granted these days. Long story short, differential calculus was first motivated by the study of motion. Our goal in this chapter is to give a precise meaning to such nebulous phrases as "instant" of time. The limits of chapter 3 will aid us in this description.

Generally, the derivative of a function describes how the function changes with respect to its independent variable. When the independent variable is time then it is a time-rate of change. But, that need not always be the case. I believe that Newton first thought of things changing with respect to time, he had physics on the brain. In contrast, Leibniz considered more abstract rates of change and the modern approach probably is closer to his work. We use Leibniz' notation for the most part. Anyway, I digress as usual.

Finally, I cannot overstate the importance of this chapter. The derivative forms the core of the calculus sequence. And it describes much more than velocity, that is just one application. Basically, if something changes then a derivative can be used to model it. Its ubiquitous.

4.1. DEFINITION OF DERIVATIVE

Let a be a fixed number throughout this discussion. Let h be an number which we allow to vary. Then a ***secant line*** at $(a, f(a))$ is simply a line which connects that point to another point $(a + h, f(a + h))$ on the graph of the function. I have pictured a particular secant line below,



You can imagine that as *h* increases or decreases we will get a different secant line. In fact, there are infinitely many secant lines. Notice that the slope of the pictured secant line is just the rise over the run, that is

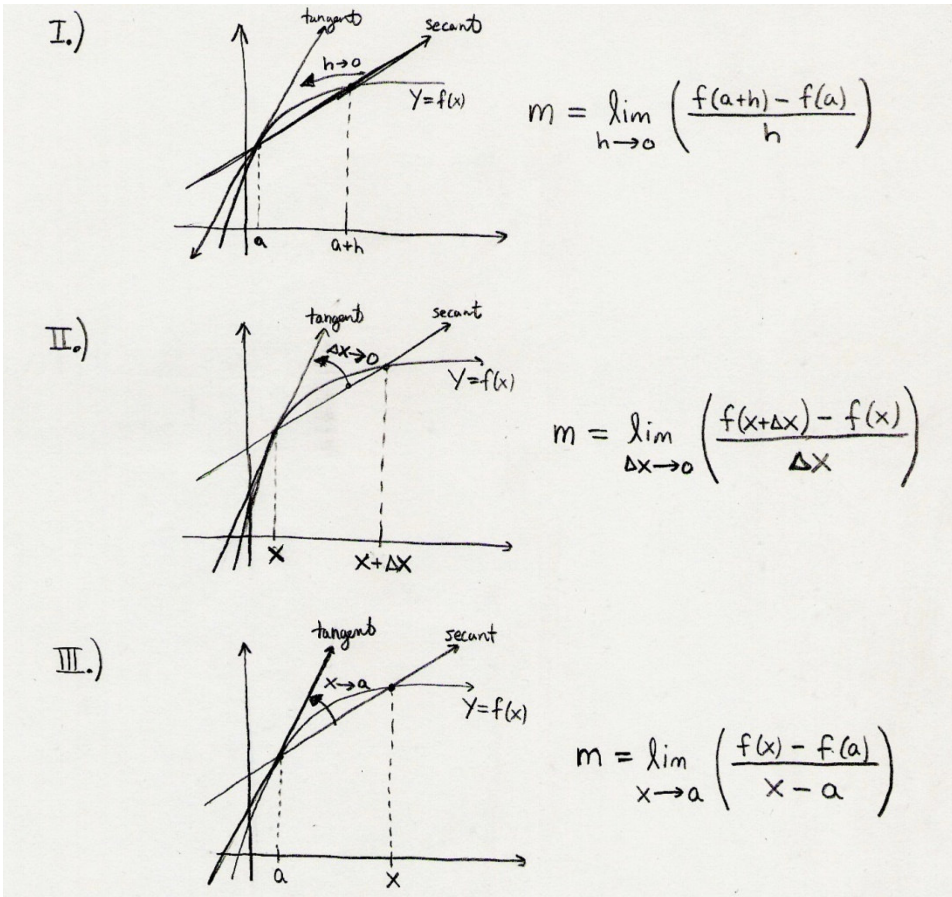
$$m = \frac{\triangle y}{\triangle x} = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}.$$

this may be familiar to you, it is the so-called *"difference quotient"* some of you may have seen in precalculus.

Now imagine that *h* goes to zero. As we take that limit we will get the ***tangent line*** which just kisses the function at the point $(a, f(a))$ (it may however intersect the graph elsewhere depending on how the graph curves away from the point of tangency).

> *Definition 4.1.1:* The tangent line to $y = f(x)$ is the line that passes through $(a, f(a))$ and has the slope
>
> $$m = \lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h} \right)$$
>
> if the limit exists, otherwise we say there is no tangent line at that point. If there is a tangent line through $(a, f(a))$ to the curve $y = f(x)$ then the equation for the tangent line is
>
> $$\boxed{y = f(a) + m(x-a)}$$

The tangent line is unique when it exists because limits are unique when they exist. There are other equivalent ways of looking at the limit which gives the slope of the tangent line. For examples:



The slope of the tangent line characterizes how quickly y is changing with respect to x. The slope of the tangent line gives us the *instantaneous rate of change of* y *is with respect to* x. Let us give the slope of the tangent line a new name, let's call it ***<u>"the derivative at a point"</u>***

> <u>*Definition 4.1.2:*</u> The derivative of f at $x = a$ is the slope of the tangent line through $(a, f(a))$ when it exists. We denote it by
> $$f'(a) = \lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h} \right)$$

this terminology becomes clearer in the next section. Sorry to not give any explicit examples so far, stick with me we will get to them soon.

You may be wondering, when does the derivative at a point fail to exist? What sort of function would make that happen? The example that follows illustrates one culprit, a "kink" or "corner" in the graph.

Example 4.1.1: The *absolute value* function is $f(x) = |x|$. As we have discussed it is really a piece-wise defined function. We have

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}.$$

It turns out that this function has a kink at zero where it changes from a negative slope to a positive slope. This means that the *difference quotient* has different left and right limits at zero. In particular,

$$\lim_{h \to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1.$$

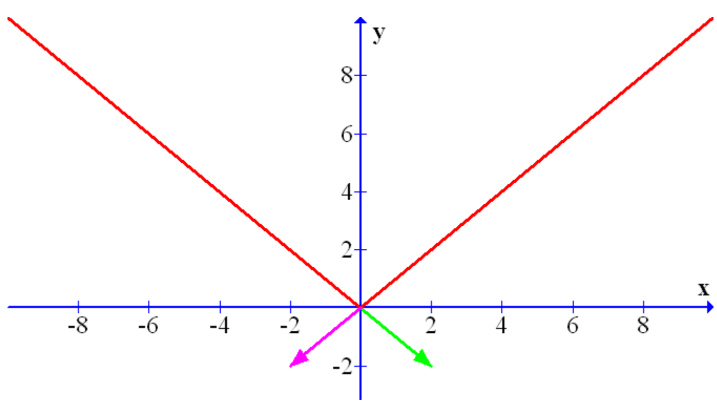
Notice that we replace $|h|$ with $-h$ because in this left limit we allow values to the left of zero on the number line, those are negative numbers. Similarly,

$$\lim_{h \to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} (1) = 1.$$

Therefore we can conclude,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \lim_{h \to 0} \frac{|h|}{h} = d.n.e.$$

Geometrically this is evidenced in our inability to pick a unique tangent line at the origin. Which should we choose, the positive (purple) or the negative (green) sloped tangent line?



We define the ***slope of a function at a point*** to be the derivative of the function at that point (when it exists). We see that a function does not have a well-defined slope at a kink or corner in the graph because the left and right limits disagree. Another way the derivative at a point can fail to exist is for the function to have a vertical tangent. A popular example of that is $f(x) = \sqrt{x}$, if you look at the graph the tangent line is vertical. Vertical lines do not have a well-defined slope.

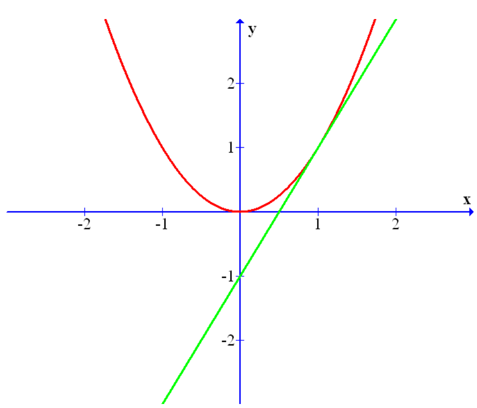
Example 4.1.2: Find the slope of the parabola $y = x^2$ at $x = 1$. In other words, find the slope of the graph $y = f(x) = x^2$ when $x = 1$. We defined the slope of the graph at a point to be the slope of the tangent line at that point. So we calculate,

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\left(\frac{f(1+h) - f(1)}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{(1+h)^2 - 1^2}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{1 + 2h + h^2 - 1}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{2h + h^2}{h} \right) \\ &= \lim_{h\to 0}(2 + h) \\ &= 2. \end{aligned}$$

I have listed more steps than I typically do for such limits. Notice the critical thing here is that once the 1 cancels with -1 then all terms have a factor of h so it cancels with the h in the denominator. We see that the slope of the parabola at the point $(1, 1)$ is 2. Moreover, we can even find the equation of the tangent line as Definition 4.1.1 described,

$$y = f(1) + f'(0)(x - 1) = 1 + 2(x - 1) = 2x - 1.$$

It is possible to find the tangent line approximately through drawing a careful graph and using a ruler and graph paper. But, our results are not approximate. We found the exact result using calculus. Here is what it looks like,



4.2. DERIVATIVE AS FUNCTION & POWER RULE

Definition 4.2.1: If a function f is differentiable at each point in $U \subseteq \mathbb{R}$ then we define a new function denoted f' which is called the ***derivative*** of f. It is defined point-wise by,

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right).$$

We also may use the notation $f' = df/dx = \frac{df}{dx}$. When a function is has a continuous derivative on $U \subseteq \mathbb{R}$ we say that $f \in C^1(U)$. If the derivative has a continuous derivative f'' on $U \subseteq \mathbb{R}$ then $f \in C^2(U)$. If we can take arbitrarily many derivatives which are continuous on $U \subseteq \mathbb{R}$ then we say that f is a ***smooth function*** and $f \in C^\infty(U)$.

