Complex Integration:

Integration of functions to and from a complex domain can be described by a Riemann-type sum. See §4.2 of Saff and Snider if curious. We'll use the known integral on $\mathbb{R}$ to formulate integrals of $\mathbb{R} \xrightarrow{f} \mathbb{C}$ and then $\mathbb{C} \xrightarrow{f} \mathbb{C}$.

Def^{n}/ Suppose $u + iv : [a,b] \to \mathbb{R}$ with u, v integrable on $[a,b]$,

$$\int_a^b \big(u(t) + iv(t)\big)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

$\boxed{E54}$

$$\int_0^1 (t + it)\,dt = \int_0^1 t\,dt + i\int_0^1 t\,dt$$
$$= \tfrac{1}{2}t^2\Big|_0^1 + \tfrac{i}{2}t^2\Big|_0^1$$
$$= \underline{\tfrac{1}{2}(1+i)}.$$

Theorem: let $f, g : [a,b] \to \mathbb{C}$ and $c \in \mathbb{C}$,

1.) $\displaystyle\int_a^b \big(f(t) + g(t)\big)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

2.) $\displaystyle\int_a^b c f(t)\,dt = c\int_a^b f(t)\,dt$

3.) $\displaystyle\int_a^b \frac{df}{dt}\,dt = f(b) - f(a)$

Proof: let $f = u + iv$ then $\frac{df}{dt} = \frac{du}{dt} + i\frac{dv}{dt}$ thus

$$\int_a^b \frac{df}{dt}\,dt = \int_a^b \frac{du}{dt}\,dt + i\int_a^b \frac{dv}{dt} \quad : \text{def}^{n} \text{ of integral.}$$
$$= u(b) - u(a) + i\cdot\big(v(b) - v(a)\big) \quad : \text{FTC for real \& imaginary parts.}$$
$$= f(b) - f(a).$$

The proofs of 1.) and 2.) are not difficult, I leave to you. //

Remark: $\boxed{E54}$ is easier if we factor out $(1+i)$ sina

$$\int_0^1 (t + it)\,dt = \int_0^1 (1+i)t\,dt = (1+i)\int_0^1 t\,dt = (1+i)\tfrac{1}{2}.$$

$\boxed{\text{ESS}}$ $$\int e^{(a+ib)t}\,dt = \frac{1}{a+ib}e^{(a+ib)t} + c_1 + ic_2$$
$$= \frac{a-ib}{a^2+b^2}e^{at}(\cos bt + i\sin bt) + c_1 + ic_2$$
$$= \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt) + c_1 + i\left[\frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt) + c_2\right]$$

But, $\int e^{(a+ib)t}dt = \int e^{at}\cos bt\,dt + i\int e^{at}\sin bt$ hence, equating the real and imaginary parts we find:

$$\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt) + c_1$$
$$\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt) + c_2 .$$

Compare with IBP twice in calculus II.

//

Integration for functions of a real domain is taken over closed intervals $[a,b]$ or in the improper case $(-\infty,\infty)$, $(-\infty, a]$, $[a,\infty)$. As we consider $\mathbb{C} \xrightarrow{f} \mathbb{C}$ the question of what to integrate over arises. The natural object is a contour. I pause to introduce a few geometric / topological terms.

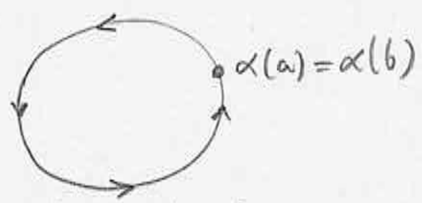
Def$^{\text{n}}$/ A path is a continuous map $\alpha : [a,b] \longrightarrow \mathbb{C}$. We say $\alpha(a)$ is starting point and $\alpha(b)$ is the endpoint. The path is smooth if it is continuously differentiable. If $\exists$ a partition $a = a_0 < a_1 < a_2 < \cdots < a_n = b$ such that
$$\alpha_\nu = \alpha\big|_{[a_\nu, a_{\nu+1}]} \qquad 0 \le \nu < n$$
are smooth paths then α is a piecewise smooth path. If $\alpha(a) = \alpha(b)$ then α is a closed path. If $\alpha|_{[a,b)}$ is 1-1 then α is a simple path.

