Harmonic Functions and their conjugates

Def^{n}/ If $\mathbb{C} \xrightarrow{u} \mathbb{R}$ solves $\underbrace{u_{xx} + u_{yy} = 0}_{\text{Laplace's Eq}^{\underline{n}} \text{ in two-dimensions}}$ $\forall z \in D$ then u is <u>harmonic</u> on D.

Th^{m}/ If $f = u + iv$ is complex-differentiable on $D \subseteq \mathbb{C}$ and u, v are cont. diff. on D then u, v are harmonic on D.

<u>Proof:</u> Suppose $f'(z)$ exists then $u_x = v_y$ and $u_y = -v_x$. A theorem of advanced calculus states continuous diff. allows us to exchange the order of partial diff. ($g_{xy} = g_{yx}$) thus

$$\begin{aligned} u_{xx} + u_{yy} &= (u_x)_x + (u_y)_y \\ &= (v_y)_x + (-v_x)_y \\ &= v_{yx} - v_{xy} \\ &= 0. \end{aligned}$$

Likewise, we can show $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0.$ //

$\boxed{E32}$ $f(z) = z = x + iy \Rightarrow u(x,y) = x$, $v(x,y) = y$ <u>harmonic on $\mathbb{C}$.</u>

$f(z) = z^2 = x^2 - y^2 + 2ixy \Rightarrow \underbrace{u(x,y) = x^2 - y^2, \ v(x,y) = 2xy}_{\text{harmonic on } \mathbb{C}.}$

$$\begin{aligned} f(z) = \sin(x+iy) &= \sin(x)\cos(iy) + \cos(x)\sin(iy) \\ &= \sin(x)\cosh(y) - i\cos(x)\sinh(y) \\ &\Rightarrow \underbrace{u(x,y) = \sin x \cosh y, \ v(x,y) = -\cos x \sinh y}_{\text{harmonic on } \mathbb{C}.} \end{aligned}$$

- each complex-differentiable function gives us a pair of harmonic functions. Can we reverse this? Given harmonic function u ($u_{xx} + u_{yy} = 0$) can we find a "conjugate" function v which makes $u + iv$ complex-diff.? Note it automatically follows, if this is possible, that $v_{xx} + v_{yy} = 0$.

How to calculate the harmonic conjugate

Th$^{\underline{m}}$ (I.5.11 of Freitag & Busam) Let $D \subseteq \mathbb{C}$ be an open rectangle whose sides are parallel to the coordinate axes. Let $u : D \to \mathbb{R}$ be harmonic fnct. Then there is a complex-differentiable fnct $f = u + iv$.

<u>Proof</u>: let $D = (a,b) \times (c,d)$ and let $x_0 \in (a,b)$, $y_0 \in (c,d)$. We seek to <u>construct</u> v such that $u_x = v_y$ and $u_y = -v_x$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt + g_1(x)$$

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow v(x,y) = \int_{x_0}^{x} -\frac{\partial u}{\partial y}(t,y)\,dt + g_2(y)$$

This suggests we define,

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)\,dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(t,y_0)\,dt$$

With v defined as above we observe v is continuously differentiable and by construction, $v_y = u_x$ and $v_x = -u_y$ hence $u + iv$ is complex-diff. //

[<u>Remark</u>: the Th$^{\underline{m}}$ above was stated for a rectangle for convenience of proof. It is true for connected regions without holes, aka, <u>simply</u> <u>connected</u> domain.]

$\boxed{E33}$ Consider $u(x,y) = \frac{1}{2}\ln(x^2+y^2)$ we calculate,

$$u_x = \frac{x}{x^2+y^2} \quad \text{and} \quad u_{xx} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Likewise, $u_{yy} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ thus $u_{xx} + u_{yy} = 0$ for $x+iy \neq 0$.

Observe that $\text{Re}(\text{Log}(z)) = \ln|z| = \frac{1}{2}\ln(x^2+y^2)$ thus a complex-diff. fnct $u+iv$ with $u = \frac{1}{2}\ln(x^2+y^2)$ will match $\text{Log}(z)$ upto an additive constant. Hence v on $\mathbb{C} - \{0\}$ harmonic is impossible as we know Log misses a branch. $\mathbb{C}_-$ is it. Of course we could use $\text{Log}_\alpha(z)$ to pick-up other slit-plane.

Point of E33 ?

We cannot hope to always find global harmonic conjugates. A single pt. missing from domain of u can spoil the goal of finding v to make $u+iv$ complex-diff.
(Remark 1 of p. 57 of Freitag should cover later) ← note to reader/me.

$\boxed{E34}$ Let $u(x,y) = x^3 - 3xy^2$ (p. 58 Freitag)

using the proof construction,

$$v(x,y) = \int_0^y (3x^2 - 3t^2)\,dt - \int_0^x -6t\cancel{y}_0\,dt$$

$$\Rightarrow \underline{v(x,y) = 3x^2y - y^3}$$

Thus $f(x,y) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ has $u_x = v_y = 3x^2 - 3y^2$ and $u_y = -6xy = -v_x$.

Discussion: Suppose $f = u + iv$ and constrast $C_1 : u(x,y) = k_1$ and $C_2 : v(x,y) = k_2$ where $f'(z)$ exists.

Parametrize the level curves,

$$C_1 : \frac{\partial u}{\partial x}\frac{dx_1}{dt} + \frac{\partial u}{\partial y}\frac{dy_1}{dt} = 0 \Rightarrow u_x \dot{x}_1 + u_y \dot{y}_1 = 0 \quad \text{I}$$

$$C_2 : \frac{\partial v}{\partial x}\frac{dx_2}{dt} + \frac{\partial v}{\partial y}\frac{dy_2}{dt} = 0 \Rightarrow v_x \dot{x}_2 + v_y \dot{y}_2 = 0 \quad \text{II}$$

For $u_x = v_y$ and $u_y = -v_x$ we find that II yields,

$$-u_y \dot{x}_2 + u_x \dot{y}_2 = 0 \quad \text{vs.} \quad u_x \dot{x}_1 + u_y \dot{y}_1 = 0$$

Geometrically:

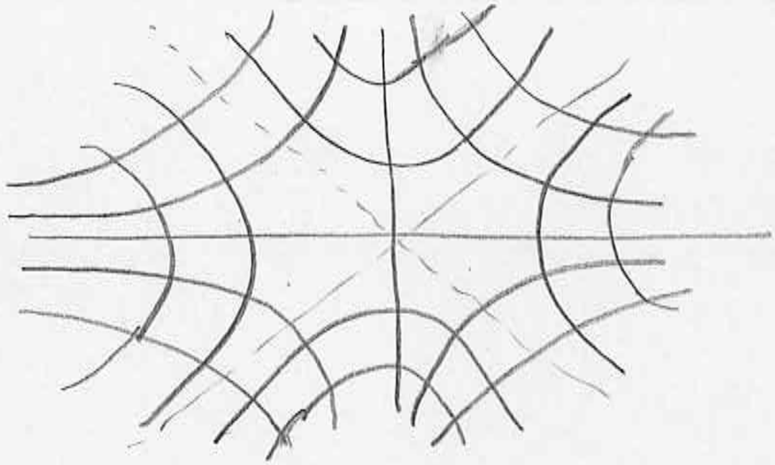
$$\langle \dot{x}_2, \dot{y}_2 \rangle \perp \langle -u_y, u_x \rangle$$
$$\langle \dot{x}_1, \dot{y}_1 \rangle \perp \langle u_x, u_y \rangle$$

$\perp$ to each other

Hence $\langle \dot{x}_2, \dot{y}_2 \rangle \perp \langle \dot{x}_1, \dot{y}_1 \rangle \therefore$ C_1 & C_2 are orthogonal trajectories !

