Theory of differentiation for $\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m$

We treat this in-depth in Advanced Calculus. We're mostly concerned with $n = m = 2$ as $\mathbb{C} = \mathbb{R}^2$. However, I give a few generalities for context,

$$F = (F^1, F^2, \dots, F^m)$$

$$F^j : \text{dom}(F) \subseteq \mathbb{R}^n \longrightarrow \mathbb{R} \quad \text{for } j = 1, 2, \dots, m$$

When F is differentiable at $x_o \in \mathbb{R}^n$ then

$$\boxed{F(x_o + h) = F(x_o) + F'(x_o)h + \eta(h)}$$

where $\dfrac{\|\eta(h)\|}{\|h\|} \longrightarrow 0$ as $h \longrightarrow 0$. The <u>derivative</u> $F'(x_o)$ is <u>Jacobian</u> <u>matrix</u>

$$F' = \left[\frac{\partial F}{\partial x_1} \middle| \frac{\partial F}{\partial x_2} \middle| \cdots \middle| \frac{\partial F}{\partial x_n} \right] = \begin{bmatrix} \nabla F^1 \\ \hline \nabla F^2 \\ \hline \vdots \\ \hline \nabla F^m \end{bmatrix}$$

where $\nabla g = \langle \partial_1 g, \partial_2 g, \dots, \partial_n g \rangle$ and $\partial_j g = \dfrac{\partial g}{\partial x_j}$.

$\boxed{E22}$ $F(x, y) = (x^2 + y^2, x - y) = \begin{bmatrix} x^2 + y^2 \\ x - y \end{bmatrix}$ ← <u>convention</u> I take $\mathbb{R}^n$ to be <u>column</u> vectors.

$$F'(x, y) = \begin{bmatrix} 2x & 2y \\ 1 & -1 \end{bmatrix}$$

$$F(a+h, b+k) = (a^2 + b^2, a - b) + \begin{bmatrix} 2a & 2b \\ 1 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$$

$$= \underbrace{(a^2 + b^2 + 2ah + 2bk, \; a - b + h - k)}_{}$$

Best ~~linear~~ affine approximation to F near (a, b).

(inverse function theorem)

Th^m/ If $\mathbb{R}^n \xrightarrow{F} \mathbb{R}^n$ has $F'(x_o)$ invertible then $\exists$ a local inverse $F^{-1} = (F|_U)^{-1}$ for some U containing x_o

Proof Sketch! Note: $F(x) \approx F(x_o) + F'(x_o)(x-x_o)$

$$y \cong F(x_o) + F'(x_o)(x-x_o)$$

Solve for x,

$$x \cong (F'(x_o))^{-1}\left[y - F(x_o) + F'(x_o)x_o\right]$$

$$\Rightarrow \underline{F^{-1}(y) = x_o + (F'(x_o))^{-1}(y - F(x_o))}.$$

- For $f : \mathbb{R} \to \mathbb{R}$ the criteria $f'(x_0)^{-1}$ exists amounts to the obvious condition $f'(x_o) \neq 0$. The existence of a local inverse is tied to $y = f(x)$ being strictly increasing or decreasing near x_o. If f' is <u>continuous</u> at x_o then $f'(x_o) \neq 0 \Rightarrow f'(x) \neq 0$ for all x in some open nbhd of x_o

- For $F : \mathbb{R}^n \to \mathbb{R}^n$ the criteria $(F'(x_o))^{-1}$ exist is captured by the <u>determinant</u>; $\det(F'(x_o)) \neq 0$. This det. condition paired with continuity of the partial derivatives $\partial_i F^j$ for $i,j = 1,2,\dots,n$ imply the existence of a local inverse.

Th^m/ If $F : \mathrm{dom}(F) \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $\partial_i F^j$ are likewise continuous at x_o then F is said to be <u>continuously</u> <u>differentiable</u>.

- Subtle Point: existence of $\partial_i F^j$ $\forall i j$ is <u>not</u> enough to $\Rightarrow F'$ exist.

Real Differentiability of mappings on $\mathbb{R}^2$ (I narrow the focus to $\mathbb{R}^2$ and add detail...)

We consider $F : \text{dom}(F) \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$. Notice that $F = (F_1, F_2)$ where $F_i : \text{dom}(F) \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ are real-valued functions on the plane. Recall we defined <u>partial</u> <u>differentiation</u> in calculus III, for $g : \text{dom}(g) \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\frac{\partial g}{\partial x} = \lim_{h \to 0} \left[\frac{g(x+h, y) - g(x,y)}{h} \right]$$

$$\frac{\partial g}{\partial y} = \lim_{h \to 0} \left[\frac{g(x, y+h) - g(x,y)}{h} \right]$$

$\boxed{\text{E23}}$ If $g(x,y) = x$ then $\frac{\partial g}{\partial y} = \lim_{h \to 0} \left[\frac{x - x}{h} \right] = 0$.
Likewise $\frac{\partial g}{\partial x} = \lim_{h \to 0} \left[\frac{x+h-x}{h} \right] = \lim_{h \to 0} [1] = 1$.
You should recall that $\frac{\partial x}{\partial y} = 0$ whereas $\frac{\partial x}{\partial x} = 1$.
Similar calculation shows $\frac{\partial y}{\partial x} = 0$ and $\frac{\partial y}{\partial y} = 1$.