Now don't worry too much about the *higher derivatives* like f'' quite yet. We'll come back to those once we are more experienced with the *first derivative* f'. I should warn you that we will proceed logically. Eventually after more work on my part we will have tools to treat many of the same problems with much less work. That said, we need to start at the beginning. Our goal in the remainder of this section is to derive the *power rule*. If you already know these things from high school then keep in mind that I do still expect you to learn why these things are true. Don't be too worried though, these proofs are fairly benign and I will give fair warning if I plan to ask a tricky one on a test.

1.) *Derivative of constant function:* $f(x) = c$. This is a very boring function, no matter what the input the output is just the fixed number c.

$$\begin{aligned} f'(x) &= \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right) \\ &= \lim_{h \to 0}\left(\frac{c - c}{h}\right) \\ &= \lim_{h \to 0}(0) \\ &= 0. \end{aligned}$$

In the *operator notation* we can write this result,

$$\boxed{\frac{d}{dx}\big(\ c\ \big) = 0}$$

Here we think of the *operator* $\frac{d}{dx}$ acting on a constant to return zero.

2.) <u>*Derivative of identity function:*</u> $f(x) = x$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{x+h-x}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{h}{h} \right) \\ &= \lim_{h\to 0}(\ 1\ ) \\ &= 1. \end{aligned}$$

In the *operator notation* we can write this result,

$$\boxed{\frac{d}{dx}(\ x\ ) = 1}$$

Yet another way to write this result is that $\frac{dx}{dx} = 1$.

3.) <u>*Derivative of quadratic function:*</u> $f(x) = x^2$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{(x+h)^2-x^2}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{x^2+2xh+h^2-x^2}{h} \right) \\ &= \lim_{h\to 0}(\ 2x+h\ ) \\ &= 2x. \end{aligned}$$

In the *operator notation* we can write this result,

$$\boxed{\frac{d}{dx}(\ x^2\ ) = 2x}$$

4.) <u>*Derivative of cubic function:*</u> $f(x) = x^3$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\left(\frac{(x+h)^3-x^3}{h} \right) \\ &= \lim_{h\to 0}\left(\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} \right) \\ &= \lim_{h\to 0}(\ 3x^2+3xh+h^2\ ) \\ &= 3x^2. \end{aligned}$$

In the *operator notation* we can write this result,

$$\boxed{\frac{d}{dx}\left(\ x^3\ \right) = 3x^2}$$

Remark: we start to see a few patterns here. It would seem that the derivative always has a one power less than the function being differentiated. We can see a pattern if we examine the derivatives calculated thus far:

1.) $\frac{d}{dx}(1) = \frac{d}{dx}(x^0) = 0x^{0-1} = 0.$
2.) $\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1x^{1-1} = 1.$
3.) $\frac{d}{dx}(x^2) = 2x^{2-1} = 2x^1 = 2x.$
4.) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2.$

I bet most of you could guess that $\frac{d}{dx}(x^4) = 4x^3$ (and you would be correct). We can summarize:

$$\boxed{\frac{d}{dx}\left(\ x^n\ \right) = nx^{n-1}}$$

this is the so-called ***Power Rule***. I will give examples of how to apply this formula in addition to those we have seen so far, but first I owe you a proof of this fundamental rule. The proof I give now is for the case that $n \in \mathbb{N}$ so $n = 1, 2, 3, \ldots$. We begin by recalling the *binomial theorem,*

$$(x+h)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}h^k = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + h^n.$$

The symbol $\binom{n}{k} \equiv \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)\cdots 3\cdot 2\cdot 1}$ is read *"n choose k"* due to its application and interpretation in basic counting theory, they are also called the *"binomial coefficients"* . There is a neat thing called Pascal's triangle which allows you to calculate the binomial coefficients w/o using the formula.

$$\begin{aligned}
\frac{d}{dx}\left(\ x^n\ \right) &= \lim_{h\to 0}\left(\ \frac{(x+h)^n - x^n}{h}\ \right) \\
&= \lim_{h\to 0}\left(\ \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-1}h^2 + \cdots h^n - x^3}{h}\ \right) \\
&= \lim_{h\to 0}\left(\ nx^{n-1} + \frac{n(n-1)}{2}x^{n-1}h + \cdots h^{n-1}\ \right) \\
&= nx^{n-1}.
\end{aligned}$$

This proof is no good if $n = 1/2$, we have no binomial theorem in that case. We will learn in a later calculus course that the binomial expansion has infinitely many terms when $n \notin \mathbb{N}$. That said, the power rule is still true in the case that $n \notin \mathbb{N}$, we just need another method of proof. See § 4.10. I hope you will forgive me for using the power rule in the case $n \notin \mathbb{N}$, you'll just have to trust me for now.

Example 4.2.1: How to use the power rule. Most of this example is actually just a lesson in notation for power functions. In each case below we must express the function as x^n in order to apply the power rule

$$\frac{d}{dx}\left(\ \sqrt{x}\ \right) = \frac{d}{dx}(x^{0.5}) = \frac{1}{2}x^{-0.5} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$$
$$\frac{d}{dx}(\ 1/x\ ) = \frac{d}{dx}(x^{-1}) = -1x^{-2} = \frac{-1}{x^2}$$
$$\frac{d}{dx}\left(\ \sqrt[3]{x}\ \right) = \frac{d}{dx}(\ x^{1/3}\ ) = (1/3)x^{-2/3} = \frac{1}{3x^{2/3}}.$$

We should also be able to apply this rule when x is not the independent variable. For examples:

$$\frac{d}{dt}(\ ttt\ ) = \frac{d}{dt}(t^3) = 3t^2$$
$$\frac{d}{dc}(c) = 1$$
$$\frac{d}{d\mu}(\mu^k) = (k-1)\mu^{k-1}$$
$$\frac{d}{dy}(y^{\pi+2}) = (\pi+2)y^{\pi+1} \approx 5.142y^{4.142}$$

Naturally we most often choose either x or t are the independent variable, but we should be able to generalize the pattern of the product rule where appropriate.

4.3. LINEARITY OF THE DERIVATIVE

I use both the "d/dx" and the *"prime"* notations.

Proposition 4.2.1: The derivative d/dx is a *linear operator*. If $c \in \mathbb{R}$ and the functions f and g are differentiable then

$$\frac{d}{dx}(\ cf\ ) = c\frac{d}{dx}(\ f\ ) = c\frac{df}{dx}$$
$$\frac{d}{dx}(\ f+g\ ) = \frac{d}{dx}(\ f\ ) + \frac{d}{dx}(\ g\ ) = \frac{df}{dx} + \frac{dg}{dx}$$

We also can write $f'(x) = \frac{df}{dx}$

$$(cf)'(x) = cf'(x) \qquad (f+g)'(x) = f'(x) + g'(x).$$

The proof follows easily from the definition of the derivative.

$$\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0}\left(\frac{(f+g)(x+h)-(f+g)(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} \right) + \lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h} \right) \\
&= f'(x)+g'(x).
\end{aligned}$$

Likewise,

$$\begin{aligned}
(cf)'(x) &= \lim_{h\to 0}\left(\frac{(cf)(x+h)-(cf)(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\frac{cf(x+h)-cf(x)}{h} \right) \\
&= c\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} \right) \\
&= cf'(x).
\end{aligned}$$

While proofs may not excite you, I hope you can see that these are really very simple proofs. We didn't do anything except apply the properties of the limit itself (namely $\lim(f+g)=\lim f+\lim g$ and $\lim(cf)=c\lim f$) to the definition of the derivative for the functions $f+g$ and cf respective.

<u>Example 4.3.1:</u> Using the power rule with linearity

$$\frac{d}{dx}(x+x^2+3)=\frac{d}{dx}(x)+\frac{d}{dx}(x^2)+\frac{d}{dx}(3)=1+2x.$$

prove linearity works for three objects and I'll grant a bonus point.

$$\frac{d}{dt}(\ 3\sqrt{t}\ )=3\frac{d}{dt}(\ \sqrt{t}\ )=\frac{3}{2\sqrt{t}}.$$

And most often I will not show all my steps (but you should show your steps on the test, especially if I say justify each step)

$$\frac{d}{dx}(\ x^2+\frac{1}{3}x^3-\frac{1}{x}+1\ )=2x+x^2+\frac{1}{x^2}.$$

We will find other ways to do the next one later, but now algebra is our only hope.

$$\frac{d}{dx}(\ x(\ x-\sqrt{x}\ )\ )=\frac{d}{dx}(\ x^2-x^{1.5}\ )=2x-1.5\sqrt{x}.$$

Example 4.3.2: What is the slope of the line $y = mx + b$ at the point $(x_o, mx_o + b)$? Consider that,

$$\frac{d}{dx}(\ mx + b\ ) = m\frac{dx}{dx} + 0 = m.$$

We find that the slope of the function $f(x) = mx + b$ is the same at all points along the line, it is simply m. That is good news, it verifies that there is no disagreement between our new calculus based definition of the slope and the old standard definition we used in algebra and precalculus. Guess what the tangent line to the line is?

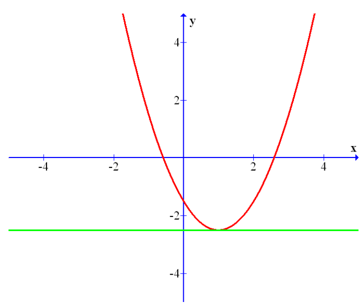
$$y = f(x_o) + f'(x_o)(x - x_o) = mx_o + b + m(x - x_o) = mx + b.$$

Of course graphically this is obvious, but it is nice to see the algebra works out.

Example 4.3.3: What is the slope of $y = f(x) = ax^2 + bx + c$ at the point $(x, ax^2 + bx + c)$? Lets calculate the derivative at x,

$$f'(x) = \frac{d}{dx}(\ ax^2 + bx + c\ ) = 2ax + b.$$

We see that a parabola will have different slopes at different points. Where is the slope zero ? Well we can just set $2ax + b = 0$ and solve to find $x = -b/2a$. If you are familiar with the formulas from algebra for the vertex of a parabola you'll recall that $h = -b/2a$ which makes a lot of sense. The vertex will have a horizontal tangent line.



What is the equation of the tangent line at x_o? The derivative at x_o is $f'(x_o) = 2ax_o + bx_o$. The equation of the tangent line is

$$\begin{aligned} y &= f(x_o) + f'(x_o)(x - x_o) \\ &= ax_o^2 + bx_o + c + (2ax_o + bx_o)(x - x_o). \end{aligned}$$

Why did I avoid asking you what the tangent line was at $(x, f(x))$? (subtle)

4.4. THE EXPONENTIAL FUNCTION

Transcendental numbers cannot be defined in terms of a solution to an algebraic equation. In contrast, you could say that $\sqrt{2}$ is not a transcendental number since it is a solution to $x^2 = 2$ (it turns out $\sqrt{2}$ has a finite expansion in terms of *continued fractions*, it is a *quadratic irrational*). Mathematicians have shown that there exist infinitely many transcendental numbers, but there are precious few that are familiar to us. Probably $\pi = 3.1415...$ is the most famous. Next in popularity to π we find the number $e = 2.718...$ named in honor of Euler. I can think of at least four seemingly distinct ways of defining $e = 2.718...$ We choose a definition which has the advantage of not using any mathematics beyond what we have so far discussed.

