Sequences and Series

Freitag pushes us to think about this now. I'll try to keep it brief and relegate some review to your homework. Crucial pts.

- A sequence is a function from $\mathbb{N}$ to something. (or $U \subseteq \mathbb{Z}$ such that U has a least member and the immediate successor property.)

  a complex sequence can be written as

  $$\{z_n\}_{n=n_0}^{\infty} = \{z_{n_0}, z_{n_0+1}, \dots\}$$

- The limit of a complex sequence is defined with the modulus in the same way as a real sequence with absolute-value.

  Def^n/ We say $\lim_{n\to\infty} z_n = L$ or $z_n \to L$ as $n \to \infty$ iff for each $\epsilon > 0$, $\exists N \in \mathbb{Z}$ such that $n > N \Rightarrow |z_n - L| < \epsilon$.

- A series of complex numbers is defined just as in the real case. The series $z_0 + z_1 + \dots = \sum_{k=0}^{\infty} z_k$ converges iff the sequence of partial sums

  $$\{z_0, z_0 + z_1, z_0 + z_1 + z_2, \dots\} = \left\{\sum_{k=0}^{n} z_k\right\}_{n=1}^{\infty}$$

  converges. Otherwise the series is said to diverge.

- A series $\sum_{k=0}^{\infty} z_n$ is absolutely convergent iff $\sum_{k=0}^{\infty} |z_n|$ converges. (again, just as in $\mathbb{R}$-case. Many of same Th^{m}'s persist.)

Remark: I realize most students are a little rusty on conv/div. theory from calculus II. Moreover, proofs of the conv. tests are not usually given in the standard calculus II course. In the interest of solidifying your foundations I wrote a summary of the theory of convergence/div. theory based on the excellent treatment of Apostol. That is a bit of a detour, so I avoid including it here. Instead we assume the theory on $\mathbb{R}$ is known to the reader. (Math 431 will dig into that)

Th$^{\underline{m}}$/ Suppose $z_n \to z$ and $w_n \to w$ as $n \to \infty$.

1.) $c \to c$ as $n \to \infty$ aka $\lim_{n\to\infty} (c) = c$.

2.) $z_n + w_n \to z + w$ as $n \to \infty$,

3.) if $c \in \mathbb{C}$ then $cz_n \to cz$ as $n \to \infty$,

4.) $z_n - w_n \to z - w$ as $n \to \infty$,

5.) $z_n w_n \to zw$ as $n \to \infty$,

6.) if $w \neq 0$ then $z_n/w_n \to z/w$ as $n \to \infty$.

Proof: 1.) is not hard! Let $\epsilon > 0$ and choose $N = 1$ if $n \geq 1$ then $|c - c| = 0 < \epsilon$ hence $\lim_{n\to\infty} c = c$.

Continuing, for 2.) suppose $\epsilon > 0$ and choose $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow |z_n - z| < \epsilon/2$ whereas $n \geq N_2 \Rightarrow |w_n - w| < \epsilon/2$. Observe $N = \max(N_1, N_2)$ imposes both constraints at once. If $n \in \mathbb{N}$ and $n \geq N$ then note:

$$
\begin{aligned}
|z_n + w_n - z - w| &\leq |z_n - z| + |w_n - w| \qquad \Delta\text{-ineq.} \\
&\leq \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}
$$

Therefore, $z_n + w_n \to z + w$ as $n \to \infty$. Continue

To prove 3, if $c=0$ then $cz_n = 0$ thus by 1.) we're done. Assume $c \neq 0$ and let $\epsilon > 0$. Suppose $N \in \mathbb{N}$ such that $n \geq N \Rightarrow |z_n - z| < \frac{\epsilon}{|c|}$. Consider, if $n \geq N$ then

$$\begin{aligned} |cz_n - cz| &= |c(z_n - z)| \\ &= |c|\,|z_n - z| \\ &< |c|\left(\epsilon/|c|\right) < \epsilon. \end{aligned}$$

Thus 3. follows.

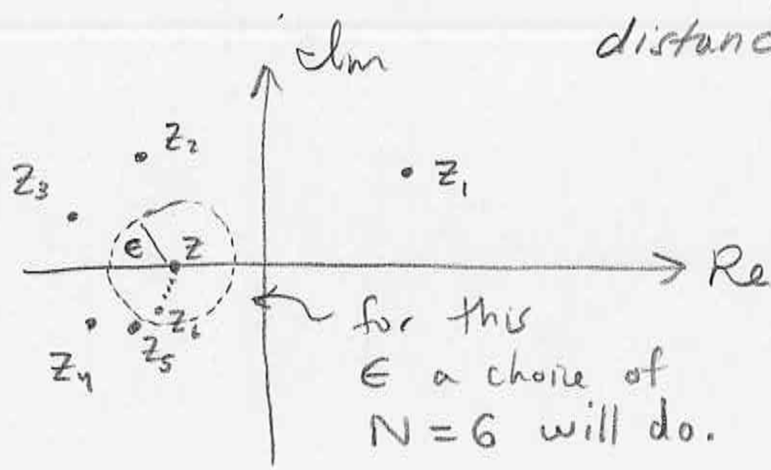
To prove 4. we use the fact $z_n - w_n = z_n + (-1 \cdot w_n)$ paired with results 2. and 3. I'll use the lim notation:

$$\begin{aligned} \lim_{n\to\infty}(z_n - w_n) &= \lim_{n\to\infty}(z_n) + \lim_{n\to\infty}(-w_n) \quad \text{by 2.} \\ &= \lim_{n\to\infty}(z_n) - \lim_{n\to\infty}(w_n) \quad \text{by 3, } (c=-1) \\ &= z - w. \end{aligned}$$