<u>Remark</u>: the partial derivatives measure the change of a function along the respective coordinate-directions. If we have some technical criteria satisfied then we'll see the <u>best</u> <u>linear</u> <u>approximation</u> to the function can be constructed from $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$: For $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$f(x,y) \cong f(a,b) + \frac{\partial f}{\partial x}(a,b)\,(x-a) + \frac{\partial f}{\partial y}(a,b)\,(y-b)$$

We wish to make this idea of <u>best linear approximation</u> a bit more precise, and we extend to $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ where no direct geometric picture is available.

Let $\|V\| = \sqrt{V \cdot V} = \sqrt{V_1^2 + V_2^2}$ for $V = \langle V_1, V_2 \rangle \in \mathbb{R}^2$.
We say $F : \text{dom}(F) \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is <u>differentiable</u> at $(a,b) \in \text{dom}(F)$ iff there exists a linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{(h,k)\to(0,0)} \frac{\| F(a+h, b+k) - F(a,b) - L(h,k) \|}{\| (h,k) \|} = 0$$

If such an L exists then we call L the <u>differential</u> of F at (a,b) and we define,

$$dF_{(a,b)}(h,k) = L(h,k).$$

The <u>standard</u> <u>matrix</u> of $dF_{(a,b)}$ is called the <u>Jacobian</u> of F at (a,b) or, following Edward's Advanced Calculus, the <u>derivative</u> of F.

$$\begin{aligned} F'(a,b) &= \left[dF_{(a,b)} \right] \\ &= \left[dF_{(a,b)}(e_1) \;\middle|\; dF_{(a,b)}(e_2) \right] \\ &= \left[\begin{array}{c|c} \partial_x F_1(a,b) & \partial_y F_1(a,b) \\ \partial_x F_2(a,b) & \partial_y F_2(a,b) \end{array} \right] \end{aligned}$$

Given the differential exists at (a,b) we have:

$$dF\left(\begin{bmatrix} h \\ k \end{bmatrix} \right) = \left[\frac{\partial F}{\partial x} \;\middle|\; \frac{\partial F}{\partial y} \right] \begin{bmatrix} h \\ k \end{bmatrix}$$

$\boxed{\text{E24}}$ Let $F(x,y) = \left(\begin{cases} x+y & \text{for } xy = 0 \\ 1 & \text{for } xy \neq 0 \end{cases}, \begin{cases} x+y & \text{for } xy = 0 \\ 1 & \text{for } xy \neq 0 \end{cases} \right)$

$$\frac{\partial F^1}{\partial x} = 1 \quad \text{and} \quad \frac{\partial F^1}{\partial y} = 1 \qquad \frac{\partial F^2}{\partial x} = 1, \quad \frac{\partial F^2}{\partial y} = 1$$

However, $F'(a,b)$ d.n.e. (differentiability $\Rightarrow$ continuity) and clearly F is <u>not</u> continuous along xy-axes.

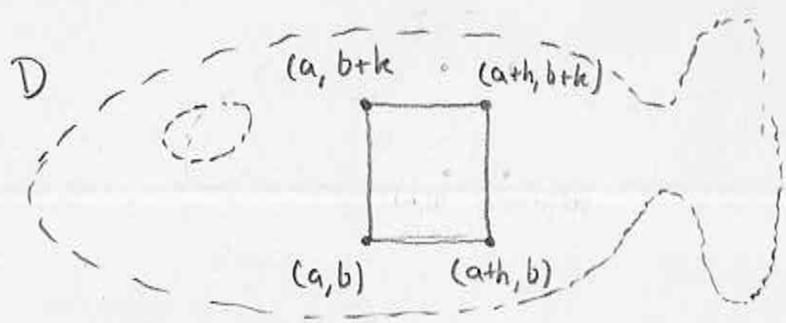
In $\boxed{E24}$ the $F : \mathbb{R}^2 \to \mathbb{R}^2$ was <u>not</u> continuously differentiable. We <u>need</u> continuity of the partial derivatives in order to obtain: $F(a+h, b+k) \approx F(a,b) + F'(a,b)\begin{bmatrix} h \\ k \end{bmatrix}$ as a good approximation. This subtlety persists for us as we study $\mathbb{C} \xrightarrow{f} \mathbb{C}$. We should prove the following: (<u>domain</u> = open connected set)

Th$^{\underline{m}}$/ If $f = (u,v) : \mathrm{dom}(f) \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ has continuous partial derivatives u_x, v_x, u_y, v_y in a <u>domain</u> D containing (a,b) f is differentiable at (a,b) meaning: $\exists\, df_{(a,b)}$ s.t.

$$\lim_{(h,k)\to(0,0)} \frac{\| f(a+h, b+k) - f(a,b) - df_{(a,b)}(h,k) \|}{\|(h,k)\|} = 0.$$

Moreover, in this case $df_{(a,b)}(h,k) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}$.