- notice, I use the term path to reflect a particular choice of parametrizing a subset of $\mathbb{C}$. We'll call $\alpha([a,b])$ the contour or oriented curve (my calculus III term)

<u>Warning</u>: as you study math of curves you'll notice the terms path, curve, arc, contour used somewhat interchangeably. We should mind the context.

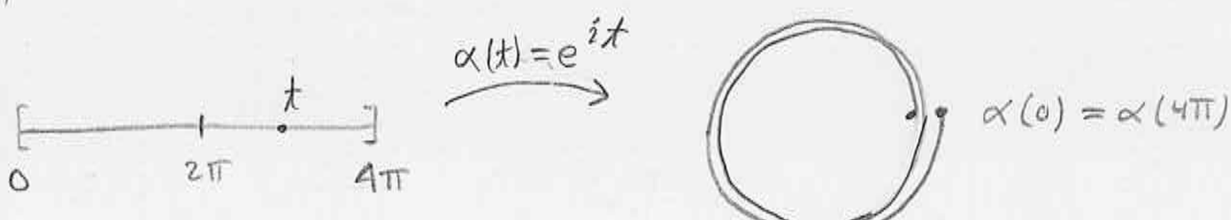
not simple, has non-terminal self-intersection

simple closed contour with CW-orientation.



simple closed contour with CCW-orientation

A path can cover a particular point-set many times:



unit-circle, covered twice in the CCW fashion.

If we used mere point-sets then capturing the idea above would be harder to communicate, but with paths we could define multiple covering in terms of the number of elements in each fiber of the path:

Defn/ A closed path has <u>winding</u> <u>number</u> <u>k</u> if the path $\alpha : [a,b] \to C \subseteq \mathbb{C}$ has k-elements in the fiber of each point in C; for <u>most</u> $z \in C$, $\#\{\alpha^{-1}\{z\}\} = k \in \mathbb{N}$. By <u>most</u> we simply allow the terminal points to have $k+1$.

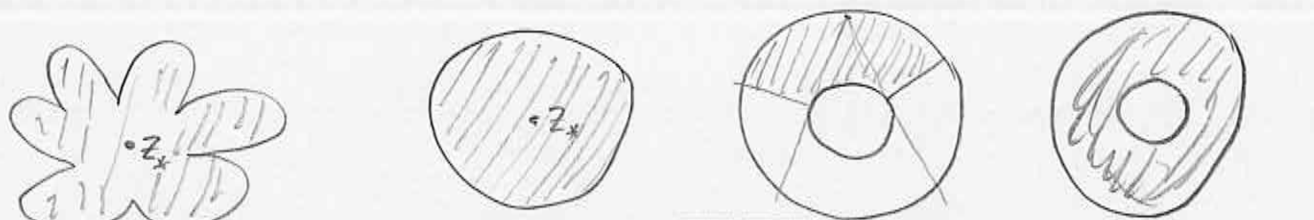
The unit-circle above has winding # 2. Of course, you can just as well give $S_1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$ winding # 42 etc... you just need the appropriate path to cover it 42 times.

SIMPLY CONNECTED, ARC-CONNECTED, etc..

Previously we said path connectedness of $S \subseteq \mathbb{C}$ means $\forall p, q \in S$ there exists α such that $p \xrightarrow{\alpha} q$. The term polygonal-path connected is less common, but has the obvious meaning. Freitag uses arcwise connected (II.2.1 pg. 77) to indicate all pairs of points in S can be connected by a piecewise smooth path inside S. A domain in Chapter II of Freitag is an arcwise connected open set. Usually authors want the concept to allow path-connected, but they chop-off technical subtleties by restricting the class of paths used to study connectivity. For Saff/Snider the use polygonal-paths. Freitag does something similar with his use of star-shaped

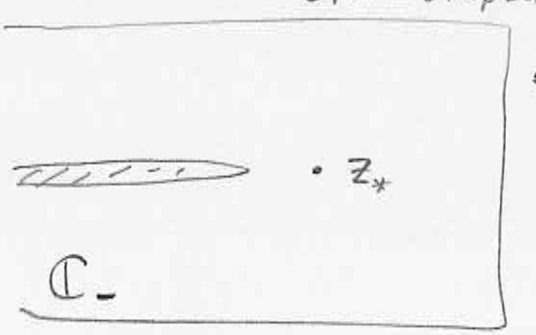
> **Def^n/** A star-shaped domain is an open set $D \subseteq \mathbb{C}$ with the following property: $\exists$ a point $z_* \in D$ such that each point $z \in D$ the whole line segment joining z_* and z is contained in D; $\{ z_* + t(z - z_*) \mid t \in [0,1] \} \subset D$. Any such point z_* is called a star-center of D