$\boxed{E35}$ $f(z) = z^2 = x^2 - y^2 + 2ixy$

$x^2 - y^2 = k_1$ vs. $2xy = k_2$

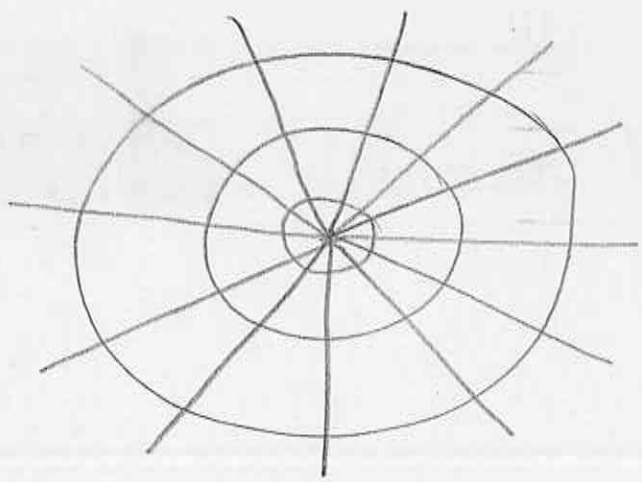


actually, both families of hyperbolas.

$\boxed{E36}$ Inverse question: what $f = u + iv$ has

$u(x,y) = k_1$
$v(x,y) = k_2$

looks like

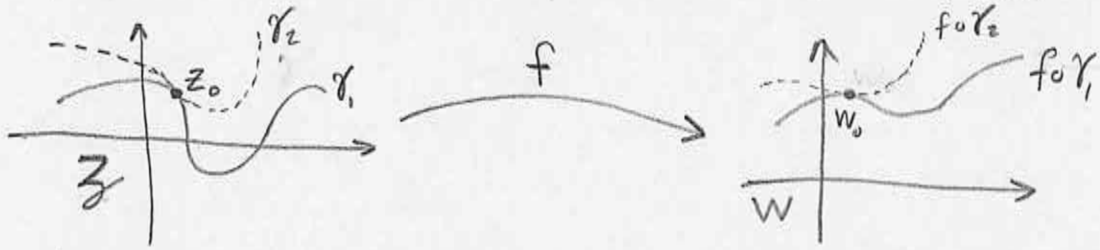


?

<u>Remark</u>: in view of the result we derive on 66 → 67 the u, v-level curves have inverse images in the z-plane which are likewise orthogonal. Let's look for this in the examples following 67 ...

CONCERNING THE GEOMETRY OF COMPLEX DIFFERENTIABLE MAPS

Suppose $\gamma : \mathbb{R} \to \mathbb{C}$ is smooth with $\gamma'(0) \neq 0$ then we say γ is a <u>regular</u> curve based at $\gamma(0)$. We wish to study the image of such curves under a complex-diff. mapping f with $f'(z_0) \neq 0$ where $z \longmapsto w = f(z)$. Let $w_0 = f(z_0)$



Consider,

$$\frac{d}{dt}(f\circ\gamma)(t) = f'(\gamma(t))\frac{d\gamma}{dt}$$

Thus, if γ_1, γ_2 are regular then $f\circ\gamma_1$, $f\circ\gamma_2$ are also regular and $(f\circ\gamma_1)'(0) = f'(z_0)\gamma_1'(0)$ and $(f\circ\gamma_2)'(0) = f'(z_0)\gamma_2'(0)$. We can compare the <u>oriented</u> <u>angle</u> between $\gamma_1'(0)$ and $\gamma_2'(0)$ with that of $(f\circ\gamma_1)'(0)$ and $(f\circ\gamma_2)'(0)$. Let $\gamma_j'(0) = a_j + ib_j$ then,

$$\gamma_1'(0) \cdot \gamma_2'(0) = a_1 b_1 + a_2 b_2$$

Let $f'(z_0) = c + id$ then $f'(z_0)(a+ib) = \underbrace{\overbrace{\begin{bmatrix} c & -d \\ d & c \end{bmatrix}}^{J_f}\begin{bmatrix} a \\ b \end{bmatrix}}_{\text{nice to use here we explained this in Prob. Set 1.}}$

$$\begin{aligned}
(f\circ\gamma_1)'(0)\cdot(f\circ\gamma_2)'(0) &= \left(J_f\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right)\cdot\left(J_f\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) \\
&= \left[J_f\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right]^T J_f\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \\
&= [a_1, b_1]\, J_f^T J_f \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \\
&= (c^2+d^2)(a_1 b_1 + a_2 b_2) \\
&= |f'(z_0)|^2\, \gamma_1'(0)\cdot\gamma_2'(0). \qquad (\star)
\end{aligned}$$

dot-product of two-vectors

<u>Remark</u>: $\vec{v}\cdot\vec{w} = \underset{(1\times n)}{v^T}\underset{(n\times 1)}{w} = [v_1, \dots, v_n]\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = v_1 w_1 + \cdots + v_n w_n$ (how to do dot-prod. with matrices)

Continuing: we found ☆

$$(f\circ\gamma_1)'(0)\cdot(f\circ\gamma_2)'(0) = |f'(z_0)|^2\,\gamma_1'(0)\cdot\gamma_2'(0)$$

Likewise, we can show

$$(f\circ\gamma_1)'(0)\cdot(f\circ\gamma_1)'(0) = |f'(z_0)|^2\,\gamma_1'(0)\cdot\gamma_1'(0)$$

$$(f\circ\gamma_2)'(0)\cdot(f\circ\gamma_2)'(0) = |f'(z_0)|^2\,\gamma_2'(0)\cdot\gamma_2'(0)$$

Which shows that:

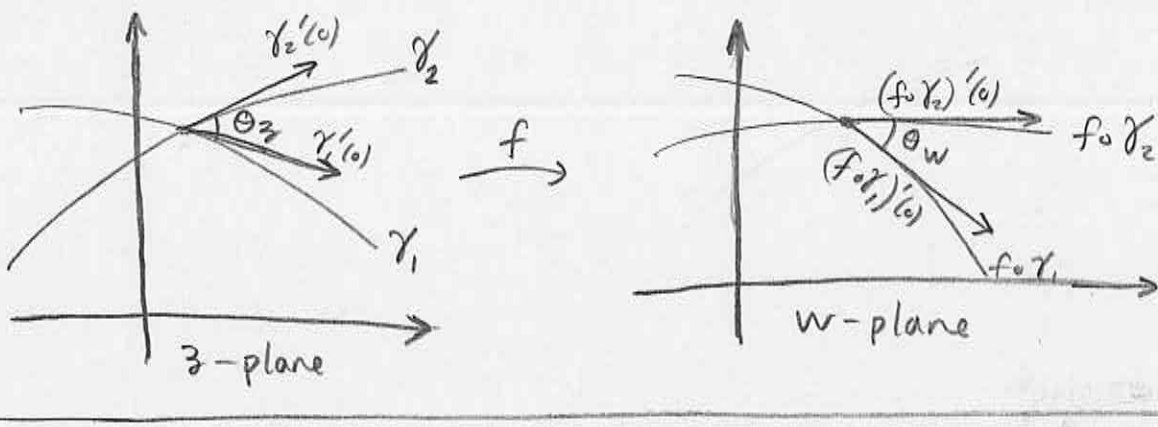
$$|(f\circ\gamma_1)'(0)|^2 = |f'(z_0)|^2\,|\gamma_1'(0)|^2$$