<u>Proof</u>: Let $\Delta f = f(a+h, b+k) - f(a,b)$ and consider $(h,k) \in \mathbb{R}^2$ such that the rectangle pictured below is interior to D.



Since each $(a,b) \in D$ is an interior point we can find h,k sufficiently small in magnitude to fit the rectangle inside D.

Observe, for $h, k \neq 0$,

$$\begin{aligned} \Delta f^j &= f^j(a+h, b+k) - f^j(a+h, b) + f^j(a+h, b) - f^j(a,b) \\ &= \left(\frac{f^j(a+h, b+k) - f^j(a+h, b)}{k}\right)k + \left(\frac{f^j(a+h, b) - f^j(a,b)}{h}\right)h \\ &= \frac{\partial f^j}{\partial y}(a+h, \bar{b})k + \frac{\partial f^j}{\partial x}(\bar{a}, b)h \end{aligned}$$

for some $\bar{a} \in (a-|h|, a) \cup (a, a+|h|)$ and $\bar{b} \in (b-|k|, b) \cup (b, b+|k|)$ by the Mean Value Th$^{\underline{m}}$ applied to f^j with respect to the x or y partial derivative ($f^1 = u$, $f^2 = v$ and u_x, u_y, v_x, v_y are all continuous in D by assumption)

Proof Continued:

Define $df_{(a,b)}(h,k) = \begin{bmatrix} u_x(a,b) & u_y(a,b) \\ v_x(a,b) & v_y(a,b) \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}$ from which it follows $\left(df_{(a,b)}(h,k)\right)^j = \frac{\partial f^j}{\partial x}(a,b)\,h + \frac{\partial f^j}{\partial y}(a,b)\,k$ for $j=1,2$.

Consider that, using the notation discussed on 50,

$$\Delta f^j - \left(df_{(a,b)}(h,k)\right)^j = \underbrace{\left(\frac{\partial f^j}{\partial x}(\bar{a},b) - \frac{\partial f^j}{\partial x}(a,b)\right)}_{g_1^j} h + \underbrace{\left(\frac{\partial f^j}{\partial y}(a+h,\bar{b}) - \frac{\partial f^j}{\partial y}(a,b)\right)}_{g_2^j} k$$

As $(h,k) \longrightarrow (0,0)$ clearly $g_1^j, g_2^j \longrightarrow 0$ by continuity of the partial derivative functions u_x, u_y, v_x, v_y. With this notation settled consider,

$$\frac{\|\Delta f - df_{(a,b)}(h,k)\|}{\|(h,k)\|} \leq \frac{|g_1^1 h + g_2^1 k|}{\sqrt{h^2+k^2}} + \frac{|g_1^2 h + g_2^2 k|}{\sqrt{h^2+k^2}}$$

$$\leq \frac{|g_1^1||h|}{\sqrt{h^2+k^2}} + \frac{|g_2^1||k|}{\sqrt{h^2+k^2}} + \frac{|g_1^2||h|}{\sqrt{h^2+k^2}} + \frac{|g_2^2||k|}{\sqrt{h^2+k^2}}$$

$$\leq |g_1^1| + |g_2^1| + |g_1^2| + |g_2^2|$$

As $\frac{|h|}{\sqrt{h^2+k^2}} \leq \frac{|h|}{\sqrt{h^2}} = \frac{|h|}{|h|} = 1$ and likewise $\frac{|k|}{\sqrt{h^2+k^2}} \leq 1$.

Observe, for (h,k) sufficiently close to $(0,0)$ we have

$$0 \leq \frac{\|f(a+h,b+k) - f(a,b) - df_{(a,b)}(h,k)\|}{\|(h,k)\|} \leq |g_1^1| + |g_2^1| + |g_1^2| + |g_2^2|$$

And as $(h,k) \longrightarrow (0,0)$ the outside expressions tend to zero thus by the squeeze Th$^{\underline{m}}$ the middle expression has a limit which <u>exists</u> and is <u>zero</u>. This proves f is differentiable at (a,b). Moreover, $df_{(a,b)}(h,k) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix}$ as claimed. //

<u>Remark</u>: this page is what is need to change "sketch of proof" to "proof". (we just sketched this proof in lecture)