Let $f(x) = a^x$ for some $a > 0$. Lets calculate the derivative of this exponential function, we'll use this calculation to define e in a somewhat indirect manner.

$$\begin{aligned} \frac{d}{dx}(\ a^x\ ) &= \lim_{h \to 0}\left(\ \frac{a^{x+h} - a^x}{h}\ \right) \\ &= \lim_{h \to 0}\left(\ \frac{a^x a^h - a^x}{h}\ \right) \\ &= \lim_{h \to 0}\left(\ \frac{a^x(a^h - 1)}{h}\ \right) \\ &= a^x \lim_{h \to 0}\left(\ \frac{a^h - 1}{h}\ \right) \end{aligned}$$

We will learn that this limit is finite for any $a > 0$. Thus the derivative of an exponential function is proportional to the function itself. We can define $a = e$ to be the case where the derivative is equal to the function.

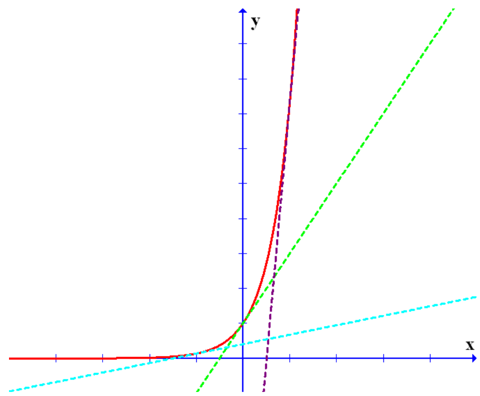
> *<u>Definition 4.4.1:</u>* The number e is the number such that
>
> $$\lim_{h \to 0}\left(\ \frac{e^h - 1}{h}\ \right) = 1.$$

I'll give you a bonus point if you can use the definition of derivative and the number e to calculate the precise value of $lim_{h \to 0}(\ \frac{a^h - 1}{h}\ )$. *This is a limit of type 0/0 but the solution is trickier than those limits we've done. Later we'll find an easier way to calculate the limit, but by then we'll also have found other tricks to calculate derivatives. As always bonus points are not required so if this all seems entirely opaque and/or obtuse feel free to turn the page.*

Then given that $\frac{d}{dx}(a^x) = a^x \lim_{h\to 0}(\frac{a^h-1}{h})$ and $\lim_{h\to 0}\frac{e^h-1}{h} \equiv 1$ we find

$$\boxed{\frac{d}{dx}(\ e^x\ ) = e^x.}$$

The exponential function $f(x) = e^x = f'(x)$ is a very special function, it has the unique property that its output is the same as the slope of its tangent line at that point. I have pictured a few representative tangents along with $y = e^x$.



By the way, I sometimes use the alternate notation $e^x = exp(x)$.

● *THE REMAINDER OF THIS SECTION IS TRIVIA. I WILL USE SOME IDEAS WE HAVE YET TO INTRODUCE, I JUST WANT THIS INFORMATION IN THE SAME PLACE AS A REFERENCE.*

1.) We could define e^x to be the function such that $\frac{d}{dx}(e^x) = e^x$ then the number *e* would be defined by the function: $e \equiv e^1$. This is essentially what we did in this section.

2.) The following limit is a more direct description of what the value of *e* is,

$$\boxed{e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n}$$

notice that this limit is type 1^∞ we have yet to discuss the tools to deal with such limits. Many folks take this as the definition of *e*, so be warned. It turns out that L'Hopital's Rule connects this definition and our definition.

3.) The natural logarithm $f(x) = \ln(x)$ arises in the study of integration in a very special role. You could define $f^{-1}(x) = e^x$ and then $e = f^{-1}(1)$.

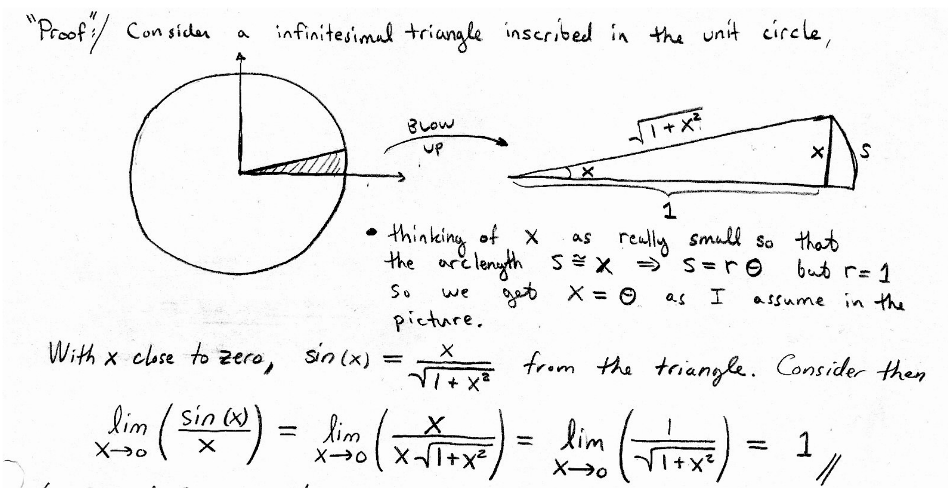
4.) The exponential could be defined by $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$ and again we could just set $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$, perhaps this is the easiest to find *e* since with just the terms listed we get $e = 1 + 1 + 0.5 + 0.1\bar{6} + \cdots \approx 2.66$ not too far off the real $e = 2.71...$ This definition probably raises more questions than it answers so we'll just leave it at that until we discuss Taylor series.

4.5. DERIVATIVES OF SINE AND COSINE

There are a few basic nontrivial limits which we need to derive in order to calculate the derivatives of sine and cosine. Most calculus books will show you some rather formal and elegant geometric proof. I instead give a heuristic proof since I think it gets to the heart of why these limits hold. You are of course welcome to look up other proofs if you find mine too common.

$$\boxed{\lim_{x \to 0}\left(\frac{\sin(x)}{x}\right) = 1}$$

The proof that follows is copied from my first ed. notes,



Next we show that,

$$\boxed{\lim_{x \to 0}\left(\frac{\cos(x)-1}{x}\right) = 0.}$$

Observe,

$$\begin{aligned}
\lim_{x \to 0}\left(\frac{\cos(x)-1}{x}\right) &= \lim_{x \to 0}\left(\frac{\cos(x)-1}{x}\cdot\frac{\cos(x)+1}{\cos(x)+1}\right) \\
&= \lim_{x \to 0}\left(\frac{\cos^2(x)-1}{x(\cos(x)+1)}\right) \\
&= \lim_{x \to 0}\left(\frac{-\sin^2(x)}{x(\cos(x)+1)}\right) \\
&= \lim_{x \to 0}\left(\frac{\sin(x)}{x}\right)\cdot\lim_{x \to 0}\left(\frac{-\sin(x)}{\cos(x)+1}\right) \\
&= 1\cdot\frac{-\sin(0)}{\cos(0)+1} \\
&= 0.
\end{aligned}$$

We now have all the tools we need to derive the derivatives of sine and cosine. I should mention that I assume you know the “adding angles” formulas for sine and cosine. If you are rusty you can take a look back in my notes where I show how to derive those trig. identities from the imaginary exponentials; just to be clear which identities I mean to use shortly :
$\sin(a \pm b) = \sin(a)\cos(b) \pm \sin(b)\cos(a)$, $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

$$\begin{aligned}
\frac{d}{dx}\big(\sin(x) \big) &= \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\sin(x)\cdot\frac{\cos(h)-1}{h} + \cos(x)\cdot\frac{\sin(h)}{h} \right) \\
&= \sin(x)\cdot\lim_{h\to 0}\left(\frac{\cos(h)-1}{h} \right) + \cos(x)\cdot\lim_{h\to 0}\left(\frac{\sin(h)}{h} \right) \\
&= \sin(x)\cdot 0 + \cos(x)\cdot 1 \\
&= \cos(x).
\end{aligned}$$

I think it is interesting that we had to use both of the limits we just found.

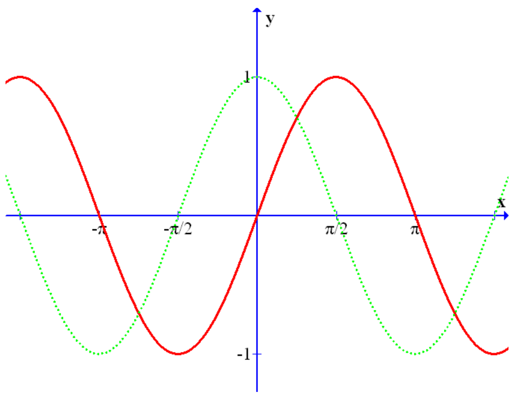
$$\begin{aligned}
\frac{d}{dx}\big(\cos(x) \big) &= \lim_{h\to 0}\left(\frac{\cos(x+h)-\cos(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\frac{\cos(x)\cos(h)-\sin(h)\sin(x)-\cos(x)}{h} \right) \\
&= \lim_{h\to 0}\left(\cos(x)\cdot\frac{\cos(h)-1}{h} - \sin(x)\cdot\frac{\sin(h)}{h} \right) \\
&= \cos(x)\cdot\lim_{h\to 0}\left(\frac{\cos(h)-1}{h} \right) - \sin(x)\cdot\lim_{h\to 0}\left(\frac{\sin(h)}{h} \right) \\
&= \cos(x)\cdot 0 - \sin(x)\cdot 1 \\
&= -\sin(x).
\end{aligned}$$

I think you will agree with me that these were harder to derive than the power rule. The neat thing is that armed with the few basic derivatives we have derived so far we will be able to differentiate just about anything once we learn a few more tools such as the *product, quotient and chain rules.* Barring the derivation of those rules this will be one of the last times we use the definition of the derivative to calculate a derivative. You see ultimately our goal is to calculate things without doing these tiresome limits. What I find really interesting is that after we get further into the subject we can make the limits disappear. Now, don’t misunderstand me here. The limiting concept is important. There are even certain applications where you don’t even have a formula for the function, all you have is raw data from some experiment. In those sort of cases you might need to apply the definition directly through some numerical methods. In this course we are mostly interested with those less interesting problems which allow pen and paper solutions. So-called analytic problems. Ok, enough philosophy of calculus, let's get back to work.