$$|(f\circ\gamma_2)'(0)|^2 = |f'(z_0)|^2\,|\gamma_2'(0)|^2$$

Thus $|(f\circ\gamma_j)'(0)| = |f'(z_0)|\,|\gamma_j'(0)|$ for $j=1,2$. Recall that $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$ hence $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$. Observe

$$\cos\theta_w = \frac{(f\circ\gamma_1)'(0)\cdot(f\circ\gamma_2)'(0)}{|(f\circ\gamma_1)'(0)|\,|(f\circ\gamma_2)'(0)|} = \frac{|f'(z_0)|^2\,\gamma_1'(0)\cdot\gamma_2'(0)}{\left(|f'(z_0)||\gamma_1'(0)|\right)\left(|f'(z_0)||\gamma_2'(0)|\right)}$$

$$= \frac{\gamma_1'(0)\cdot\gamma_2'(0)}{|\gamma_1'(0)|\,|\gamma_2'(0)|} = \cos\theta_z$$



$$\boxed{\text{Def}^{\underline{n}}/\text{ A mapping from } \mathbb{C} \longrightarrow \mathbb{C} \text{ which preserves oriented angles between curves is called a conformal mapping.}}$$

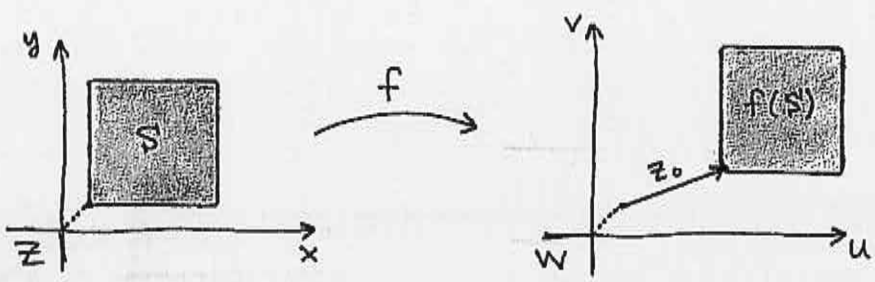
the calculations of the past two pages show complex-diff. at z_0 $\Rightarrow$ conformality at z_0. Exception to this rule is when $f'(z_0)=0$.

2.5 transformations and mappings

The examples given in this section are by no means comprehensive. Mostly this section is just for fun. Notice that most of the transformations are given by functions with the exception of the square root transformation. The transformation $z \to w = z^{1/2}$ is called a **multiply-valued function.** We could say it is a 1 to 2 function, technically this means it is not a function in the strict sense of the term common to modern mathematics. We ought to say it is a **relation**. However, it is customary to refer to such relations as multiply-valued functions. We begin with a few simple transformations: in each case we picture the domain and range as separate complex planes. The domain is called the z-plane whereas the range is in the w-plane.

2.5.1 translations

E37 **Example 2.5.1.** . . Let $f(z) = z + z_o$. Then if $S \subseteq \mathbb{C}_z$ we'll find $f(S) = z_o + S \subseteq \mathbb{C}_w$

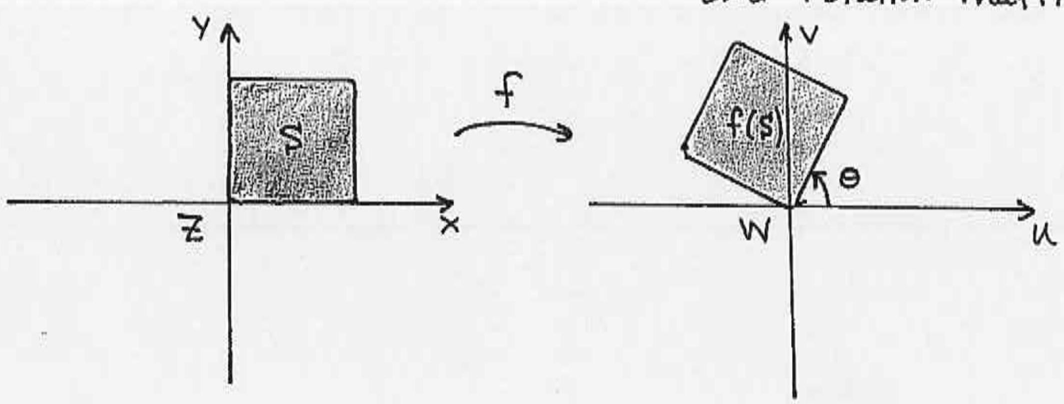


E38

2.5.2 rotations

Example 2.5.2. . . Let $f(z) = e^{i\theta} z$. Note that this is same as $f(x+iy) = (\cos\theta + i\sin\theta)(x+iy) = \cos\theta x - \sin\theta y + i(\sin\theta x + \cos\theta y)$

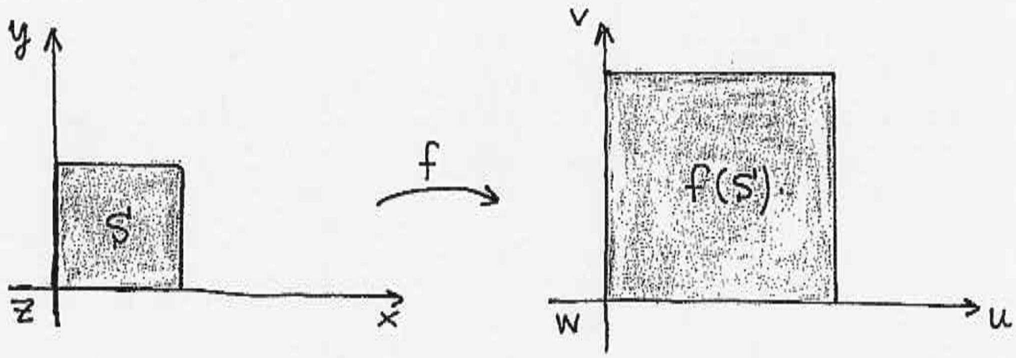
$$\Rightarrow \quad f(x,y) = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{2\times 2 \text{ rotation matrix.}} \begin{bmatrix} x \\ y \end{bmatrix}$$



E39

2.5.3 magnifications

Example 2.5.3. . . Let $f(z) = cz$ for some $c \in \mathbb{R}$.