Remark: the limit concept of calculus III and the concept we defined in this course are essentially the same. For this reason I freely exchange $z \to z_o = x_o + iy_o$ with $(x,y) \longrightarrow (x_o, y_o)$. It's also clear (we proved it back on 36)

$$\lim_{z \to z_o} (u+iv) = a+ib \iff \lim_{z \to z_o} u = a \text{ \& } \lim_{z \to z_o} v = b$$

The length of a two-dim'l vector is just the modulus of the corresponding complex number. Thus,

$$\lim_{z \to z_o} \|(u,v)\| = \lim_{z \to z_o} |u+iv|.$$

Applying this to the limit which implicitly defines df_{z_o}, we find real-differentiability of $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is given by

$$\boxed{\lim_{H \to 0} \frac{|f(z_o+H) - f(z_o) - df_{z_o}(H)|}{|H|} = 0} \quad (\star)$$

where it is assumed $df_{z_o} : \mathbb{C} \longrightarrow \mathbb{C}$ is a linear transformation over the real numbers,

① $df_{z_o}(z_1+z_2) = df_{z_o}(z_1) + df_{z_o}(z_2) \quad \forall z_1, z_2 \in \mathbb{C}$

② $df_{z_o}(cz) = c\, df_{z_o}(z) \quad \forall z \in \mathbb{C}$ and $c \in \mathbb{R}$.

Question: how are the criteria of complex-differentiability at z_o for $f = u+iv$ and real-differentiability of $f = u+iv$ at $z_o \in \mathbb{C} = \mathbb{R}^2$ related?

$$\text{complex-diff at } z_o \iff f'(z_o) = \lim_{H \to 0}\left(\frac{f(z_o+H)-f(z_o)}{H}\right) \in \mathbb{C}.$$

$$\text{real-diff at } z_o \iff (\star) \text{ above holds.}$$

How to connect these concepts?

It's convenient to reformulate the condition of complex-diff. at z_o so it resembles the $(\star)$ of 52. Recall,

$$f'(z_o) = \lim_{H \to 0}\left(\frac{f(z_o+H)-f(z_o)}{H}\right) \in \mathbb{C}$$

$$\Rightarrow \lim_{H\to 0}\left(\frac{f(z_o+H)-f(z_o)}{H}\right) - \lim_{H \to 0}\left(\frac{Hf'(z_o)}{H}\right) = 0$$

$$\Rightarrow \lim_{H\to 0}\left(\frac{f(z_o+H)-f(z_o)-Hf'(z_o)}{H}\right) = 0$$

$$\Rightarrow \lim_{H\to 0}\frac{|f(z_o+H)-f(z_o)-Hf'(z_o)|}{|H|} = 0$$

$\Rightarrow$ f is <u>real</u>-differentiable at z_o with $df_{z_o}(H) = Hf'(z_o)$.

Th^{m}/ If $f'(z_o) = a+ib \in \mathbb{C}$ then f is real-differentiable as a function on $\mathbb{R}^2$ and if $f = u+iv$ then $u_x(z_o) = v_y(z_o) = a$ and $v_x(z_o) = -u_y(z_o) = b$.

<u>Proof</u>: the discussion preceding the Th^{m} proves real-diff. at z_o. Real differentiability $\Rightarrow$ $J_f = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ and $df_{z_o}(H) = J_f(z_o)H$.

But, we also have $df_{z_o}(H) = Hf'(z_o)$ by the discussion above. Hence, letting $H = H_1 + iH_2$ we find

① $f'(z_o)H = (a+ib)(H_1+iH_2) = (aH_1 - bH_2,\ bH_1 + aH_2) = \begin{bmatrix} aH_1 - bH_2 \\ bH_1 + aH_2 \end{bmatrix}$

② $J_f(z_o)H = \begin{bmatrix} u_x(z_o) & u_y(z_o) \\ v_x(z_o) & v_y(z_o)\end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} u_x(z_o)H_1 + u_y(z_o)H_2 \\ v_x(z_o)H_1 + v_y(z_o)H_2 \end{bmatrix}$.

Suppose $H_1 = 1$, $H_2 = 0$ and find $u_x(z_o) = a$, $v_x(z_o) = b$.
Suppose $H_1 = 0$, $H_2 = 1$ and find $u_y(z_o) = -b$, $v_y(z_o) = a$.
It follows that $u_x(z_o) = v_y(z_o)$ and $u_y(z_o) = -v_x(z_o)$. //

Observe $f(x+iy) = x - iy$ is real-differentiable on $\mathbb{R}^2$ however f is not complex-differentiable by $\boxed{E21}$. Thus, real-differentiability alone $\not\Rightarrow$ complex differentiability. We need to add the conditions that $\underbrace{u_x = v_y \text{ and } u_y = -v_x}_{\text{CAUCHY RIEMANN EQ}^{\text{n}}\text{'s}}$ at z_o. More precisely,