To summarize this section so far it's pretty simple,

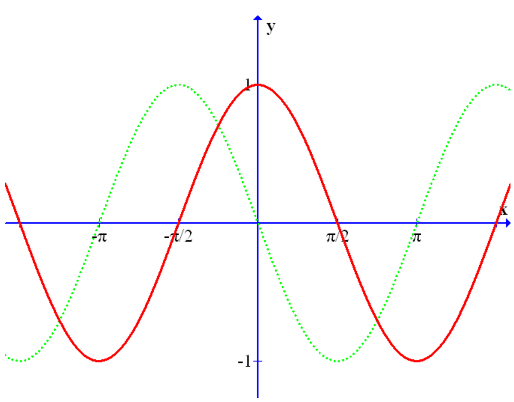
$$\frac{d}{dx}(\ \sin(x)\ ) = \cos(x) \qquad \frac{d}{dx}(\ \cos(x)\ ) = -\sin(x)$$

Let's examine how this plays out graphically,



I have graphed in red $y = \sin(x)$ and in green $y = \cos(x)$. Can you see that where the sine has a horizontal tangent the cosine function is zero? On the other hand whenever sine crosses the x-axis the cosine function is at either one or minus one. Question, what is the quickest that sine can possibly change? Notice that the slope of the sine function characterizes how quickly the sine function is changing.

The graph below has $y = \cos(x)$ in red and its derivative $y = -\sin(x)$ in green.



Can you see how the derivative and the function are related ?

4.6. PRODUCT RULE

It is often claimed by certain students that $\frac{d}{dx}(\ fg\ ) = \frac{df}{dx}\frac{dg}{dx}$ but this is almost never the case. Instead, you should use the ***product rule***.

> *Proposition 4.6.1:* Let f and g be differentiable functions then
>
> $$\frac{d}{dx}(\ fg\ ) = \frac{df}{dx}g + f\frac{dg}{dx}$$
>
> which can also be written $(fg)' = f'g + fg'$.

This is known as the *product rule* for derivatives. I owe you a proof of why this works, we will start with the definition of the derivative and then after a sneaky step or two we'll have it.

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0}\left(\ \frac{(fg)(x+h)-(fg)(x)}{h}\ \right) \\
&= \lim_{h\to 0}\left(\ \frac{f(x+h)g(x+h)-f(x)g(x)}{h}\ \right) \\
&= \lim_{h\to 0}\left(\ \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\ \right) \\
&= \lim_{h\to 0}\left(\ \left[\frac{f(x+h)-f(x)}{h}\right]\cdot g(x+h)\ +\ f(x)\cdot\left[\frac{g(x+h)-g(x)}{h}\right]\ \right) \\
&= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\cdot\lim_{h\to 0}(\ g(x+h)\ )\ +\ f(x)\cdot\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

I added zero in the third line, a very sneaky move. Then in the next to last step I pulled out $f(x)$ which is sensible since it does not depend on h. Then in the very last step I used that $\lim_{h\to 0}(\ g(x+h)\ ) = g(x)$ which is true since g is a continuous function. How do I know that g is continuous given that it is differentiable ? *That sounds like a good bonus point question.*

> *Example 4.6.1:* Lets derive the derivative of x^2 a new way,
>
> $$\frac{d}{dx}(x^2) = \frac{d}{dx}(xx) = \frac{dx}{dx}x + x\frac{dx}{dx} = 2x.$$
>
> We derived this fact from the definition before, I think this way is easier. Anyway, I always recommend knowing more than one way to understand a mathematical truth, it helps when doubt ensues.

Example 4.6.2: Identify that in the problem that follows $f(x) = x$ and $g(x) = e^x$ thus by the product rule,

$$\frac{d}{dx}(xe^x) = \frac{dx}{dx}e^x + x\frac{d(e^x)}{dx} = e^x + xe^x$$

Example 4.6.3: observe that $f(x) = \sin(x)$ and $g(x) = \cos(x)$ so by the product rule,

$$\begin{aligned}\frac{d}{dx}(\sin(x)\cos(x)) &= \frac{d(\sin(x))}{dx}\cos(x) + \sin(x)\frac{d(\cos(x))}{dx} \\ &= \cos^2(x) - \sin^2(x).\end{aligned}$$

You might wonder what happens if we have a product of three things, suppose that f, g, h are differentiable then,

$$\begin{aligned}\frac{d}{dx}(fgh) &= \frac{d(fg)}{dx}h + fg\frac{dh}{dx} \\ &= (\frac{df}{dx}g + f\frac{dg}{dx})h + fg\frac{dh}{dx} \\ &= \frac{df}{dx}gh + f\frac{dg}{dx}h + fg\frac{dh}{dx}\end{aligned}$$

so the rule for products of three things follows from the product rule for two things. You can derive $(fghj)' = f'ghj + fg'hj + fgh' + fghj'$ by the same logic.

Example 4.6.4:

$$\begin{aligned}\frac{d}{dx}(x^2\sin(x)e^x) &= \frac{d(x^2)}{dx}\sin(x)e^x + x^2\frac{d(\sin(x))}{dx}e^x + x^2\sin(x)\frac{d(e^x)}{dx} \\ &= 2x\sin(x)e^x + x^2\cos(x)e^x + x^2\sin(x)e^x\end{aligned}$$

Example 4.6.5: You can combine the product rule with linearity,

$$\begin{aligned}\frac{d}{dx}(\sqrt{x} + 3x^3e^x) &= \frac{d}{dx}(\sqrt{x}) + 3\frac{d}{dx}(x^3e^x) \\ &= \frac{1}{2\sqrt{x}} + 3(\frac{d(x^3)}{dx}e^x + x^3\frac{d(e^x)}{dx} \\ &= \frac{1}{2\sqrt{x}} + 9x^2e^x + x^3e^x\end{aligned}$$

the possibilities are endless.

4.7. QUOTIENT RULE

Proposition 4.7.1: let f, g be differentiable functions with $g \neq 0$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$$

this is called the ***quotient rule***. In prime notation; $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$.

This rule actually follows from the product rule. Let $Q(x) = f(x)/g(x)$ then since $g(x) \neq 0$ it follows that $f(x) = Q(x)g(x)$. That's a product so we can use the product rule; $f' = (Qg)' = Q'g + Qg'$. Solve this for Q',

$$Q' = \frac{f' - Qg'}{g} = \frac{f' - \frac{f}{g}g'}{g} = \frac{f'g - fg'}{g^2}.$$

Example 4.7.1: We already know the derivatives of sine and cosine, with the help of the quotient rule we can differentiate the tangent function.

$$\begin{aligned}
\frac{d}{dx}(\tan(x)) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \frac{\frac{d}{dx}(\sin(x))\cos(x) - \sin(x)\frac{d}{dx}(\cos(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x).
\end{aligned}$$

This is the *secant* function squared. I expect you to *remember* this derivative. You are of course free to derive it if you have time.

Example 4.7.2:

$$\frac{d}{dx}\left(\frac{x^3}{x^2+7}\right) = \frac{3x^2(x^2+7) - x^3(2x)}{(x^2+7)^2} = \frac{x^4 + 21x^2}{(x^2+7)^2}.$$

Example 4.7.3:

$$\frac{d}{dx}\left(\frac{1}{3x+5}\right) = \frac{0(3x+5) - 1(3)}{(3x+5)^2} = \frac{-3}{(3x+5)^2}.$$

Example 4.7.4: The reciprocal trigonometric functions' derivatives all follow from the quotient rule,

$$\begin{aligned}\frac{d}{dx}(\sec(x)) &= \frac{d}{dx}\left(\frac{1}{\cos(x)}\right)\\ &= \frac{\frac{d}{dx}(1)\cos(x) - 1\frac{d}{dx}(\cos(x))}{\cos^2(x)}\\ &= \frac{\sin(x)}{\cos^2(x)}\\ &= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}\\ &= \sec(x)\tan(x).\end{aligned}$$

Likewise the derivative of the *cosecant* follows from the quotient rule

$$\begin{aligned}\frac{d}{dx}(\csc(x)) &= \frac{d}{dx}\left(\frac{1}{\sin(x)}\right)\\ &= \frac{\frac{d}{dx}(1)\sin(x) - 1\frac{d}{dx}(\sin(x))}{\sin^2(x)}\\ &= \frac{-\cos(x)}{\sin^2(x)}\\ &= -\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)}\\ &= -\csc(x)\cot(x).\end{aligned}$$

Example 4.7.5: the quotient rule is used in conjunction with other rules sometimes, here I use linearity to start,

$$\begin{aligned}\frac{d}{dx}\left(e^x + \frac{x+x^2}{3-x}\right) &= \frac{d}{dx}(e^x) + \frac{d}{dx}\left(\frac{x+x^2}{3-x}\right)\\ &= e^x + \frac{\frac{d}{dx}(x+x^2)(3-x) - (x+x^2)\frac{d}{dx}(3-x)}{(3-x)^2}\\ &= e^x + \frac{(1+2x)(3-x) - (x+x^2)(-1)}{(3-x)^2}\\ &= e^x + \frac{3 - x + 6x - 2x^2 + x + x^2}{x^2 - 6x + 9}\\ &= e^x + \frac{3 + 6x - x^2}{x^2 - 6x + 9}\end{aligned}$$

The last couple lines were just algebraic simplification, the most important thing here was that you understood how the quotient rule was applied.