Proofs of 5. and 6. are a bit more subtle. I'll leave them for a bonus homework problem. //

Geometry of $z_n \to z$

For each $\epsilon > 0$ we need to be able to choose $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \underbrace{|z_n - z| < \epsilon}_{\text{distance } d(z_n, z) < \epsilon}$.



Proofs are a bit tricky since we need to argue for arbitrary $\epsilon > 0$.

The theorem below is also true for $\mathbb{R}^n$.

$$\boxed{\text{Th}^{m}/ \ \lim_{n\to\infty}(u_n + i v_n) = a+ib \iff \lim_{n\to\infty} u_n = a \text{ and } \lim_{n\to\infty} v_n = b.}$$

(we intend u_n, v_n, a, b to denote $\mathbb{R}$-values)

<u>Proof</u>: ($\Rightarrow$) Assume $u_n + i v_n \longrightarrow a+ib$. Observe that:

$$|u_n - a| \leq |u_n - a + i(v_n - b)| \quad \text{I}$$

$$|v_n - b| = |i(v_n - b)| \leq |i(v_n - b) + u_n - a| \quad \text{II}$$

Therefore, both $|u_n - a|$ and $|v_n - b|$ are controlled by $|u_n + i v_n - (a+ib)|$ which may be made as small as we wish by the given data. To be explicit, if $\epsilon > 0$ choose $N \in \mathbb{N}$ s.t. $n \geq N$ $\Rightarrow |u_n + i v_n - (a+ib)| < \epsilon$. It follows by I & II that $|u_n - a| < \epsilon$ and $|v_n - b| < \epsilon$ hence $u_n \longrightarrow a$ and $v_n \longrightarrow b$ as $n \longrightarrow \infty$.

($\Leftarrow$) Assume $u_n \longrightarrow a$ and $v_n \longrightarrow b$ as $n \longrightarrow \infty$. Suppose $\epsilon > 0$ and choose $N \in \mathbb{N}$ s.t. $n \geq N$ $\Rightarrow |u_n - a| < \epsilon/2$ and $|v_n - b| < \epsilon/2$.

Consider,

$$\begin{aligned} |u_n + i v_n - (a+ib)| &= |u_n - a + i(v_n - b)| \\ &\leq |u_n - a| + |i(v_n - b)| \\ &= |u_n - a| + |v_n - b||i| \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Therefore, $u_n + i v_n \longrightarrow a+ib$ as $n \longrightarrow \infty$. //

<u>Remark</u>: the proof for $\mathbb{R}^n$ is very much the same. See my Math 332 notes or Edwards' Advanced Calc.

Proposition: If $\lim_{n\to\infty} z_n = z$ then

① $\lim_{n\to\infty} \bar{z}_n = \bar{z}$

② $\lim_{n\to\infty} |z_n| = |z|$

Proof: for ① let $z_n = u_n + iv_n$ and $z = a+ib$ and by the Th^{m} on ㉔ we find, $u_n \to a$ and $v_n \to b$

also, $\lim_{n\to\infty}(u_n - iv_n) = \lim_{n\to\infty} u_n - i\lim_{n\to\infty} v_n = a - ib$.

by the Th^{m} on ㉒ (ok I should add $\alpha z_n + \beta w_n \to \alpha z + \beta w$ to that Th^{m} to make it complete)

We shown $\lim_{n\to\infty} \bar{z}_n = a - ib = \bar{z}$ thus ① is true.

I leave the proof of ② for your homework. //

Prop. If $|z| < 1$ then $\lim_{n\to\infty} |z|^n = 0$

Proof: Let $z = re^{i\theta}$ then $|z| = |r|$ hence
$\lim_{n\to\infty} |z|^n = \lim_{n\to\infty} r^n = 0$ as $|r| < 1$. //

Remark: to show $r^n \to 0$ as $n\to\infty$ we can use the bounded monotonic sequence is convergent Th^{m}. Suppose $0 < r < 1$.
$\{r, r^2, r^3, \dots\}$ has $r^n - r^{n+1} = r^n(1-r) \geq 0$
thus $r^{n+1} \leq r^n$. Moreover $0 < r^n < 1$ hence the sequence is bounded and decreasing $\therefore r^n \to x \in \mathbb{R}$. To find x note

$$rx = r\lim_{n\to\infty} r^n = \lim_{n\to\infty}(r^{n+1}) = \lim_{n\to\infty}(r^n) = x$$

thus $rx = x \Rightarrow (r-1)x = 0$ where $r \neq 1 \Rightarrow x = 0$.

