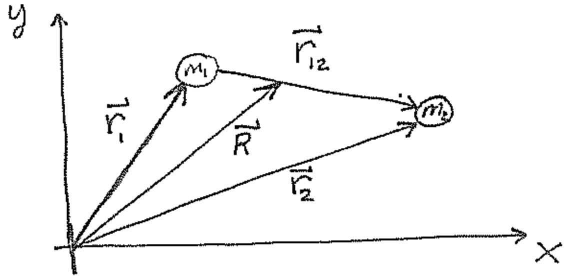
Central Force Problem



m_1 at $\vec{r}_1$

m_2 at $\vec{r}_2$

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = \vec{r}$$

$$\vec{R} = \frac{1}{m_1+m_2}\left(m_1\vec{r}_1 + m_2\vec{r}_2\right)$$

We suppose the only force is an interaction whose strength depends only on the distance $|\vec{r}_{12}| = |\vec{r}_2 - \vec{r}_1|$.

<u>Define</u> $\vec{r} = \vec{r}_2 - \vec{r}_1$, the relative coordinate. We have two fundamental transformations:

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad\qquad \vec{r}_1 = \vec{R} - \frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{R} = \frac{m_1}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}_2 \qquad\qquad \vec{r}_2 = \vec{R} + \frac{m_1}{m_1+m_2}\vec{r}$$

We also define velocities of the com (center of mass) $\vec{R}$ and relative coordinate $\vec{r}$,

$$\vec{V} = \frac{d\vec{R}}{dt} \qquad\qquad \vec{V}_1 = \frac{d\vec{r}_1}{dt} = \frac{d\vec{R}}{dt} - \frac{m_2}{m_1+m_2}\frac{d\vec{r}}{dt} = \vec{V} - \frac{m_2}{m_1+m_2}\vec{v}$$

$$\vec{v} = \frac{d\vec{r}}{dt} \qquad\qquad \vec{V}_2 = \frac{d\vec{r}_2}{dt} = \vec{V} + \frac{m_1}{m_1+m_2}\vec{v}$$

We compute the Lagrangian in the com/relative coordinate system as follows,

$$L = \frac{1}{2}m_1\vec{V}_1\cdot\vec{V}_1 + \frac{1}{2}m_2\vec{V}_2\cdot\vec{V}_2 - U(|\vec{r}_2 - \vec{r}_1|)$$

$$= \frac{1}{2}m_1\left(\vec{V} - \frac{m_2}{m_1+m_2}\vec{v}\right)\cdot\left(\vec{V} - \frac{m_2}{m_1+m_2}\vec{v}\right) + \frac{1}{2}m_2\left(\vec{V} + \frac{m_1}{m_1+m_2}\vec{v}\right)\cdot\left(\vec{V} + \frac{m_1}{m_1+m_2}\vec{v}\right) - U(r)$$

$$= \frac{1}{2}(m_1+m_2)\vec{V}\cdot\vec{V} + \frac{1}{2}\frac{m_1m_2^2 + m_2m_1^2}{(m_1+m_2)^2}\vec{v}\cdot\vec{v} - U(r)$$

$$= \frac{1}{2}(m_1+m_2)V^2 + \frac{1}{2}\left(\frac{m_1m_2}{m_1+m_2}\right)v^2 - U(r).$$

Continuing, the Lagrangian in the $\vec{r}, \vec{R}$ coordinates,

$$L = \frac{1}{2}(m_1+m_2)V^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)v^2 - U(r)$$

$$\Rightarrow \boxed{L = \frac{1}{2}MV^2 + \frac{1}{2}\mu v^2 - U(r)}$$

Where we define:

$M = m_1+m_2$ the total mass

$\mu = \dfrac{m_1 m_2}{m_1+m_2}$ the reduced mass

A typical application is $m_1 = M_{SUN}$, $m_2 = M_{EARTH}$. In this case $\mu = \dfrac{M_E M_S}{M_E+M_S} \approx \dfrac{M_E M_S}{M_S} = M_E$. It is approximately true to say $\vec{R} = \vec{r}_{sun}$, the com is close to center of sun. However, the math here allows for an <u>exact</u> description of the motion. We need not assume the sun is the com.

<u>Equations of Motion</u>:

$\vec{R}$: Let $\vec{R} = (X, Y, Z)$ then using $L = \frac{1}{2}M(\dot{X}^2+\dot{Y}^2+\dot{Z}^2) + \frac{1}{2}\mu(\dot{x}^2+\dot{y}^2+\dot{z}^2) - U(r)$

letting $\vec{r} = (x,y,z)$.

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) &= \frac{\partial L}{\partial X} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Y}}\right) &= \frac{\partial L}{\partial Y} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Z}}\right) &= \frac{\partial L}{\partial Z} \end{aligned}\right\} \longrightarrow M\ddot{X} = M\ddot{Y} = M\ddot{Z} = 0.$$

$\hookrightarrow$ the center of mass moves with constant velocity motion.

$$\boxed{\vec{R}(t) = (X_o, Y_o, Z_o) + t(V_{xo}, V_{yo}, V_{zo})}$$

<u>Remark</u>: we've proved previously that if $\vec{F}$ is directed along $\vec{r}$ then the resulting motion falls onto a particular plane. Without loss of generality we suppose $z = 0$ and the motion is described by just x & y. Thus,

$$L = \frac{1}{2}M(\dot{X}^2+\dot{Y}^2+\dot{Z}^2) + \frac{1}{2}\mu(\dot{x}^2+\dot{y}^2) - U(r)$$

Continuing: (we choose $z=0$ for clarity of notation)