($c > 1$ magnifies whereas $c < 1$ shrinks shapes)

2.5.4 linear mappings

E40 **Example 2.5.4.** . . $f(z) = mz + b$ is actually an affine mapping since $f(0) = b \neq 0$ generally speaking. Now, $m \in \mathbb{C}$ can be written in polar form as $m = ce^{i\beta}$ thus

$$f(z) = ce^{i\beta}z + b$$

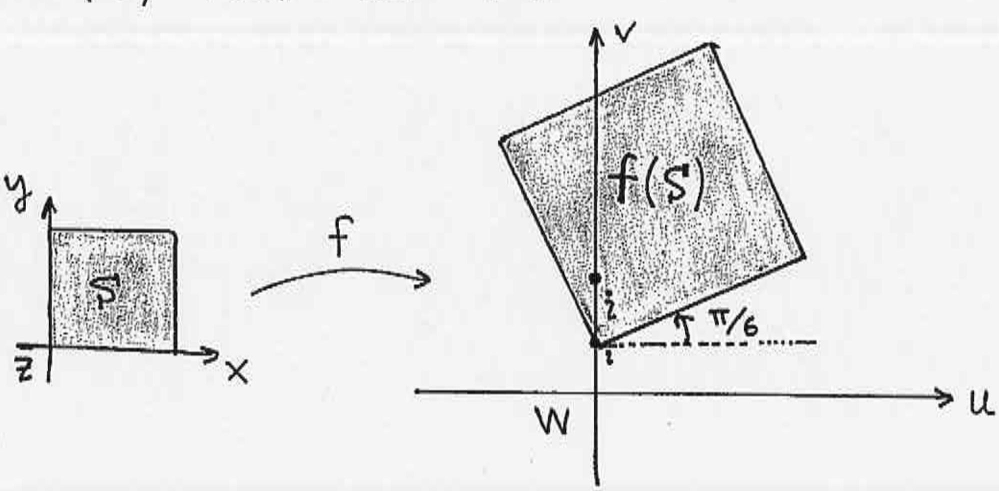
$$\Rightarrow f = (T_b \circ M_c \circ R_\beta)(z)$$

$T_b(z) = z + b$: translation.

$M_c(z) = cz$: magnification.

$R_\beta(z) = e^{i\beta}z$: rotation.

For example, $f(z) = 2e^{i\pi/6} + i$



If $|m| = 1$ then $f(z) = mz + b$ gives rigid motion on plane.

2.5.5 the $w = z^2$ mapping

Example 2.5.5. . .

E41 $W = z^2 := (x+iy)^2 = x^2 - y^2 + 2ixy = u + iv = f(z)$

This gives $u = x^2 - y^2$ and $v = 2xy$.

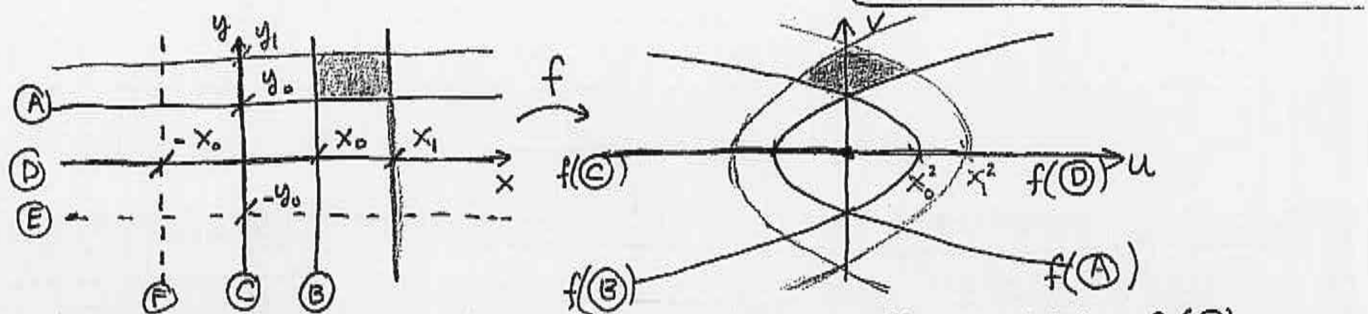
① $x = x_o$ maps to $u = x_o^2 - y^2$ and $v = 2x_o y$

Hence $y = v/2x_o \Rightarrow \underline{u = x_o^2 - v^2/4x_o^2}$ sideways parabola opens leftward, has v-intercept $u = x_o^2$.

② $y = y_o$ maps to $u = x^2 - y_o^2$ and $v = 2xy_o$

Hence $x = v/2y_o \Rightarrow \underline{u = v^2/4y_o^2 - y_o^2}$ sideways parabola opens rightward, has v-intercept $u = -y_o^2$

Of course $x = 0$ and $y = 0$ are special cases



$f(A) = f(E)$ & $f(B) = f(F)$.

2.5.6 the $w = z^{1/2}$ mapping

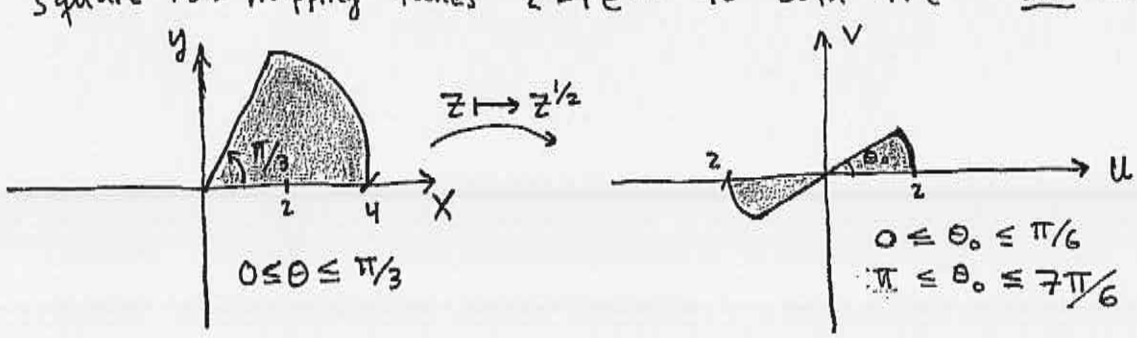
Example 2.5.6. . .

E42 $W = z^{1/2} = \{ z_o \in \mathbb{C} \mid z_o^2 = z \}$

$$f(z) = f(re^{i\theta}) = \{ z_o \in \mathbb{C} \mid r_o e^{i\theta_o} = z_o,\ r_o^2 e^{2i\theta_o} = re^{i\theta} \}$$
$$= \{ r_o e^{i\theta_o} \mid r_o^2 = r,\ 2\theta_o = \theta + 2\pi k,\ k \in \mathbb{Z} \}$$
$$= \{ \sqrt{r}\, e^{i\theta_o} \mid \theta_o = \theta/2 \pm \pi k,\ k \in \mathbb{Z} \}$$
$$= \{ \sqrt{r}\, e^{i\theta/2},\ \sqrt{r}\, e^{i(\theta/2 + \pi)} \}$$
$$= \{ \sqrt{r}\, e^{i\theta/2},\ -\sqrt{r}\, e^{i\theta/2} \}$$

$\underline{e^{i\pi} = -1}$.

The square root mapping takes $z = re^{i\theta}$ to both $\sqrt{r}e^{i\theta/2}$ and $-\sqrt{r}e^{i\theta/2}$.