4.8. CHAIN RULE

If I were to pick a name for this rule it would be the *composite function rule* because the "chain rule" actually just tells us how to differentiate a composite function. Of all the rules so far this one probably requires the most practice. So be warned. Also, let me warn you about notation.

$$f'(x) = \frac{df}{dx} = \frac{df}{dx}(x) = \left.\frac{df}{dx}\right|_x$$

We have suppressed the (x) up to this point, reason being that it was always the same so we'd get tired of writing the (x) everywhere. Now we will find that we need to evaluate the derivative at things other than just (x). For example suppose that $f(x) = x^2$ so we have $f'(x) = 2x$ then

$$\frac{df}{dx}(x^3 + 7) = \left.\frac{df}{dx}\right|_{(x^3+7)} = 2(x^3 + 7)$$

We substituted $x^3 + 7$ in the place of x. I sometimes avoid the notation $\frac{df}{dx}(x)$ because it might be confused with multiplication by x. The difference should be clear from the context of the equation. Sometimes the substitution could be more abstract, again suppose $f(x) = x^2$ so we have $f'(x) = 2x$ then

$$\frac{df}{dx}(u) = \left.\frac{df}{dx}\right|_u = 2u$$

Proposition 4.8.1: The ***Chain Rule*** states that if $h = f \circ u$ is a composite function such that f is differentiable at $u(x)$ and u is differentiable at x then

$$\begin{aligned}\frac{d}{dx}(f \circ u) = (f \circ u)'(x) &= f'(u(x))u'(x) \\ &= \frac{df}{dx}(u(x))\frac{du}{dx} \\ &= \left.\frac{df}{dx}\right|_u \frac{du}{dx} \\ &= \frac{df}{du}\frac{du}{dx}\end{aligned}$$

In words, the derivative of a composite function is the product of the derivative of the outside function (f) evaluated at the inside function (u) with the derivative of the inside function.

Please don't worry too much about all the notation, you are free to just use one that you like (provided it is correct of course). Anyway, let's look at an example or two before I give a proof.

Example 4.8.1: consider $h(x) = (3x+7)^5$ we can identify that this is a composite function with inside function $u = 3x + 7$ and outside function $f(x) = x^5$.

$$\begin{aligned} \frac{d}{dx}(3x+7)^5 &= \frac{df}{dx}\bigg|_{3x+7}\frac{d}{dx}(3x+7) \\ &= 5x^4\bigg|_{3x+7}\cdot 3 \\ &= 15(3x+7)^4 \end{aligned}$$

I could also have written my work in the last example as follows,

$$\frac{d}{dx}(3x+7)^5 = \frac{d}{dx}(u^5) = 5u^4\frac{du}{dx} = 5(3x+7)^4\cdot 3 = 15(3x+7)^4.$$

Or you could even suppress the u notation all together and just write

$$\frac{d}{dx}(3x+7)^5 = 5(3x+7)^4\frac{d}{dx}(3x+7) = 15(3x+7)^4.$$

I just recommend writing at least one middle step, if you try to do it all at once in your head you are likely to miss something generally speaking.

Example 4.8.2:

$$\begin{aligned} \frac{d}{dx}(\sin(x^2)) &= \frac{d}{dx}(\sin(u)) \\ &= \cos(u)\frac{du}{dx} \\ &= \cos(x^2)\frac{d}{dx}(x^2) \\ &= 2x\cos(x^2). \end{aligned}$$

Example 4.8.3:

$$\begin{aligned} \frac{d}{dx}(exp(3x^2+x)) &= \frac{d}{dx}(exp(u)) \\ &= exp(u)\frac{du}{dx} \\ &= \exp(3x^2+x)\frac{d}{dx}(3x^2+x) \\ &= (6x+1)\exp(3x^2+x). \end{aligned}$$

Proof of the Chain Rule: The proof I give here relies on approximating the function by its tangent line, this is called the *linearization of the function*. Observe that $u'(x) = \lim_{h\to 0}(\frac{u(x+h)-u(x)}{h})$ and we can rewrite the l.h.s. in terms of a matching limit $u'(x) = \lim_{h\to 0}(\frac{u'(x)h}{h})$. Thus

$$\lim_{h\to 0}\left(\frac{u'(x)h}{h}\right) = \lim_{h\to 0}\left(\frac{u(x+h)-u(x)}{h}\right).$$

This shows that if $h \to 0$ then $u'(x)h \approx u(x+h) - u(x)$ which says that $u(x+h) \approx u(x) + u'(x)h$. We can make the same argument to show that $f(u+\delta) \approx f(u) + f'(u)\delta$ for small δ (*the* $\delta = u'(x)h$ *which is small in the argument below since* $u'(x)$ *is finite and* $h \to 0$). Consider then,

$$\begin{aligned}
\frac{d}{dx}(f\circ u) &= \lim_{h\to 0}\left(\frac{(f\circ u)(x+h)-(f\circ u)(x)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{f(u(x+h))-f(u(x))}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{f(u(x)+u'(x)h))-f(u(x))}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{f(u(x))+u'(x)hf'(u(x))-f(u(x))}{h}\right)\\
&= \lim_{h\to 0}\big(u'(x)f'(u(x))\big)\\
&= f'(u(x))u'(x).
\end{aligned}$$

So the proof of the chain rule relies on approximating both the inside and outside function by their tangent line. Let's get back to the examples.

Example 4.8.4:

$$\frac{d}{dx}(e^{\sqrt{x}}) = \frac{d}{dx}(e^u) = e^u\frac{du}{dx} = e^{\sqrt{x}}\frac{d}{dx}(\sqrt{x}) = e^{\sqrt{x}}\frac{1}{2\sqrt{x}}$$

Example 4.8.5: let a be a constant,

$$\frac{d}{dx}(\sin(ax)) = \frac{d}{dx}(\sin(u)) = \cos(u)\frac{du}{dx} = \cos(ax)\frac{d}{dx}(ax) = a\cos(ax)$$

Example 4.8.6: let a be a constant,

$$\frac{d}{dx}(e^{ax}) = \frac{d}{dx}(e^u) = e^u\frac{du}{dx} = e^{ax}\frac{d}{dx}(ax) = ae^{ax}$$

Example 4.8.7: let a be a constant,

$$\frac{d}{dx}(f(ax)) = \frac{d}{dx}(f(u)) = f'(u)\frac{du}{dx} = f'(ax)\frac{d}{dx}(ax) = af'(ax).$$

I let the function be arbitrary , it follows the same pattern as the last two examples. This is a common type of example.

I will neglect the extra u notation past this point unless I think it is helpful,

Example 4.8.8: let a, b, c be constants,

$$\begin{aligned} \frac{d}{dx}\left(\sqrt{ax^2 + bx + c} \right) &= \frac{1}{2\sqrt{ax^2+bx+c}} \cdot \frac{d}{dx}(ax^2 + bx + c) \\ &= \frac{2ax + b}{2\sqrt{ax^2+bx+c}} \end{aligned}$$

I admit that all the examples up to this point have been fairly mild. The remainder of the section I give examples which combine the chain rule with itself and the product or quotient rules.

Example 4.8.9:

$$\begin{aligned} \frac{d}{dx}\left(\sqrt{x^2 + \sqrt{x^2+3}} \right) &= \frac{1}{2\sqrt{x^2+\sqrt{x^2+3}}} \cdot \frac{d}{dx}\left(x^2 + \sqrt{x^2+3}\right) \\ &= \frac{1}{2\sqrt{x^2+\sqrt{x^2+3}}}\left(2x + \frac{1}{2\sqrt{x^2+3}}\frac{d}{dx}(x^2+3)\right) \\ &= \frac{1}{2\sqrt{x^2+\sqrt{x^2+3}}}\left(2x + \frac{x}{\sqrt{x^2+3}}\right) \end{aligned}$$

Example 4.8.10: let a, b, c be constants,

$$\begin{aligned} \frac{d}{dx}\big(\cos(a\sin(bx+c)) \big) &= -\sin(a\sin(bx+c)) \cdot \frac{d}{dx}\big(a\sin(bx+c)\big) \\ &= -\sin(a\sin(bx+c)) \cdot a\cos(bx+c)\frac{d}{dx}(bx+c) \\ &= -ab\sin(a\sin(bx+c))\cos(bx+c) \end{aligned}$$

We have to work outside in, one step at a time. Both of these examples followed the pattern $(f \circ g \circ h)(x) = f(g(h(x)))$ which has the derivative $(f \circ g \circ h)'(x) = f'(g(h(x)))g'(h(x))h'(x)$. Of course, in practice I do not try to remember that formula, I just apply the chain rule repeatedly until the problem boils down to basic derivatives.

Example 4.8.11:

$$\frac{d}{dx}(x^3e^{2x}\cos(x^2)) = \frac{d}{dx}(x^3)e^{2x}\cos(x^2) + x^3\frac{d}{dx}(e^{2x})\cos(x^2) + x^3e^{2x}\frac{d}{dx}(\cos(x^2))$$
$$= 3x^2e^{2x}\cos(x^2) + x^3e^{2x}\frac{d(2x)}{dx}\cos(x^2) + x^3e^{2x}(-\sin(x^2)\frac{d(x^2)}{dx})$$
$$= 3x^2e^{2x}\cos(x^2) + 2x^3e^{2x}\cos(x^2) - 2x^4e^{2x}\sin(x^2)$$

And we can rearrange this expression using $\sin(x^2) = 1 - \cos^2(x^2)$.

$$\frac{d}{dx}(x^3e^{2x}\cos(x^2)) = x^2e^{2x}\bigg(\cos(x^2)[3 + 2x + 2x^2] - 2x^2\bigg)$$

Example 4.8.12:

$$\frac{d}{dx}(e^xx^2)^3 = 3(e^xx^2)^2\frac{d}{dx}(e^xx^2)$$
$$= 3(e^xx^2)^2\left(\frac{d(e^x)}{dx}x^2 + e^x\frac{d(x^2)}{dx}\right)$$
$$= 3(e^xx^2)^2\left(x^2e^x + 2xe^x\right)$$

The better way to think about this one is that $(e^xx^2)^3 = e^{3x}x^6$ then the differentiation is prettier in my opinion,

$$\frac{d}{dx}(e^{3x}x^6) = \frac{d(e^{3x})}{dx}x^6 + e^{3x}\frac{d(x^6)}{dx}$$
$$= 3e^{3x}x^6 + 6x^5e^{3x}$$

Can you see that these answers are the same?