Absolute Convergence, an important concept

$\boxed{\text{Def}^{\underline{n}} \text{ (again) } \sum_{k=0}^{\infty} z_k \text{ is absolutely convergent if the corresponding series } \sum_{k=0}^{\infty} |z_k| \text{ converges.}}$

The same definition is given in real case where modulus is replaced with absolute value. The property of absolute convergence allows <u>rearrangement</u>. In contrast, <u>conditionally convergent</u> series converge but are not absolutely convergent. Riemann proved that conditionally convergent series can be rearranged to converge to <u>any</u> value! Apostol has a nice outline of Riemann's argument centered on the identity $\sum_{k=1}^{n} \frac{1}{k} \leq \log(n+1)$.

$\boxed{\text{Th}^{\underline{m}} / \text{ If } \sum_{k=0}^{\infty} z_n \text{ is absolutely convergent then } \sum_{k=0}^{\infty} z_n \text{ converges.}}$

<u>Proof</u>: Absolute convergence $\Rightarrow \lim_{n\to\infty} \sum_{k=0}^{n} |z_k| = L$.

Let $z_n = x_n + i y_n$ to see that we're given $\sum_{k=0}^{n} \sqrt{x_n^2 + y_n^2} \longrightarrow L$ as $n \longrightarrow \infty$. Consider $\sum_{k=0}^{\infty} x_n$.

we claim $\sum_{k=0}^{\infty} |x_n|$ converges. Note that

$$\sqrt{x_n^2 + y_n^2} \geq \sqrt{x_n^2} = |x_n|$$

thus, by comparison $\text{Th}^{\underline{m}}$ it follows $\sum_{k=0}^{\infty} |x_n|$ converges. Likewise for $\sum_{k=0}^{\infty} |y_n|$. But, absolute convergence in $\mathbb{R}$ implies convergence thus $\sum x_n$ and $\sum y_n$ converge. By $\text{Th}^{\underline{m}}$ on (24) it follows $\sum (x_n + i y_n)$ converges. //

<u>Remark</u>: I'm certain this proof can be improved!

Geometric Series:

Suppose $|z| < 1$ and consider

$$1 + z + z^2 + \cdots = \sum_{k=0}^{\infty} z^k$$

Let us calculate the n^{th} partial sum, I'll use the standard trick,

$$S_n = 1 + z + z^2 + \cdots + z^{n-2} + z^{n-1}$$

$$z S_n = z + z^2 + \cdots\cdots + z^{n-1} + z^n$$

Thus, $(1-z) S_n = 1 - z^n$ as the other terms cancel.

Observe $|z| < 1 \Rightarrow z \neq 1$ hence $1 - z \neq 0$ and we can solve for $S_n = \dfrac{1-z^n}{1-z}$. Consider,

$$\lim_{n\to\infty} \left| \frac{z^n}{1-z} \right| = \lim_{n\to\infty} \frac{|z|^n}{|1-z|} = \frac{\lim_{n\to\infty} |z|^n}{|1-z|} = \frac{0}{|1-z|} = 0.$$

Thus, $\lim_{n\to\infty}\left(\dfrac{z^n}{1-z}\right) = 0$ by <u>Lemma</u> in homework ☺.

$\boxed{\text{If } \lim_{n\to\infty} |w_n| = 0 \text{ then } \lim_{n\to\infty} w_n = 0.}$

It follows that

$$\lim_{n\to\infty} S_n = \frac{1}{1-z} - \underbrace{\cancel{\lim_{n\to\infty}\left(\frac{z^n}{1-z}\right)}}_{0} = \frac{1}{1-z}.$$

$$\boxed{\text{Th}^{m}/\ \ 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{provided } |z| < 1}$$

Notice we can easily extend this result for $a \in \mathbb{C}$

$$\boxed{a + az + az^2 + \cdots = \sum_{n=0}^{\infty} az^n = \frac{a}{1-z}}$$

<u>Remark</u>: this result is most of why I'm discussing series at this juncture. We'll use it <u>dozens</u> of times.

How to define e^z, $\cos z$, $\sin z$ with series techniques:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

these are Freitag's definitions. We took a different approach.

To see these are well-defined one should check their convergence. Notice that

$$\sum_{n=0}^{\infty} \left| \frac{z^n}{n!} \right| = \sum_{n=0}^{\infty} \frac{|z|^n}{n!} = e^{|z|}$$

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(2n+1)!} z^{2n+1} \right| = \sin(|z|)$$

$$\sum_{n=0}^{\infty} \left| \frac{(-1)^n}{(2n)!} z^{2n} \right| = \cos(|z|)$$

Using calculus II where we argued by ratio test these real series converged absolutely on $\mathbb{R}$.

We may return to this discussion (pg. 26 – 27) after Test 1 when we study series sol$^{\underline{n}}$'s in greater depth. For now, let's adopt the def$^{\underline{n}}$ given in these notes as primary so we need not prove the Cauchy Multiplication Th$^{\underline{m}}$ until later. (no sequences / series on Test 1 our focus will be elsewhere.)