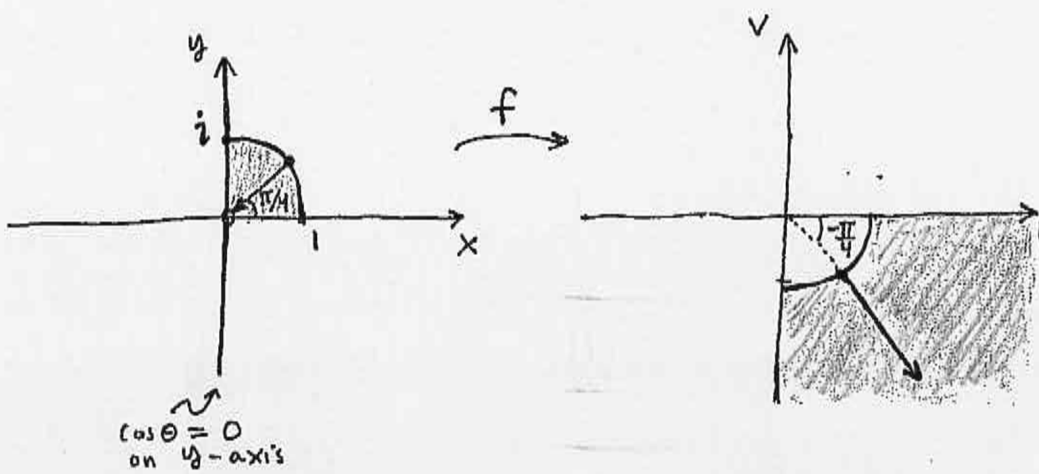


2.5.7 reciprocal mapping

E43 **Example 2.5.7.** . . .

Let $f(z) = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2+y^2} = u + iv$ $\longrightarrow u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

Polar coordinates nice here, $f(r,\theta) = \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$.
This means $u = \cos\theta/r$ and $v = -\sin\theta/r$ we can eliminate r w/o much trouble; $v/u = -\tan\theta$
Thus, for $r \neq 0$ and $\cos\theta \neq 0$ we have $\underline{v = -\tan\theta\, u}$.



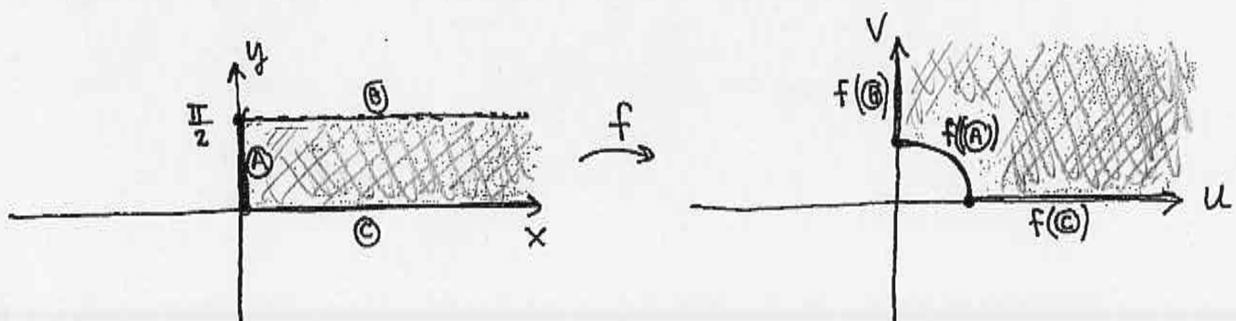
2.5.8 exponential mapping

E44 **Example 2.5.8.** . . .

$$f(z) = e^z = e^{x+iy} = e^x\cos y + ie^x\sin y = u + iv$$

We have $v = e^x\sin y$ and $u = e^x\cos y \Rightarrow \frac{v}{u} = \sin y/\cos y$

Thus, $v = (\tan y)u$ for $\cos y \neq 0$. Let $w = e^z$, note that $|w| = |e^z| = |e^x e^{iy}| = e^x$ thus $|w| \neq 0$ whereas $-\infty < x \leq 0$ maps to $0 < |w| < e^0 = 1$ and $x \geq 0$ maps to $1 \leq |w| < \infty$.



- If $S = \{(x,y) \mid x \in \mathbb{R} \text{ and } y_0 \leq y \leq y_0 + 2\pi\}$ then $f(S) = \mathbb{C} - \{0\}$.
- The exponential is 1-1 with respect to a 2π-width horizontal strip.

2.6 branch cuts

The inverse mappings of $w = z^n$ and $w = e^z$ are $w = z^{1/n}$ or $w = log(z)$. Technically these are not functions since the mappings $w = z^n$ and $w = e^z$ are not injective. If we cut down the domain of $w = z^n$ or $w = e^z$ then we can gain injectivity. The process of selecting just one of the many values of a multiply-valued function is called a **branch cut**. If a particular point is common to all the branch cuts for a particular mapping then the point is called a **branch point**. I don't attempt a general definition here. We'll see how the branch cuts work for the root and logarithm in this section.

2.6.1 the principal root functions

$$(re^{i\theta})^{1/n} = \{\underbrace{\sqrt[n]{r}\, e^{i\theta/n}}_{c_o}, \underbrace{\sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{2\pi}{n})}}_{c_o\omega_n}, \dots, \underbrace{\sqrt[n]{r}\, e^{i(\frac{\theta}{n}+\frac{2\pi(n-1)}{n})}}_{c_o\omega_n^{n-1}}\}$$

$f(z) = z^n$ is not injective on all of $\mathbb{C}$, we need to restrict the dom(f) to a sector. Consider

$$f(z_1) = f(z_2)$$
$$z_1^n = z_2^n$$
$$(r_1e^{i\theta_1})^n = (r_2e^{i\theta_2})^n$$
$$r_1^n e^{i\theta_1 n} = r_2^n e^{i\theta_2 n} \Rightarrow n\theta_1 = n\theta_2 + 2\pi k \text{ for } k \in \mathbb{Z}.$$

If we restrict θ_1, θ_2 to the range $(\frac{2\pi}{n}k, \frac{2\pi}{n}(k+1))$ then $\frac{2\pi k}{n} < \theta_1, \theta_2 < \frac{2\pi(k+1)}{n}$ thus,

$$n\theta_1 = n\theta_2 + 2\pi k \longrightarrow \theta_1 = \theta_2 + \frac{2\pi k}{n}$$

If $k \geq 1$ and $n > 1$ then we'd have

$$\frac{2\pi k}{n} < \theta_1, \theta_2 < \frac{2\pi k}{n} + \frac{2\pi}{n} \quad \text{and} \quad \theta_1 = \theta_2 + \frac{2\pi k}{n}$$
$$\hookrightarrow \theta_1 - \theta_2 = \frac{2\pi k}{n} \geq \frac{2\pi}{n}$$

But, this contradicts the inequality above hence $k = 0$ and $\theta_1 = \theta_2$.

$$\boxed{\text{Observation: } f(z) = z^n \text{ is injective on any sector with } \theta_o \leq \theta < \theta_o + \frac{2\pi}{n}}$$

A branch cut of $z^{1/n}$ is a selection of a single root from the set of outputs. This makes the branch cut a local inverse for $f(z) = z^n$. A branch cut makes a multiply-valued map into a function.