> **Th$^{\text{m}}$/** If $u, v : \mathbb{C} \to \mathbb{R}$ have continuous partial derivatives u_x, v_x, u_y, v_y on a domain containing z_o and if $u_x(z_o) = v_y(z_o)$ and $u_y(z_o) = -v_x(z_o)$ then $f = u + iv$ is complex-differentiable at z_o with $f'(z_o) = u_x(z_o) + i v_x(z_o)$

Proof: Given the continuous differentiability of u and v we have from the Th$^{\text{m}}$ on 50 that $f = u+iv$ is real-differentiable and

$$df_{z_o}(H) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \overset{*}{=} \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}\begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} u_x H_1 - v_x H_2 \\ v_x H_1 + u_x H_2 \end{bmatrix} \quad \left(\begin{array}{l} * \text{ applied the} \\ u_x = v_y \\ u_y = -v_x \end{array}\right)$$

where I have omitted the z_o-dependence for brevity. Note that

$$df_{z_o}(H_1 + iH_2) = (u_x + i v_x)(H_1 + iH_2)$$

At this point, we'd like to reverse the implications from pg. 53. Consider, by real differentiability, let $C = u_x(z_o) + i v_x(z_o)$,

$$\lim_{H \to 0} \frac{|f(z_o+H) - f(z_o) - df_{z_o}(H)|}{|H|} = 0$$

$$\Rightarrow \lim_{H \to 0} \left[\frac{f(z_o+H) - f(z_o) - CH}{H}\right] = 0$$

$$\Rightarrow \lim_{H \to 0} \left(\frac{f(z_o+H) - f(z_o)}{H}\right) - \lim_{H \to 0}\left(\frac{CH}{H}\right) = 0$$

By Lemma to follow proof on 55.

$$\Rightarrow \lim_{H \to 0} \left(\frac{f(z_o) - f(z_o)}{H}\right) = C$$

Hence $C = f'(z_o) \Rightarrow f'(z_o) = u_x(z_o) + i v_x(z_o)$ as claimed. //

Lemma: If $\lim_{z \to z_o} (f - g) = 0$ and $\lim_{z \to z_o} g$ exists then $\lim_{z \to z_o} f = \lim_{z \to z_o} g$.

Proof:
$$\lim_{z \to z_o} f = \lim_{z \to z_o} (f - g + g)$$
$$= \lim_{z \to z_o} (f-g) + \lim_{z \to z_o} g$$
$$= 0 + \lim_{z \to z_o} g$$
$$= \lim_{z \to z_o} g. //$$

Remark: the formula $df_{z_o}(H) = cH$ is very special. For a merely real-differentiable function at z_o we have

$$df_{z_o}(H) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}$$
$$= \begin{bmatrix} u_x H_1 + u_y H_2 \\ v_x H_1 + v_y H_2 \end{bmatrix}$$

$H = H_1 + iH_2$, $\bar{H} = H_1 - iH_2$ $\Rightarrow$ $H_1 = \frac{1}{2}(H + \bar{H})$, $H_2 = \frac{1}{2i}(H - \bar{H})$

$$= u_x\left(\tfrac{1}{2}(H+\bar{H})\right) + u_y\left(\tfrac{1}{2i}(H-\bar{H})\right) + i\left[v_x\left(\tfrac{1}{2}(H+\bar{H})\right) + v_y\left(\tfrac{1}{2i}(H-\bar{H})\right)\right]$$

$$= \underbrace{\left[\tfrac{1}{2}(u_x + v_y) + \tfrac{i}{2}(v_x - u_y)\right]}_{\text{with CR-eq}^{\text{n}}\text{'s } u_x = v_y,\ v_x = -u_y \text{ this simplifies to } (u_x + iv_x)} H + \underbrace{\left[\tfrac{1}{2}(u_x - v_y) + \tfrac{i}{2}(v_x + u_y)\right]}_{\text{becomes zero with application of the CR-eq}^{\text{n}}\text{'s.}} \bar{H}$$

Details aside, generically, $df_{z_o}(H) = cH + \tilde{c}\bar{H}$. When we attempt to follow the argument on (54) we will get stuck on the $\lim_{H \to 0}\left(\frac{\tilde{c}\bar{H}}{H}\right)$ which d.n.e. <u>unless</u> $\tilde{c} = 0$ which gives us exactly the CR-eq$^{\text{n}}$'s. We can look at $f(x,y) = f(z, \bar{z})$ (abusing notation considerably) as a complex-notation for a fnct. of two-real variables:

$$\boxed{df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

I have not made this precise in this course, ... maybe later

The CR-eq$^{\text{n}}$'s are compactly phrased $\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow f = f(z)$. This dovetails nicely with our discussion back on (45).