Example 4.8.13:

$$\frac{d}{d\theta}\left(\frac{\sin(3\theta)}{\sqrt{\theta+4}}\right) = \frac{3\cos(3\theta)\sqrt{\theta+4} - \sin(3\theta)\frac{1}{2\sqrt{\theta+4}}}{(\sqrt{\theta+4})^2}$$
$$= \frac{3\cos(3\theta)\sqrt{\theta+4}\sqrt{\theta+4} - \sin(3\theta)\frac{\sqrt{\theta+4}}{2\sqrt{\theta+4}}}{(\sqrt{\theta+4})^3}$$
$$= \frac{6(\theta+4)\cos(3\theta) - \sin(3\theta)}{2(\theta+4)^{\frac{3}{2}}}$$

Example 4.8.14: observe we can find the power rule from the product rule.

$$\frac{d}{dx}(x^n) = \frac{d}{dx}(xx\cdots x) = \frac{dx}{dx}x^{n-1} + x\frac{dx}{dx}x^{n-2} + \cdots + x^{n-1}\frac{dx}{dx}$$
$$= x^{n-1} + x^{n-1} + \cdots + x^{n-1}$$
$$= nx^{n-1}.$$

Example 4.8.15:

$$\begin{aligned}\frac{d}{dt}\big(\sin(\sqrt{2t-1})\big) &= \cos(\sqrt{2t-1})\frac{d(\sqrt{2t-1})}{dt} \\ &= \cos(\sqrt{2t-1})\frac{1}{2\sqrt{2t-1}}\frac{d(2t-1)}{dt} \\ &= \frac{\cos(\sqrt{2t-1})}{\sqrt{2t-1}}\end{aligned}$$

Example 4.8.16:

$$\begin{aligned}\frac{d}{dt}(t^2\cos(\sin(t)) &= 2t\cos(\sin(t)) + t^2\big(-\sin(\sin(t))\frac{d}{dt}(\sin(t))\big) \\ &= 2t\cos(\sin(t)) - t^2\sin(\sin(t))\cos(t)\end{aligned}$$

In most of the examples we have been able to reduce the answer into some expression involving no derivatives. This is generally not the case. As the next couple of examples illustrate, we can have expressions that once differentiated yield a new expressions which still contain derivatives.

Example 4.8.17: Suppose that c and f are functions of t then,

$$\frac{d}{dt}\big(cf\big) = \frac{dc}{dt}f + c\frac{df}{dt}$$

Notice that if c is a constant then $\frac{dc}{dt} = 0$ so in that case we have that $\frac{d}{dt}\big(cf\big) = c\frac{df}{dt}$.

Example 4.8.18: Suppose that a particle travels on a circle of radius R centered at the origin. The particle has coordinates (x, y) that satisfy the equation of a circle; $x^2 + y^2 = R^2$. Moreover, both x and y are functions of time t. What can we say about dx/dt and dy/dt ?

$$\frac{d}{dt}\big(x^2 + y^2\big) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Notice since the radius R is constant it follows that R^2 is also constant thus $\frac{d}{dt}(R^2) = 0$. Apparently the derivatives dx/dt and dy/dt must satisfy

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Now this says that $\frac{dx}{dt} = \frac{-y}{x}\frac{dy}{dt}$ (for points with $x \neq 0$). The position vector is $\vec{r} = (x, y)$ and velocity vector is $\vec{v} = (\frac{dx}{dt}, \frac{dy}{dt})$. The *dot-product* is

$$\vec{r}\cdot\vec{v} = (x, y)\cdot(\frac{dx}{dt}, \frac{dy}{dt}) = x\frac{dx}{dt} + y\frac{dy}{dt}$$

We will learn that when $\vec{r}\cdot\vec{v} = 0$ the vectors $\vec{r}$ and $\vec{v}$ are perpendicular. So the equation we found involving dx/dt and dy/dt expresses that particles traveling in a circle have velocity which is tangent to the circle. (Tangents to a circle meet radial vectors at right angles)

4.9. IMPLICIT DIFFERENTIATION

Up to this point we have primarily dealt with expressions where it is convenient to just differentiate what we are given directly. We just wrote down our $f(x)$ and proceeded with the tools at our disposal, namely linearity, the product, quotient and chain rules. For the most part this direct approach will work, but there are problems which are best met with a slightly indirect approach. We call the thing we want to find y then we'll differentiate some equation which characterizes y and typically we get an equation which *implicitly* yields dy/dx. This technique will reward us with the formulas for the derivatives of all sorts of inverse functions. Before we get to the inverse functions let's start with a few typical implicit derivatives.

Example 4.9.1: Observe that the equation $x^2 + y^3 = e^y$ implicitly defines y as a function of x. Let's find $\frac{dy}{dx}$. Differentiate the given equation on both sides.

$$\frac{d}{dx}(x^2 + y^3) = \frac{d}{dx}(e^y)$$

now differentiate and use the chain rule where appropriate,

$$2x + 3y^2\frac{dy}{dx} = e^y\frac{dy}{dx}$$

Now solve for $\frac{dy}{dx}$,

$$(e^y - 3y^2)\frac{dy}{dx} = 2x \quad \Longrightarrow \quad \boxed{\frac{dy}{dx} = \frac{2x}{e^y - 3y^2}}$$

Notice that this equation is a little unusual in that the derivative involves both x and y.

Example 4.9.2: Observe that the equation $xy + \sin(x) = e^{xy}$ implicitly defines y as a function of x. Let's find $\frac{dy}{dx}$.

$$\frac{d}{dx}\big(xy + \sin(x)\big) = \frac{d}{dx}(e^{xy})$$

$$\Longrightarrow \quad \frac{dx}{dx}y + x\frac{dy}{dx} + \cos(x) = e^{xy}\frac{d}{dx}(xy)$$

$$\Longrightarrow \quad y + x\frac{dy}{dx} + \cos(x) = e^{xy}\left(y + x\frac{dy}{dx}\right)$$

Now solve for $\frac{dy}{dx}$,

$$y + \cos(x) - ye^{xy} = (xe^{xy} - x)\frac{dy}{dx} \quad \Longrightarrow \quad \boxed{\frac{dy}{dx} = \frac{y + \cos(x) - ye^{xy}}{xe^{xy} - x}}.$$

You might question why such differentiation is interesting. One good reason is that it is what we use to solve related rates problems.

Example 4.9.3: Suppose that we know the radius of a spherical hot air balloon is expanding at a rate of 1 meter per minute due to an inflating fan. At what rate is the volume increasing if the radius R is at 10 meters ? To begin we need to recall that the volume V is related to the radius according to $V = \frac{4\pi}{3}R^3$ for sphere. Then,

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4\pi}{3}R^3\right) = 4\pi R^2 \frac{dR}{dt} = 4\pi(10m)^2\frac{m}{min} \approx 1200\frac{m^3}{min}.$$

We'll do more of these in a later section.

So I hope you get the idea about these sort of problems. I'm going to shift back to the other type of problem that implicit differentiation is great for. That is the problem of calculating the inverse function's derivative. We know the derivatives of $e^x, \cos(x), \sin(x), \tan(x), \sec(x)$. I will now systematically derive the derivatives of $\ln(x), \cos^{-1}(x), \sin^{-1}(x), \tan^{-1}(x), \sec^{-1}(x)$ using essentially the same technique every time.

Example 4.9.4: let $y = \ln(x)$ we wish to find $\frac{d}{dx}(\ln(x))$. To begin we take the exponential of both sides of $y = \ln(x)$ to obtain

$$e^y = e^{\ln(x)} = x$$

Now differentiate with respect to x and solve for $\frac{dy}{dx}$

$$e^y\frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{e^y}$$

Now remember that we found $e^y = x$ so we have shown that

$$\boxed{\frac{d}{dx}\big(\ \ln(x)\ \big) = \frac{1}{x}}$$

Example 4.9.5: let $y = \cos^{-1}(x)$ we wish to find $\frac{d}{dx}(\cos^{-1}(x))$. To begin we take the cosine of both sides of $y = \cos^{-1}(x)$ to obtain

$$\cos(y) = \cos(\cos^{-1}(x)) = x$$

Now differentiate with respect to x and solve for $\frac{dy}{dx}$

$$-\sin(y)\frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin(y)}$$

Now $\sin^2(y) + \cos^2(y) = 1$ thus $\sin(y) = \sqrt{1-\cos^2(y)}$ but remember that we found $\cos(y) = x$ so $\sin(y) = \sqrt{1-x^2}$ thus we find

$$\boxed{\frac{d}{dx}\big(\ \cos^{-1}(x)\ \big) = \frac{-1}{\sqrt{1-x^2}}}$$

Example 4.9.6: let $y = \sin^{-1}(x)$ we wish to find $\frac{d}{dx}(\sin^{-1}(x))$. To begin we take the sine of both sides of $y = \sin^{-1}(x)$ to obtain

$$\sin(y) = \sin(\sin^{-1}(x)) = x$$

Now differentiate with respect to x and solve for $\frac{dy}{dx}$

$$\cos(y)\frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)}$$

Now $\sin^2(y) + \cos^2(y) = 1$ thus $\cos(y) = \sqrt{1 - \sin^2(y)}$ but remember that we found $\sin(y) = x$ so $\cos(y) = \sqrt{1 - x^2}$ thus we find

$$\boxed{\frac{d}{dx}\left(\ \sin^{-1}(x)\ \right) = \frac{1}{\sqrt{1 - x^2}}}$$

Example 4.9.7: let $y = \tan^{-1}(x)$ we wish to find $\frac{d}{dx}(\tan^{-1}(x))$. To begin we take the sine of both sides of $y = \tan^{-1}(x)$ to obtain

$$\tan(y) = \tan(\tan^{-1}(x)) = x$$

Now differentiate with respect to x and solve for $\frac{dy}{dx}$

$$\sec^2(y)\frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

Now $\sin^2(y) + \cos^2(y) = 1$ thus if we divide this equation by $\cos^2(y)$ we'll obtain the less familiar identity $\tan^2(y) + 1 = \sec^2(y)$. But we know that in this example $\tan(y) = x$ hence $\sec^2(y) = 1 + x^2$. To conclude,

$$\boxed{\frac{d}{dx}\left(\ \tan^{-1}(x)\ \right) = \frac{1}{1 + x^2}}$$

Example 4.9.8: let $y = \sec^{-1}(x)$ we wish to find $\frac{d}{dx}(\sec^{-1}(x))$. To begin we take the sine of both sides of $y = \sec^{-1}(x)$ to obtain

$$\sec(y) = \sec(\sec^{-1}(x)) = x$$

Now differentiate with respect to x and solve for $\frac{dy}{dx}$

$$\sec(y)\tan(y)\frac{dy}{dx} = 1 \quad \Longrightarrow \quad \frac{dy}{dx} = \frac{1}{\sec(y)\tan(y)}$$

The identity $\tan^2(y) + 1 = \sec^2(y)$ tells us that $\tan y = \sqrt{\sec^2(y) - 1}$. But we know that in this example $\sec(y) = x$ hence $\tan y = \sqrt{x^2 - 1}$. Thus,

$$\boxed{\frac{d}{dx}\left(\ \sec^{-1}(x)\ \right) = \frac{1}{x\sqrt{x^2 - 1}}}$$

I hope you can see the pattern in the last five examples. To find the derivative of an inverse function we simply need to know the derivative of the function plus a little algebra. The same technique would allow us to derive the derivatives of $\cosh^{-1}(x), \sinh^{-1}(x), \tanh^{-1}(x), \csc^{-1}(x), \cot^{-1}(x)$. I have not included those in these notes because we have yet to calculate the derivatives of $\cosh(x), \sinh(x), \tanh(x), \csc(x), \cot(x)$. Hmmm… maybe I'll ask those on the test. (hauntingly maniacal laugh follows here). The next examples follow the same general idea, but the pattern differs a bit.