E45 **Example:** $f(z) = \{ w \in z^{1/2} \mid 0 \leq \text{Arg}(w) \leq \pi \}$, $\text{dom}(f) = \{ z \in \mathbb{C} \mid \theta \neq \pi$

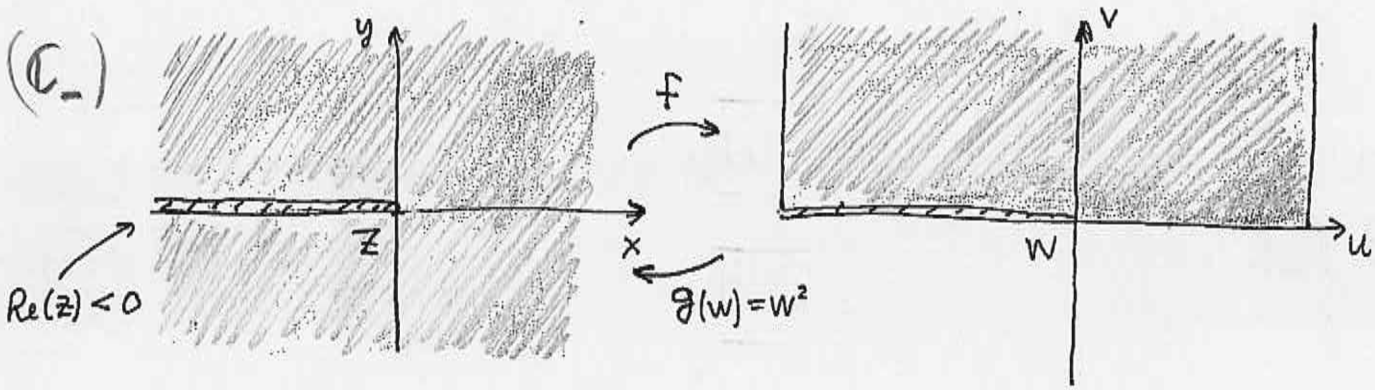
$f(1) = \{ w \in 1^{1/2} \mid 0 \leq \text{Arg}(w) \leq \pi \}$, $1^{1/2} = \{1, -1\}$

$\text{Arg}(1) = 0$, $\text{Arg}(-1) = \pi$

$\Rightarrow \underline{f(1) = 1}$.

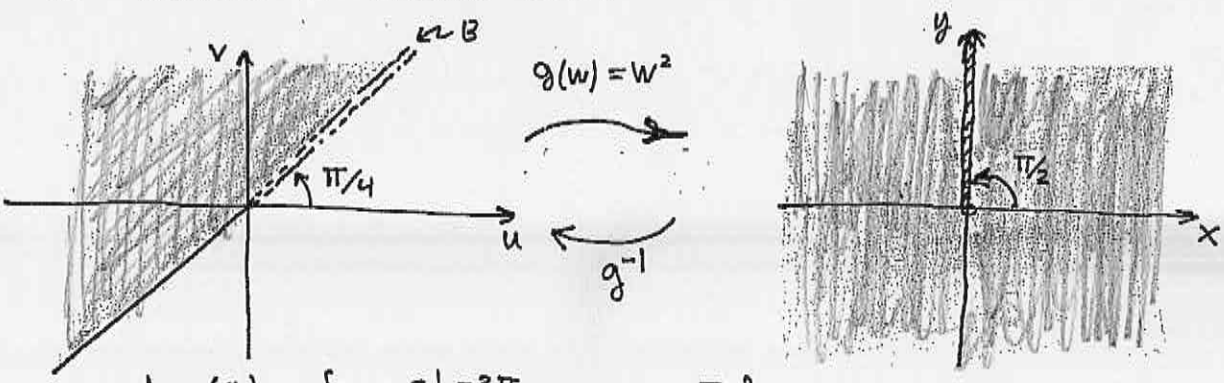
$$\begin{aligned} f(1+i) &= \{ w \in (1+i)^{1/2} \mid 0 \leq \text{Arg}\, w < \pi \} \\ &= \left\{ w \in \left\{ \tfrac{1+i}{\sqrt{2}}, \tfrac{-1-i}{\sqrt{2}} \right\} \;\middle|\; 0 \leq \text{Arg}\, w < \pi \right\} \\ &= \frac{1+i}{\sqrt{2}} \quad \text{since} \quad \text{Arg}\left(\frac{1+i}{\sqrt{2}}\right) = \pi/4 \end{aligned}$$

You can see that f is single valued because we have selected just <u>one</u> of the two-values for $z^{1/2}$ relative to the set $\text{dom}(f) = \{ z \in \mathbb{C} \mid z \neq x < 0,\ z = x+iy \}$



Let $g(w) = w^2$ for $0 \leq \text{Arg}(w) < \pi$ then we see that $g = f^{-1}$ and $f^{-1} = g$. In other words, f is a <u>local</u> inverse for the squared function $h(z) = z^2$.

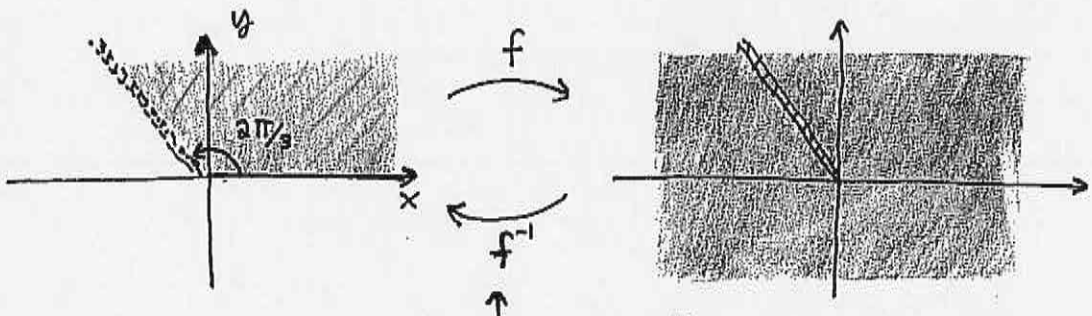
There are many other branches for $z^{1/2}$. If we restrict $h(w) = w^2$ to any half-plane then h is injective onto $\mathbb{C}$ modulo a branch cut.



$\text{dom}(g) = \{ w \in \mathbb{C} \mid -\frac{3\pi}{4} \leq \arg(z) < \frac{\pi}{4} \}$

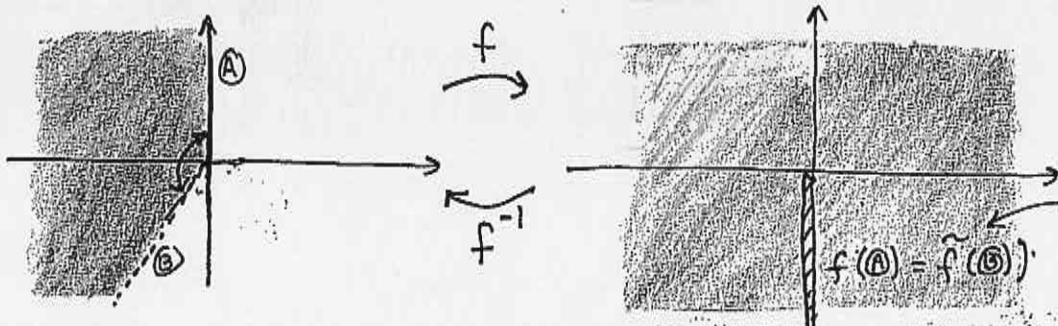
E46 **Example**: Let $f(z) = z^3$ with $\text{dom}(f) = \{z \in \mathbb{C} \mid 0 \leq \text{Arg}(z) < 2\pi/3\}$

then $f^{-1}(w) = \{\underbrace{z \in \text{dom}(f)}_{\substack{\text{this selects the cube-root} \\ \text{with } 0 \leq \text{Arg}(z) < 2\pi/3}} \mid z^3 = w\}$



↑ branch of $z^{1/3}$.