Notation, notation, notation

If $f'(z)$ exists then we write $f'(z) = \frac{df}{dz}$. We found the Cauchy Riemann (CR)-eq^{n}'s are a necessary, but insufficient, condition for complex-diff. at z. A sufficient set of conditions is given by the pair of continuous diff. of u, v and the CR-eq^{n}'s. Ok, summary over, suppressing the explicit z-dependence on some terms,

$$\frac{df}{dz} = u_x + iv_x = \frac{\partial}{\partial x}(u+iv) \Rightarrow \underline{\frac{d}{dz} = \frac{\partial}{\partial x}}.$$

$$\frac{df}{dz} = v_y - iu_y = -i\frac{\partial}{\partial y}(u+iv) \Rightarrow \underline{\frac{d}{dz} = \frac{1}{i}\frac{\partial}{\partial y}}.$$

In view of these identities,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = -i\frac{\partial f}{\partial y}$$

Remark: Minh was about to say $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = 0$. This is true in context. This operator is trivial on the set of complex-differentiable functions. However,

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(x - iy) = \frac{\partial x}{\partial x} - i^2\frac{\partial y}{\partial y} = 2 \neq 0.$$

But, $f(z) = \bar{z}$ is not complex-diff. as we have shown in $\boxed{E21}$.

Formally, the following calculation is intriguing, note $x = \frac{1}{2}(z+\bar{z})$, $y = \frac{i}{2}(\bar{z} - z)$

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}$$

Adding these yields $\partial_x = \partial_z + \partial_{\bar{z}}$ and $i\partial_y = \partial_{\bar{z}} - \partial_z$

Note also (for future reference)

$$\begin{aligned}\partial_x^2 + \partial_y^2 &= (\partial_x + i\partial_y)(\partial_x - i\partial_y) \\ &= (2\partial_{\bar{z}})(2\partial_z) \\ &= 4\partial_{\bar{z}}\partial_z\end{aligned}$$

Apparently if u is a fnct. of either z or $\bar{z}$ alone it solves $u_{xx} + u_{yy} = 0$.

<u>Remark</u>: pgs. 46–56 are an expanded version of what I presented in lecture on 1/31/13. I hope you read these. What follows was also discussed in large part on 1/31/13.

$\boxed{E25}$ $f(z) = z^2$

$$f(x+iy) = \underbrace{x^2-y^2}_{u} + i\,\underbrace{(2xy)}_{v}$$

Observe, $u_x = v_y = 2x$ and $u_y = -v_x = -2y$

Consequently, as u_x, u_y, v_x, v_y are clearly continuous we find $f'(z) = u_x + iv_x = 2x + i(2y) = 2(x+iy) = 2z$.

Of course this agrees with our early work on the power rule see pg. 42.

$\boxed{E26}$ $f(z) = e^z = e^{x+iy} = \underbrace{e^x\cos(y)}_{u} + i\,\underbrace{e^x\sin(y)}_{v}$

observe $u_x = v_y = e^x\cos(y)$ and $v_x = -u_y = e^x\sin(y)$

hence, noting continuity of partials, $f'(z) = u_x + iv_x = e^x\cos(y) + ie^x\sin y$, which shows the pleasing result $\boxed{\frac{d}{dz}(e^z) = e^z}$

> Def$^{\underline{n}}$/ If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is complex-differentiable on all of $\mathbb{C}$ then we say f is an <u>entire</u> function.

<u>Remark</u>: $\boxed{E25}$ and $\boxed{E26}$ concern entire fncts.

$\boxed{E27}$ We <u>define</u> complex sine, cosine and their hyperbolic siblings,

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \ \ \& \ \ \sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right)$$

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \ \ \& \ \ \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$$

We calculate derivatives of the above via $\boxed{E26}$ and the chain-rule we proved previously.

$$\begin{aligned}
\frac{d}{dz}\Big(\sin(z)\Big) &= \frac{d}{dz}\left[\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right] \\
&= \frac{i}{2i}e^{iz} - \left(\frac{-i}{2i}\right)e^{-iz} \\
&= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
&= \cos(z).
\end{aligned}$$

$\boxed{\text{E27 Continued}}$ By very similar calculations we can show

$$\frac{d}{dz}\Big(\cos(z)\Big) = -\sin z \quad , \frac{d}{dz}\Big(\cosh z\Big) = \sinh(z) \quad , \frac{d}{dz}\Big(\sinh z\Big) = \cosh z .$$

<u>Remark</u>: many other identities for real trig. and hyperbolic trig functions transfer over to the complex domain. One notable exception is the bounded property of $\sin(x), \cos(x)$ for $x \in \mathbb{R}$. In homework you'll discover $|\sin(z)|, |\cos(z)|$ are unbounded for $z \in \mathbb{C}$.

- We turn to the problem of inverse functions. If we knew $f^{-1}(f(z)) = z$ and that f^{-1}, f are <u>both</u> complex differentiable then the Th$^{\underline{m}}$ below is not crucial. However, if we only know f is complex-diff. and $f^{-1}(f(z)) = z$ then the Th$^{\underline{m}}$ provides complex-diff. of f^{-1}.