__Example 4.9.9:__ Suppose that $y = a^x$ we have yet to calculate the derivative of this for arbitrary $a > 0$ except the one case $a = e$. Turns out that this one case will dictate what the rest follow. Take the natural log of both sides to obtain $\ln(y) = \ln(a^x) = x\ln(a)$. Now differentiate,

$$\frac{d}{dx}\big(\ln(y)\big) = \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x\ln(a)) = \ln(a).$$

we just used Example 4.9.4 to differentiate the $\ln(y)$. Now solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \ln(a)y = \ln(a)a^x \quad \Longrightarrow \quad \boxed{\frac{d}{dx}(a^x) = \ln(a)a^x}$$

I should mention that I know another method to derive the boxed equation. In fact I prefer the following method which is based on a useful purely algebraic trick: $a^x = exp(\ln(a^x)) = exp(x\ln(a))$ so we can just calculate

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln(a)}) = e^{x\ln(a)}\frac{d(x\ln(a))}{dx} = e^{x\ln(a)}\ln(a) = \ln(a)a^x.$$

but beware the sneaky step, how did I know to insert the $exp \circ \ln$? I just did.

__Example 4.9.10:__ Suppose that $y = x^x$. This is not a function we have encountered before. It is neither a power nor an exponential function, it's sort of both. I'll admit the only place I've seen them is on calculus tests. Anyway to begin we take the natural log of both sides; $\ln(y) = \ln(x^x) = x\ln(x)$. Differentiate w.r.t x,

$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + x\frac{1}{x} \quad \Longrightarrow \quad \frac{dy}{dx} = y(\ln(x) + 1)$$

Therefore we find,

$$\boxed{\frac{d}{dx}(x^x) = (\ln(x) + 1)x^x}$$

If you have a problem with an unpleasant exponent it sometimes pays off take the logarithm. It may change the problem to something you can deal with. The process of morphing an unsolvable problem to one which is solvable through known methods is most of what we do in calculus. We learn a few basic tools then we spend most of our time trying to twist other problems back to those simple cases. I have one more basic derivative to address in this section.

Example 4.9.11: Let $y = \log_a(x)$ we can exponentiate both sides w.r.t. base a which cancels the $\log_a$ in the sense $a^{\log_a(x)} = x$,

$$a^y = x \implies \ln(a)a^y\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\ln(a)a^y}$$

But then since $a^y = x$ therefore we conclude,

$$\boxed{\frac{d}{dx}\big(\log_a(x)\big) = \frac{1}{\ln(a)x}}$$

Notice in the case $a = e$ we have $\log_e(x) = \ln(x)$ and $\ln(e) = 1$. So this result agrees with Example 4.9.4.

At this point I have derived almost every elementary function's derivative. Those which I have not calculated so far can certainly be calculated using nothing more than the strategies and methods advertised thus far.

4.10. LOGARITHMIC DIFFERENTIATION

The idea of logarithmic differentiation is fairly simple. When confronted with a product of bunch of things one can take the logarithm to convert it to a sum of things. Then you get to differentiate a sum rather than a product. This is a labor saving device.

Example 4.10.1: Find the derivative of $y = xe^{x^2+9}\sqrt{3x+7}$ using logarithmic differentiation. Take the natural log to begin,

$$\begin{aligned}\ln(y) = \ln(xe^{x^2+9}\sqrt{3x+7}) &= \ln(x) + \ln(e^{x^2+9}) + \ln(\sqrt{3x+7}) \\ &= \ln(x) + x^2 + 9 + \frac{1}{2}\ln(3x+7)\end{aligned}$$

We used the properties of the natural log to simplify as best we could before going on to the next step: differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2x + \frac{3}{2(3x+7)}$$

$$\implies \boxed{\frac{dy}{dx} = xe^{x^2+9}\sqrt{3x+7}\left(\frac{1}{x} + 2x + \frac{3}{2(3x+7)}\right)}.$$

This is much easier than the 3-term product rule for this problem.

Example 4.10.2: Find $\frac{dy}{dx}$ via logarithmic differentiation. Let.

$$y = \left(\frac{1}{2-x}\right)(x+32)^{\frac{1}{4}}(x^2-3)^4$$

Take the natural log to begin,

$$\begin{aligned}\ln(y) &= \ln(2-x)^{-1} + \ln(x+32)^{\frac{1}{4}} + \ln(x^2-3)^4 \\ &= -\ln(2-x) + \frac{1}{4}\ln(x+32) + 4\ln(x^2-3)\end{aligned}$$

We used the properties of the natural log to simplify as best we could before going on to the next step: differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{2-x} + \frac{1}{4(x+32)} + \frac{4(2x)}{x^2-3}$$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \left(\frac{1}{2-x}\right)(x+32)^{\frac{1}{4}}(x^2-3)^4\left(\frac{-1}{2-x} + \frac{1}{4(x+32)} + \frac{8x}{x^2-3}\right)}.$$

Example 4.10.3: Let a, b, c be constants. Differentiate y.

$$y = \left(\frac{1}{x-a}\right)\left(\frac{1}{x-b}\right)^2\left(\frac{1}{x-c}\right)^3$$

Take the natural log to begin,

$$\ln(y) = -\ln(x-a) - 2\ln(x-b) - 3\ln(x-c)$$

We used the properties of the natural log to simplify as best we could before going on to the next step: differentiate w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{x-a} - \frac{2}{x-b} - \frac{3}{x-c}$$

$$\Longrightarrow \boxed{\frac{dy}{dx} = \left(\frac{1}{x-a}\right)\left(\frac{1}{x-b}\right)^2\left(\frac{1}{x-c}\right)^3\left(\frac{-1}{x-a} - \frac{2}{x-b} - \frac{3}{x-c}\right)}.$$

Example 4.10.4: Differentiate y.

$$y = (x^2+1)(x-3)^2(x^3+x)^3(x-1)^4$$

Take the natural log to begin,

$$\begin{aligned}\ln(y) &= \ln(x^2+1) + 2\ln(x-3) + 3\ln(x^3+x) + 4\ln(x-1) \\ \Longrightarrow \quad \frac{1}{y}\frac{dy}{dx} &= \frac{2x}{x^2+1} + \frac{2}{x-3} + \frac{3(3x^2+1)}{x^3+x} + \frac{4}{x-1}\end{aligned}$$

$$\Longrightarrow \boxed{\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{2}{x-3} + \frac{3(3x^2+1)}{x^3+x} + \frac{4}{x-1}\right)}.$$

Example 4.10.5: Sometimes we might have a $\ln$ to start with, but the same algebraic wisdom applies, simplify products to sums then differentiate. Find $\frac{dy}{dx}$ for $y = \ln(\frac{\sin(x)\sqrt{x}}{x^2+3x-2})$.

$$\begin{aligned} y &= \ln(\frac{\sin(x)\sqrt{x}}{x^2+3x-2}) \\ &= \ln(\sin(x)) + \frac{1}{2}\ln(x) - \ln(x^2+3x-2) \end{aligned}$$

Now differentiate w.r.t. x and we're done.

$$\frac{dy}{dx} = \frac{\cos(x)}{\sin(x)} + \frac{1}{2x} - \frac{2x+3}{x^2+3x-2}.$$

Example 4.10.6: What about

$$y = \ln((x+1)^{30} + 2)$$

We **cannot** simplify this one because we do not have a product inside the natural log. Just differentiate w.r.t x

$$\frac{dy}{dx} = \frac{1}{(x+1)^{30}+2}\frac{d}{dx}\left((x+1)^{30}+2\right) = \frac{30(x+1)^{29}}{(x+1)^{30}+2}$$

I wish there was some nice simple formula to break apart $\ln(A+B)$ but as far as I know $\ln(A+B) = ?$, I mean that there is no simple formula to split it up. On the other hand we have seen that $\ln(AB) = \ln(A) + \ln(B)$ is an extremely useful property when used together with $\ln(A^c) = c\ln(A)$.

Proof of the Power Rule for any power:

Let $y = x^n$ take the natural log to obtain $\ln(y) = \ln(x^n) = n\ln(x)$. Differentiate,

$$\frac{1}{y}\frac{dy}{dx} = \frac{n}{x} \implies \frac{dy}{dx} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}.$$

This proof (in contrast to our earlier proof) works in the case that $n \notin \mathbb{N}$. Somehow these curious little logarithms have circumvented the whole binomial theorem. We conclude that for any $n \in \mathbb{R}$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}.}$$

4.11. ODDS AND ENDS

This section is mostly optional. I wanted to try a few new things. Also I have avoided a few of the hyperbolic functions, I'll take care of them here as well. Let's do that first. Recall by *definition* have that

$$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$$

So the derivatives are easy to calculate,

$$\frac{d}{dx}\big(\cosh(x)\big) = \frac{1}{2}\left(\frac{d(e^x)}{dx} + \frac{d(e^{-x})}{dx}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \sinh(x).$$
$$\frac{d}{dx}\big(\sinh(x)\big) = \frac{1}{2}\left(\frac{d(e^x)}{dx} - \frac{d(e^{-x})}{dx}\right) = \frac{1}{2}\left(e^x + e^{-x}\right) = \cosh(x).$$

In contrast to the cosine derivative in the usual case there is no minus sign here. I should mention that if we know about how to see sine and cosine in terms of imaginary exponentials then there is a similar calculation we can do to find the derivatives of sine and cosine. Logically this may be bit circular for most folks who inadvertently use the derivatives of sine and cosine to validate Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ (where $i = \sqrt{-1}$). I don't particularly care at this juncture which is the chicken and which is the egg, the point is that the calculations that follow are *consistent*. Given Euler's formula we can show that $e^{ix} + e^{-ix} = 2\cos(x)$ while $e^{ix} - e^{-ix} = 2i\sin(x)$. Consequently,

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

Then we assume that the derivatives of imaginary exponentials work the same as the derivative of real exponentials that means we ought to have the formulas $\frac{d}{dx}(e^{ix}) = ie^{ix}$ and $\frac{d}{dx}(e^{-ix}) = -ie^{ix}$. (*technically, we should go the other direction, the **known** derivatives of sine and cosine go towards proving that* $d/dx(e^x) = \lambda e^{\lambda x}$ for $\lambda = a + ib \in \mathbb{C}$). Hence,

$$\frac{d}{dx}\big(\cos(x)\big) = \frac{1}{2}\left(\frac{d(e^{ix})}{dx} + \frac{d(e^{-ix})}{dx}\right) = \frac{1}{2}\left(ie^{ix} - ie^{-ix}\right) = -\sin(x).$$
$$\frac{d}{dx}\big(\sin(x)\big) = \frac{1}{2i}\left(\frac{d(e^{ix})}{dx} - \frac{d(e^{-ix})}{dx}\right) = \frac{1}{2i}\left(ie^{ix} + ie^{-ix}\right) = \cos(x).$$

I made use of the imaginary arithmetic $i = -1/i$ which is a straight-forward consequence of the basic identity $i^2 = -1$. Personally I find the similarity of the hyperbolic and ordinary trigonometric function a fascinating analogy. There are many interrelations; clearly $\cosh(ix) = \cos(x)$ and $\sinh(ix) = i\sin(x)$.