E47 **Example**: Let $f(z) = z^3$ with $\text{dom}(f) = \{z \in \mathbb{C} \mid \frac{\pi}{2} \leq \text{Arg}(z) < \frac{\pi}{2} + \frac{2\pi}{3}\}$



note B not in domain of f. $\tilde{f}(z) = z^3 \ \forall z$.

Note $f(i) = i^3 = -i$ and

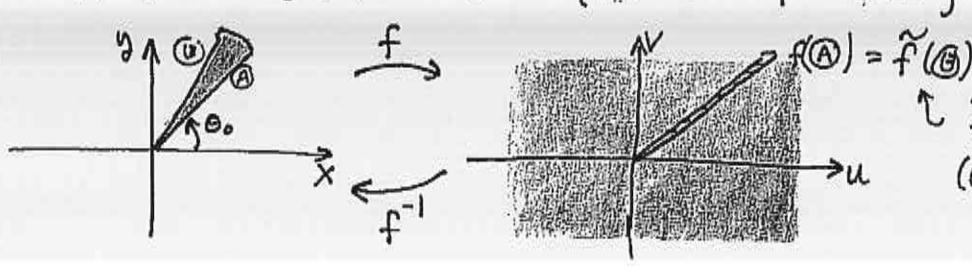
E48 **Remark**: If we restrict $f(z) = z^n$ to a domain,

$$\text{dom } f = \{z \in \mathbb{C} \mid \theta_o \leq \arg(z) < \theta_o + \tfrac{2\pi}{n}\}$$

then f will be one-one and

$$f^{-1}(w) = \{z \in \mathbb{C} \mid \theta_o \leq \arg(z) < \theta_o + \tfrac{2\pi}{n}\}$$

with $\text{dom}(f^{-1}) = \text{range}(f) = \mathbb{C} - \{re^{in\theta_o} \mid r \leq 0\}$

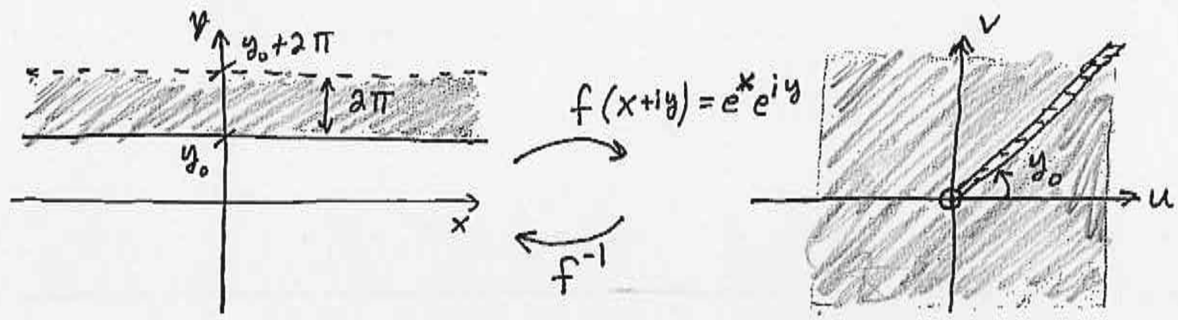


$\tilde{f}(z) = z^n \ \forall z \in \mathbb{C}$.

(can't use f on B)

2.6.2 logarithms

We defined $e^z = e^{x+iy} = e^x\cos(y) + ie^x\sin(y)$ $\forall z \in \mathbb{C}$. Notice $e^{z+2\pi i m} = e^z$ for all $m \in \mathbb{Z}$. Therefore, if $f(z) = e^z$ then we must choose dom(f) as a horizontal strip with width 2π if we want f to be injective.



The inverse functions for particular restrictions of the $z \longmapsto e^z$ function are called <u>logarithms</u>. We can define a <u>multiply-valued function</u>

$$\log(z) = \{ w \in \mathbb{C} \mid e^w = z,\ z \neq 0 \}$$

Polars,

$$\begin{aligned} \log(re^{i\theta}) &= \{ w \in \mathbb{C} \mid e^w = re^{i\theta},\ r \neq 0 \} \\ &= \{ u+iv \in \mathbb{C} \mid e^u e^{iv} = re^{i\theta},\ r \neq 0 \} \\ &= \{ u+iv \in \mathbb{C} \mid e^u = r,\ v = \theta + 2\pi k,\ k \in \mathbb{Z} \} \end{aligned}$$

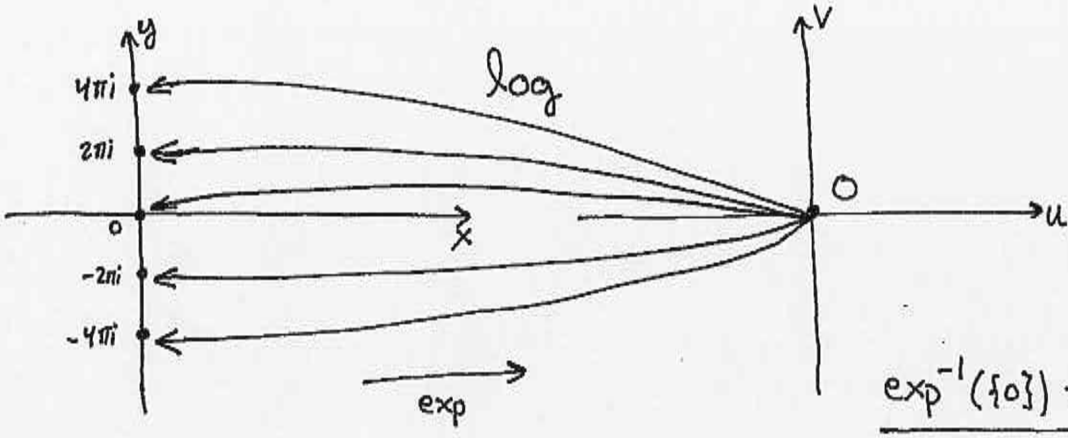
$$\therefore\ \underline{\log(re^{i\theta}) = \{ \ln(r) + i(\theta + 2\pi k) \mid k \in \mathbb{Z},\ r \neq 0 \}}.$$

Equivalently,

$$\log(z) = \underbrace{\ln|z| + i\arg(z)}_{\text{this is a set.}}$$

E49

Example: $\log(1) = \{ \ln(1) + i\theta \mid \theta \in \arg(1) \} = \{ 2\pi k i \mid k \in \mathbb{Z} \}.$



$\exp^{-1}(\{0\}) = \log(1)$.