Th^m/ If $f(z)$ is complex-differentiable on a domain D, $z_o \in D$, and $f'(z_o) \neq 0$. Then $\exists$ a (small) disk $U \subseteq D$ containing z_o such that $f|_U$ is 1-1 and $f(U) = V$ with $f^{-1}: V \longrightarrow U$ a complex-diff on V and

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)} .$$

<u>Proof</u>: see handout from Gamelin or see Freitag pg. 53-54, Theorem I.5.7 where the <u>implicit</u> <u>function</u> Th$^{\underline{m}}$ is also given. Following Gamelin, we note the existence of U, V such that $f|_U$ is injective and $f^{-1}: V \rightarrow U$ is a fnct. is granted by the inverse fnct. Th$^{\underline{m}}$ of advanced calculus (not an easy Th$^{\underline{m}}$ to prove). Let $g = f^{-1}$ and consider $w, w_1 \in U$ with $w \neq w_1$, set $z = g(w)$ and $z_1 = g(w_1)$. By injectivity of g we find $z_1 \neq z$. By def$^{\underline{n}}$ of g note $f(z) = w$ and $f(z_1) = w_1$. Thus,

$$\frac{g(w) - g(w_1)}{w - w_1} = \frac{z - z_1}{f(z) - f(z_1)} = \left(\frac{f(z) - f(z_1)}{z - z_1}\right)^{-1}$$

As $w \rightarrow w_1$ note $z \rightarrow z_1$ and the RHS above $\longrightarrow (f'(z_1))^{-1}$ thus $g'(w_1) = \dfrac{1}{f'(z_1)}$ and the Th$^{\underline{m}}$ follows. //

Remark: one of the deeper integral theorems of complex analysis later gives a proof of the preceding thm w/o reliance on the deep Thm from adv. calc. Alternatively, we could just check the CR-eqns for the inverse fnct directly ☺

E28 Consider $Log(z) = \ln|z| + i\,Arg(z)$ thus finding $u(x,y)$ and $v(x,y)$ requires some thought. Not too much though,

$$u(z) = \ln|z| \quad \& \quad v(z) = Arg(z)$$

Note: $u(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$.

and: $v(x,y) = \tan^{-1}\left(\frac{y}{x}\right) + k$ $(x \neq 0)$ where k depends on the quadrant in question, however we can agree

$$v_x = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1+y^2/x^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} = -u_y$$

$$v_y = \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = u_x$$

Now, what about continuity of u_x, v_x, u_y, v_y? Nothing too subtle for u_x, u_y, continuity clear for $\mathbb{C} - \{(0,0)\}$. However, for v_x, v_y you might worry $\exists$ discontinuity at $x = 0$, however, it is not the case. This is easier to believe if you think about $Arg(z)$, the discontinuity is found at $(-\infty, 0] + i(0)$ the negative real axis and the origin.

E29 Using the Thm, life is easy, $\exp(Log(z)) = z$ and hence Log is complex-diff at appropriate values and

$$1 = \frac{d}{dz}\left(e^{Log(z)}\right) = e^{Log(z)}\frac{d}{dz}\left[Log(z)\right] = z\frac{d}{dz}\left[Log(z)\right]$$

$$\Rightarrow \underbrace{\boxed{\frac{d}{dz}\left(Log(z)\right) = \frac{1}{z}}}_{}$$

for $z \in \mathbb{C} - \{x + i(0) \mid x \in (-\infty, 0]\}$

note: to finish E28,

$$f'(z) = u_x + iv_x = \frac{x - iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}.$$

<u>Remark</u>: in $\boxed{E29}$ the points for which $\frac{d}{dz}\left(\text{Log}(z)\right) \neq \frac{1}{z}$ we're precisely those for which $\text{Log}(z)$ is discontinuous. As we discussed $\text{Arg}(z)$ is discontinuous on $\{(x,0) \mid x \leq 0\}$. Since $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$ the Log suffers the same discontinuity. If we used another branch of $\log(z)$ then the discont. would move to the new location of the angle-jump. With this in mind, we claim

$$\frac{d}{dz}\left(\log(z)\right) = \frac{1}{z}$$

provided a guide to understand $\frac{d}{dz}\left(\text{Log}_\alpha(z)\right) = \frac{1}{z}$, for $\text{Log}_\alpha(z) = \ln|z| + i\,\text{Arg}_\alpha(z)$ where we define $\text{Arg}_\alpha(z) \in \arg(z)$ such that $\text{Arg}_\alpha(z) \in (\alpha, \alpha + 2\pi]$. In particular, $\alpha = -\pi$ gives $\text{Arg}_{-\pi} = \text{Arg}$ and we've discussed $\frac{d}{dz}\left(\text{Log}(z)\right) = \frac{1}{z}$ for z with $\text{Arg}(z) \neq -\pi + 2\pi$. Likewise $\frac{d}{dz}\left(\text{Log}_\alpha(z)\right) = \frac{1}{z}$ for z with $\text{Arg}_\alpha(z) \neq \alpha + 2\pi$.