Calculations we can do for the sine and cosine will have corresponding calculations that work for sinh and cosh. The bigger lesson here is that the distinction between sine, cosine and the exponential function is blurred as we transition to the complex case. This I suppose is not too surprising if we just think about the way algebra works over real numbers verses complex numbers. If we have a polynomial and we look for roots of the polynomial that are real numbers we may or may not be successful. Generally speaking there will be a product of linear factors which correspond to real roots and then a bunch of irreducible quadratic factors which correspond to complex roots. So there are different types of factors over the real numbers. In contrast once we allow complex roots then we can factor any polynomial into a product of linear factors so all the roots look the same from the complex perspective. In the same sense the distinction between exponentials and sines and cosines vanishes as we allow complex exponentials to enter the scene. Let me give an example of the factorization since my comments above may be needlessly opaque to you at the present (it's not really a hard idea)

$$\begin{aligned} P(x) &= (x-2)(x^2+9) \text{ factorization of } P \text{ over } \mathbb{R} \\ P(x) &= (x-2)(x+3i)(x-3i) \text{ factorization of } P \text{ over } \mathbb{C} \end{aligned}$$

Let's go on and think a little more about how the laws of exponents can tell us all sorts of things about trig. identities. It would sure be nice if

$$e^{iA}e^{iB} = e^{i(A+B)}$$

was true. So, let's assume that is the case. (not very good math logic, but hey I'm trying to show you consistency in this section so don't be too disappointed in my shallow logic). Ok, let's insert sines and cosines and see what we get,

$$\begin{aligned} e^{iA}e^{iB} &= (\cos(A) + i\sin(A))(\cos(B) + i\sin(B)) \\ &= \cos(A)\cos(B) - \sin(A)\sin(B) + i(\sin(A)\cos(B) + \sin(B)\sin(A)) \end{aligned}$$

On the other hand for the r.h.s. we observe,

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B).$$

The real and imaginary parts of the equations above have to be independently equal. So we can equate these two expressions and find

$$\begin{aligned} \cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\ \sin(A+B) &= \sin(A)\cos(B) + \sin(B)\sin(A) \end{aligned}$$

Technically, the fact that these "adding angles" trig identities hold true is the core of the proof that the complex exponential works, it is not too difficult to derive these identities from a few pictures and some basic trigonometry. Perhaps I have assigned you such a homework problem. I would argue that it is much easier to recover these formulas from the complex exponential in the event you forget them.

We can also derive things like the half-angle formulas without much trouble

$$\begin{aligned}\cos^2(x) &= \left(\frac{1}{2}(e^{ix} + e^{-ix})\right)^2 = \frac{1}{4}(e^{2ix} + 2 + e^{-2ix}) \\ &= \frac{1}{2}(1 + \frac{1}{2}(e^{2ix} + e^{-2ix})) = \frac{1}{2}(1 + \cos(2x)).\end{aligned}$$

So what? The point is that if you can just remember that $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$ then almost any trig. identity can be derived in a fashion similar to the examples I just gave on the last page. I suppose I should admit that the real justification for the complex exponentials working as they do is that all of these trigonometric identities can also be derived using other arguments. So from a foundational viewpoint I have put the cart before the horse. My point to you is that these various interrelationships I have explored in this section can be terribly useful, perhaps a good complex variables course would go through these arguments in their proper order.

Now let's think a little more about trig. identities. What can we learn from differentiating trig. identities? Will we learn new identities as a consequence ? Let's try the Pythagorean identity for sine and cosine,

$$\frac{d}{dx}(\cos^2(x) + \sin^2(x)) = \frac{d}{dx}(1) \implies 2\cos(x)(-\sin(x)) + 2\sin(x)\cos(x) = 0.$$

Well ok zero equals zero. True, but not particularly enlightening. Let's try the half-angle formula,

$$\frac{d}{dx}(\cos^2(x)) = \frac{d}{dx}(\frac{1}{2}(1 + \cos(2x)) \implies -2\cos(x)\sin(x) = -2\sin(2x)$$

Ah ha, look what we just found, one of my favorite trigonometric identities. Its used in the derivation of a pretty formula for the range of a parabolic trajectory;

$$\boxed{\sin(2x) = \sin(x)\cos(x)}.$$

One last experiment, suppose we knew just one of the adding angles formulas,

$$\sin(x + a) = \sin(x)\cos(a) + \sin(c)\cos(x)$$

It's fairly obvious that if we take a to be a constant and differentiate with respect to x the we will obtain the other adding angles formula,

$$\cos(x + a) = \cos(x)\cos(a) - \sin(c)\sin(x)$$

where the minus came from differentiating cosine. What's the point of all of this ? Simply this, the more ways you have to understand something the harder it is to forget anything. This is my personal philosophy of calculus, I want to know not just one solution, I want to know a whole arsenal of solutions for a given problem. Then when I'm faced with a new problem I have the advantage of attacking it by a number of angles. Consistency is a powerful companion in mathematics, it can get you out of a lot of corners. You just have to think outside the box a little.

4.12. KNOWN DERIVATIVES

I collect all the basic derivatives for future reference. I do expect that you memorize the derivatives of $x^n, e^x, \cos(x), \sin(x)$. The rest of these you should probably be able to derive or remember as the context suggests. If the derivative of $\tan(x)$ is just a little part of a bigger problem then later in the course it is customary to just write down that the derivative is $\sec^2(x)$. But, if I ask for the derivative of tangent as a stand-alone problem then I probably intend for you to go through how we get $\frac{d}{dx}(\tan(x)) = \sec^2(x)$ from the def$\underline{\text{n}}$ of the tangent function and the quotient rule. If you are uncertain of the level of detail I wish to see then please ask me ***before*** the test is finished.

$f(x)$	$\frac{df}{dx}$	Comments about $f(x)$	Formulas I use
c	0	constant function	
x	1	line $y = x$ has slope 1	
x^2	$2x$		
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$		
x^n	nx^{n-1}	power rule	
e^x	e^x	the exponential	
5^x	$\ln(5)5^x$		
a^x	$\ln(a)a^x$	an exponential	
$\ln(x)$	$\frac{1}{x}$	the natural log	$\ln(e^x) = x$, $e^{\ln(x)} = x$
$\log x$	$\frac{1}{\ln(10)x}$	log base 10	
$\log_a(x)$	$\frac{1}{\ln(a)x}$	log base a	$\log_a(a^x) = x$, $a^{\log_a(x)} = x$
$\sin(x)$	$\cos(x)$		$\sin^2(x) + \cos^2(x) = 1$
$\cos(x)$	$-\sin(x)$		
$\tan(x)$	$\sec^2(x)$		$\tan^2(x) + 1 = \sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$	reciprocal of cosine	$\sec(x) = 1/\cos(x)$
$\cot(x)$	$-\csc^2(x)$	reciprocal of tangent	$\cot(x) = \cos(x)/\sin(x)$
$\csc(x)$	$-\csc(x)\cot(x)$	reciprocal of sine	$\csc(x) = 1/\sin(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$	inverse sine	$\sin(\sin^{-1}(x)) = x$
$\cos^{-1}(x)$	$\frac{-1}{\sqrt{1-x^2}}$	inverse cosine	$\cos^{-1}(\cos(x)) = x$
$\tan^{-1}(x)$	$\frac{1}{x^2+1}$	inverse tangent	
$\sinh(x)$	$\cosh(x)$	hyperbolic sine	$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
$\cosh(x)$	$\sinh(x)$	hyperbolic cosine	$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
$\tanh(x)$	$sech^2(x)$	hyperbolic tangent	
$\sinh^{-1}(x)$	$\frac{1}{\sqrt{1+x^2}}$	inverse sinh	
$\cosh^{-1}(x)$	$\frac{1}{\sqrt{x^2-1}}$	inverse cosh	
$\tanh^{-1}(x)$	$\frac{1}{1-x^2}$	inverse tanh	

The formulas given in the table are not exhaustive. I know of many other useful formulas for these basic functions. You may consult Chapter 2 for more of those details. Finally, let us conclude this chapter with a list of useful rules of differentiation. These in conjunction with the basic derivatives we listed earlier in this section will allow us to differentiate almost anything you can imagine. (*this is quite a contrast to integration as we shall shortly discover)*

name of property	operator notation	prime notation
Linearity	$\frac{d}{dx}[f+g] = \frac{d}{dx}[f] + \frac{d}{dx}[g]$ $\frac{d}{dx}[cf] = c\frac{d}{dx}[f]$	$(f+g)' = f' + g'$ $(cf)' = cf'$
Product Rule	$\frac{d}{dx}[fg] = \frac{df}{dx}g + f\frac{dg}{dx}$	$(fg)' = f'g + fg'$
Quotient Rule	$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{\frac{df}{dx}g - f\frac{dg}{dx}}{g^2}$	$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$
Chain Rule	$\frac{d}{dx}[f \circ u] = \frac{df}{du}\frac{du}{dx}$	$(f \circ u)'(x) = f'(u(x))u'(x)$

Beyond these basic properties we have seen in this chapter that the technique of implicit differentiation helps extend these simple rules to cover the inverse functions. It all goes back to the definition logically speaking, but it is comforting to see that once we have established the derivatives of the basic functions and these properties we have little need of applying the definition directly. I would argue this is part of what separates modern (say the last 400 years) mathematics from ancient mathematics. We have no need to calculate limits by some exhaustive numerical method. Instead, for a wealth of examples, we can find tangents through what are essentially algebraic calculations. This is an amazing simplification. However, more recent times have shown computers can model problems which defy algebraic description. We truly have many options in present-day mathematics.