E50

Example:

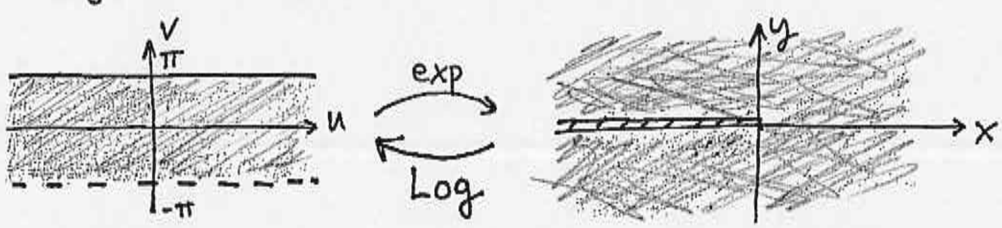
$$\log(4+5i) = \log\left(\sqrt{41}\, e^{i\tan^{-1}(5/4)}\right) = \left\{ \ln\sqrt{41} + i\left(\tan^{-1}(5/4) + 2\pi k\right) \;\middle|\; k \in \mathbb{Z} \right\}$$

$\boxed{\text{Def}^{n}/\ \ \mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z) \ \text{ for } z \in \mathbb{C} \ \& \ z \neq 0}$

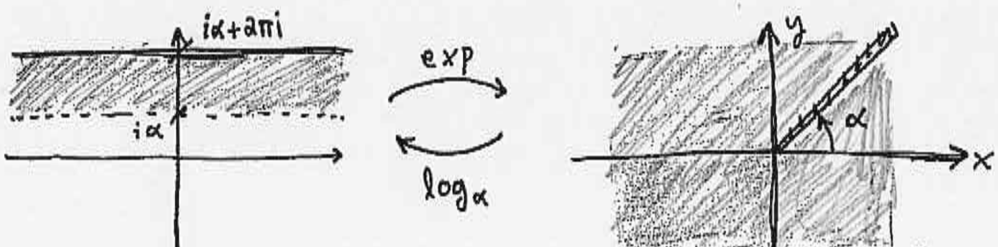
Example:

E51 $\mathrm{Log}(4+5i) = \ln\sqrt{41} + i\tan^{-1}(5/4)$.

the Log is the principal logarthim, it is a branch of log.



For Log the branch is along the negative x-axis.



$\boxed{\text{Def}^{n}/\ \ \log_\alpha(z) = \ln|z| + i\theta \ \text{ where } \theta \in \arg(z) \text{ and } \alpha < \theta \leq \alpha + 2\pi}$

(this could be presented much earlier)

3.3.2 Cauchy Riemann equations in polar coordinates

If we use polar coordinates to rewrite f as follows:

$$f(x(r,\theta),y(r,\theta)) = u(x(r,\theta),y(r,\theta)) + iv(x(r,\theta),y(r,\theta))$$

we use shorthands $F(r,\theta) = f(x(r,\theta),y(r,\theta))$ and $U(r,\theta) = u(x(r,\theta),y(r,\theta))$ and $V(r,\theta) = v(x(r,\theta),y(r,\theta))$. We derive the CR-equations in polar coordinates via the chain rule from multivariate calculus,

$$U_r = x_r u_x + y_r u_y = \cos(\theta)u_x + \sin(\theta)u_y \quad \text{and} \quad U_\theta = x_\theta u_x + y_\theta u_y = -r\sin(\theta)u_x + r\cos(\theta)u_y$$

Likewise,

$$V_r = x_r v_x + y_r v_y = \cos(\theta)v_x + \sin(\theta)v_y \quad \text{and} \quad V_\theta = x_\theta v_x + y_\theta v_y = -r\sin(\theta)v_x + r\cos(\theta)v_y$$

We can write these in matrix notation as follows:

$$\begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}\begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Multiply these by the inverse matrix: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{bmatrix}^{-1} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix}$ to find

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \frac{1}{r}\begin{bmatrix} r\cos(\theta) & -\sin(\theta) \\ r\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} U_r \\ U_\theta \end{bmatrix} = \begin{bmatrix} \cos(\theta)U_r - \frac{1}{r}\sin(\theta)U_\theta \\ \sin(\theta)U_r + \frac{1}{r}\cos(\theta)U_\theta \end{bmatrix}$$

A similar calculation holds for V. To summarize:

$$\boxed{u_x = \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta} \qquad \boxed{v_x = \cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta}$$

$$\boxed{u_y = \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta} \qquad \boxed{v_y = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta}$$

Another way to derive these would be to just apply the chain-rule directly to u_x,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial u}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial u}{\partial \theta}$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$. I leave it to the reader to show you get the same formulas from that approach. The CR-equation $u_x = v_y$ yields:

$$(A.) \quad \cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta = \sin(\theta)V_r + \tfrac{1}{r}\cos(\theta)V_\theta$$

Likewise the CR-equation $u_y = -v_x$ yields:

$$(B.) \quad \sin(\theta)U_r + \tfrac{1}{r}\cos(\theta)U_\theta = -\cos(\theta)V_r + \tfrac{1}{r}\sin(\theta)V_\theta$$

Multiply (A.) by $r\sin(\theta)$ and $(B.)$ by $r\cos(\theta)$ and subtract (A.) from (B.):

$$U_\theta = -rV_r$$

Likewise multiply (A.) by $r\cos(\theta)$ and $(B.)$ by $r\sin(\theta)$ and add (A.) and (B.):

$$rU_r = V_\theta$$

Finally, recall that $z = re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$ hence

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= (\cos(\theta)U_r - \tfrac{1}{r}\sin(\theta)U_\theta) + i(\cos(\theta)V_r - \tfrac{1}{r}\sin(\theta)V_\theta) \\ &= (\cos(\theta)U_r + \sin(\theta)V_r) + i(\cos(\theta)V_r - \sin(\theta)U_r) \\ &= (\cos(\theta) - i\sin(\theta))U_r + i(\cos(\theta) - i\sin(\theta))V_r \\ &= e^{-i\theta}(U_r + iV_r) \end{aligned}$$

Theorem 3.3.11.

If $f(re^{i\theta}) = U(r,\theta) + iV(r,\theta)$ is a complex function written in polar coordinates r, θ then the Cauchy Riemann equations are written $U_\theta = -rV_r$ and $rU_r = V_\theta$. If $f'(z_o)$ exists then the CR-equations in polar coordinates hold. Likewise, if the CR-equations hold in polar coordinates and all the polar component functions and their partial derivatives with respect to r, θ are continuous on an open disk about z_o then $f'(z_o)$ exists and $f'(z) = \underbrace{e^{-i\theta}(U_r + iV_r)}$.

$$\frac{df}{dz} = e^{-i\theta}\frac{\partial f}{\partial r}$$ (neat!)

Example 3.3.12. . .

E52

$$f(re^{i\theta}) = re^{i\theta} = r\cos\theta + ir\sin\theta$$

$U(r,\theta) = r\cos\theta$ & $V(r,\theta) = r\sin\theta$

$U_\theta = -r\sin\theta$ & $V_\theta = r\cos\theta$

$U_r = \cos\theta$ & $V_r = \sin\theta$

$\Big\}$ $U_\theta = -rV_r = -r\sin\theta$. $rU_r = V_\theta = r\cos\theta$.

$$\frac{df}{dz} = e^{-i\theta}(\cos\theta + i\sin\theta) = e^{-i\theta}e^{i\theta} = 1.$$ (is this surprising?)

Example 3.3.13. . .

E53

$$f(r,\theta) = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r} = \underbrace{\frac{1}{r}\cos\theta}_{U} - \underbrace{\frac{i}{r}\sin\theta}_{V}$$

You can check that $U_\theta = -rV_r$ and $rU_r = V_\theta$.

$$\begin{aligned} \frac{df}{dz} &= e^{-i\theta}\left(\frac{\partial U}{\partial r} + i\frac{\partial V}{\partial r}\right) = e^{-i\theta}\left(\frac{-1}{r^2}\cos\theta + \frac{i}{r^2}\sin\theta\right) \\ &= -e^{-i\theta}(\cos\theta - i\sin\theta)\frac{1}{r^2} \\ &= -e^{-i\theta}e^{-i\theta}\frac{1}{r^2} \\ &= \frac{-1}{r^2e^{2i\theta}} = \frac{-1}{(re^{i\theta})^2} \end{aligned}$$

(is this surprising?)