We introduce polar coordinates for the relative coordinate,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Then,

$$L = \frac{1}{2}M(\dot{X}^2+\dot{Y}^2+\dot{Z}^2) + \frac{1}{2}\mu(\dot{r}^2+r^2\dot{\theta}^2) - U(r)$$

We find

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow \underline{\mu\ddot{r} = -\frac{\partial U}{\partial r} + \mu r\dot{\theta}^2}.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}(\underbrace{\mu r^2\dot{\theta}}_{\text{conserved quantity!}}) = 0$$

(mechanists call θ an "ignorable" coordinate)

(angular momentum $l = \mu r^2\dot{\theta} = \text{const.}$)

The radial eq$^{\underline{n}}$ simplifies now that we know $\dot{\theta} = \frac{l}{\mu r^2}$,

$$\mu\ddot{r} = \frac{-\partial U}{\partial r} + \mu r\left(\frac{l}{\mu r^2}\right)^2$$

$$\Rightarrow \underline{\mu\frac{d^2r}{dt^2} = -\frac{d}{dr}\left[U + \frac{l^2}{2\mu r^2}\right]}.$$

Multiply by $\frac{dr}{dt}$ to obtain,

$$\mu\frac{dr}{dt}\frac{d^2r}{dt^2} + \frac{dr}{dt}\frac{d}{dr}\left[U + \frac{l^2}{2\mu r^2}\right] = 0$$

$$\frac{d}{dt}\left[\underbrace{\frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{l^2}{2\mu r^2}}_{\text{conserved quantity!}}\right] = 0$$

(energy $E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{U(r) + \frac{l^2}{2\mu r^2}}$

The effective potential is used to lump fictitious forces into the same category as the real potential U. We can analyze as if V_{eff} is an ordinary potential function. (we ignore θ)

$$\boxed{V_{eff} = U(r) + \frac{l^2}{2\mu r^2}}$$

Equations of Motion for Central Force Problem

$$\left.\begin{aligned} l &= \mu r^2 \dot{\theta} \\ E &= \frac{1}{2}\mu \dot{r}^2 + U(r) + \frac{l^2}{2\mu r^2} \end{aligned}\right\}$$

conserved quantities for central force motion where $F = -\frac{dU}{dr}$

We have two, fairly simple differential equations which are possible to solve once we specify a choice of $U(r)$.

$$\boxed{\begin{aligned} \frac{d\theta}{dt} &= \frac{l}{\mu r^2} \\ \frac{dr}{dt} &= \pm\sqrt{E - U - \frac{l^2}{2\mu r^2}}\sqrt{\frac{2}{\mu}} \end{aligned}}$$

Orbit Equation:

It turns out solving for r as fnct of θ is better replaced with problem of finding $u = \frac{1}{r}$ as fnct. of θ.

Consider, $\mu \ddot{r} - \frac{l}{\mu r^3} + \frac{dU}{dr} = 0$ where $\frac{d\theta}{dt} = \frac{l}{\mu r^2}$

Chain-rule reveals,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{l}{\mu r^2}$$

$$\Rightarrow \frac{d^2 r}{dt^2} = \frac{d}{dt}\left[\frac{l}{\mu r^2}\frac{dr}{d\theta}\right] = \frac{d}{d\theta}\left[\frac{l}{\mu r^2}\frac{dr}{d\theta}\right]\frac{d\theta}{dt} = \underbrace{\frac{l}{\mu r^2}\frac{d}{d\theta}\left[\frac{l}{\mu r^2}\frac{dr}{d\theta}\right]}_{\ddot{r}}$$

Hence,

$$\mu \frac{l}{\mu r^2}\frac{d}{d\theta}\left[\frac{l}{\mu r^2}\frac{dr}{d\theta}\right] - \frac{l}{\mu r^3} + \frac{dU}{dr} = 0$$

$$\frac{l^2}{\mu r^2}\underbrace{\frac{d}{d\theta}\left[\frac{1}{r^2}\frac{dr}{d\theta}\right]} - \frac{l}{\mu r^3} = -\frac{dU}{dr}$$

$$\frac{-l^2}{\mu r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) - \frac{l}{\mu r^3} = F(r)$$

$$\Rightarrow \underline{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{-\mu r^2}{l^2}F(r)}$$

Orbit Equation Continued

We found

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{-\mu r^2}{l^2} F(r)$$

$$\hookrightarrow \boxed{\frac{d^2 u}{d\theta^2} + u = \frac{-\mu}{u^2 l^2} F(1/u) \qquad \text{for } u = \frac{1}{r}}$$

orbit equation

Remark: I have several pages of examples based on this equation if you look in my Physics 412 notes..

Example: Newton's Universal Law of Gravitation.

Suppose $F = \frac{-Gm_1m_2}{r^2}$ write $F = \frac{-k}{r^2}$ where $k = Gm_1m_2$

Note, $F\left(\frac{1}{u}\right) = \frac{-k}{(1/u)^2} = -ku^2$ hence the orbit equation takes form:

$$\underbrace{\frac{d^2u}{d\theta^2} + u = \frac{k\mu}{l^2}}_{\text{well-known ordinary differential equation.}} \leftarrow \text{constant of motion.}$$

the sol$^{\underline{n}}$ is simply,

$$u = u_o\cos(\theta + \theta_o) + u_p$$

Note $u_p = A \Rightarrow u_p' = 0 \dashrightarrow u_p = k\mu/l^2$. therefore,

$$u = u_o\cos(\theta + \theta_o) + \frac{k\mu}{l^2} = \frac{1}{r}$$

$$\Rightarrow \boxed{r = \frac{1}{u_o\cos(\theta+\theta_o) + k\mu/l^2}}$$

this is the equation of a conic section in polar coordinates. to make it pretty it's nice to use E as a constant in the sol$^{\underline{n}}$. It's a page or three to settle notation. Ask if interested.