- We only obtain the formula for the <u>slit</u>-plane.

<u>Remark</u>: Happily Freitag has given us $\mathbb{C}_-$ as a nice notation for $\mathbb{C} - \{z \mid \text{Re}(z) \leq 0, \text{Im}(z) = 0\}$. (pg. 54)

$\boxed{E30}$ Let $f(z) = z^c = \exp(c\,\text{Log}(z))$ (Principle Power Function)

$$f'(z) = \exp(c\text{Log}(z))\frac{d}{dz}\left(c\text{Log}(z)\right) \quad \text{by chain-rule}$$

$$= \exp(c\text{Log}(z))\frac{c}{z}$$

$$= c\exp(c\text{Log}(z))\exp(-1\cdot\text{Log}(z))$$

$$= c\exp((c-1)\text{Log}(z))$$

$$= \underbrace{cz^{c-1}}!$$

for $z \in \mathbb{C}_-$

with the understanding $z^c = \exp(c\text{Log}(z))$.

(<u>Remark</u>: $z=-1$ troubling here, but $c=-1$ ok, in fact $c \in \mathbb{C}$ good.)

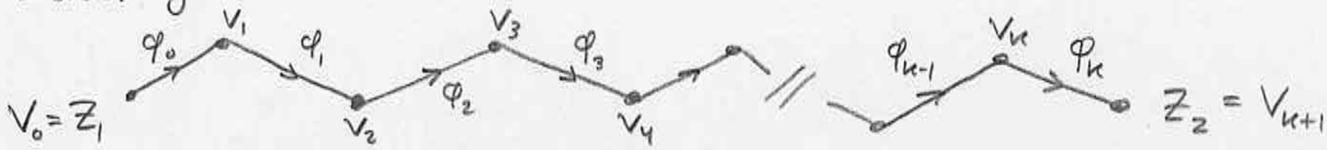
Recall: If $I \subseteq \mathbb{R}$ is an interval then $f'(x) = 0 \ \forall x \in I$ iff $f(x) = c \ \ \forall x \in I$. There is an analogue here:

$$\boxed{\text{Th}^{m}/ \text{ Let } D \text{ be an open, arc-connected, subset of } \mathbb{C}, \quad f'(z) = 0 \ \ \forall z \in D \iff f(z) = c \ \ \forall z \in D}$$

Proof: $\Leftarrow$] Clearly $\frac{d}{dz}(c) = 0$ for each $z \in D$.

$\Rightarrow$] Suppose $f'(z) = 0 \ \ \forall z \in D$. Let $z_1, z_2 \in D$ then by arc-connectedness $\exists \varphi_j : [0,1] \longrightarrow D$ for $j = 0, 1, \dots, k$ where $\varphi_j(1) = \varphi_{j+1}(0)$ for $j = 0, 1, \dots, k$ and $\varphi_0(0) = z_1$, $\varphi_k(1) = z_2$.

Labeling the vertices of the arc by $V_1, V_2, \dots, V$



Explicitly $\varphi_j(t) = V_j + t(V_{j+1} - V_j)$ for $0 \le t \le 1$ for $j = 0, 1, \dots, k$.
Clearly $\varphi_j'(t) = V_{j+1} - V_j$ for $0 \le t \le 1$ using 1-sided derivatives at $t = 0, 1$ as appropriate. Calculate,

$$\frac{d}{dt}\Big[f(\varphi_j(t))\Big] = f'(\varphi_j(t))\frac{d}{dt}\Big[\varphi_j(t)\Big] = \underbrace{f'(\varphi_j(t))}_{0}(V_{j+1} - V_j)$$

since $\varphi_j(t) \in D$

Remark: actually, apply CR-eq^{n}'s and calculus III

thus, by calculus I, $f(\varphi_j(t)) = c_j$ for $0 \le t \le 1$ and for each $j = 0, 1, 2, \dots, k$. But, the vertices are common

$$f(\varphi_1(0)) = f(\varphi_0(1)) \Rightarrow c_1 = c_0$$

and similarly $c_2 = c_1$, $c_3 = c_2$, $\dots$, $c_k = c_{k-1}$ thus $c_0 = c_k$ which shows $f(z_1) = f(z_2)$. But, z_1, z_2 were an arbitrary pair in D thus $f(D) = \{c\}$. //

$\boxed{\text{E31}}$ If $f'(z)$ exists and $f = Re(f)$ on (an arc-connected) domain D then f is constant on D

$f = Re(f) \Rightarrow v = 0$ thus $u_x = v_y = 0$ and $v_x = -u_y = 0$
Consequently $f'(z) = 0$ on $D \Rightarrow f$ constant by Th^{m} above.