I'll recite Freitag's Examples (pg. 83)



star-shaped

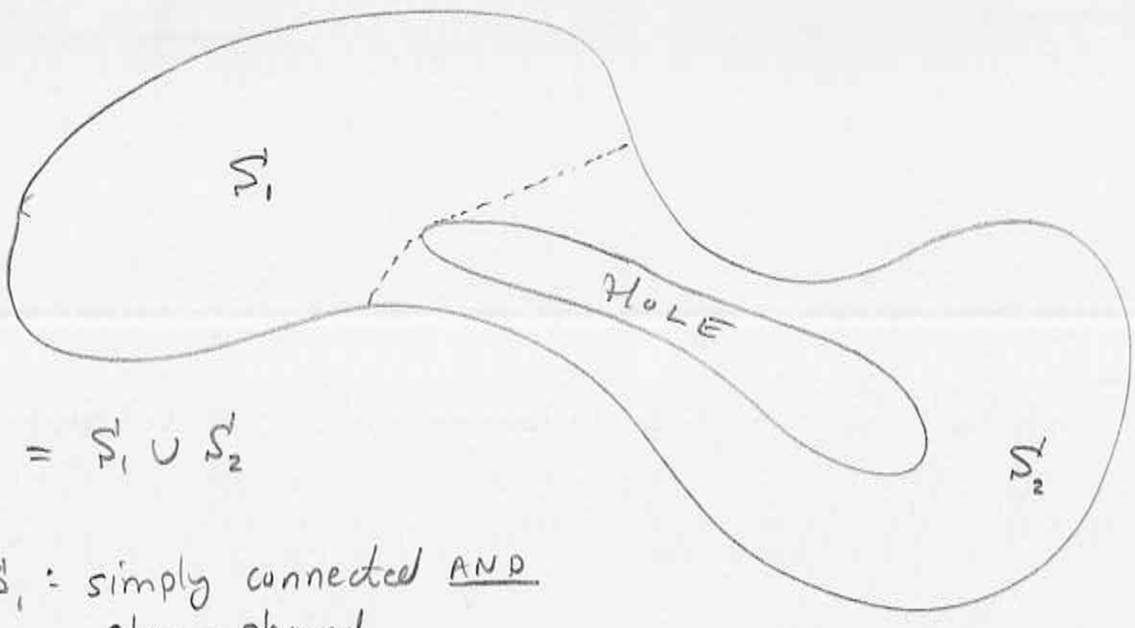
not star-shaped



Def^{n} A set $S \subseteq \mathbb{C}$ is <u>simply</u> <u>connected</u> if S allows <u>all</u> simple closed paths to continuously deform (inside S) to a point in S.

Other slogans include: it is a connected set with no holes. See pg. 233 - 244 of Freitag for a more technical discussion. For our purposes, I may at times <u>claim</u> a result for <u>simply</u> <u>connected</u> sets while I only prove it for <u>star-shaped</u>. Churchill uses "simply-connected" since at the end of the course that is the term you want to own. (I don't mean to claim we'll carefully study Homotopy ← study of deformation of loops to "see" holes, that is not needed)

<u>To summarize</u>: as a whole the set S below is connected, but it is not simply connected. However, we can cut it into two simply connected parts



$S = S_1 \cup S_2$

S_1 : simply connected <u>AND</u> star-shaped.

S_2 : simply connected, but not star-shaped.

We could cut-up S_2 into star-shaped parts. Moreover, (while I have no intention to prove this!) you can cut-up any simply connected set into star-shaped parts. Thus, the use of star-shaped suffices to lay foundation.

Def^n/ Let $f = u+iv$ and $C \subseteq \mathbb{C}$ a smooth curve (aka arc)

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\,\gamma'(t)dt$$

where $\gamma : [a,b] \longrightarrow C$ parametrizes C. Moreover, if C is a piecewise-smooth curve (aka a contour) with parametrization $\gamma : [t_0, t_n] \longrightarrow C$ such that $\gamma_j : [t_{j-1}, t_j] \to C_j$ where $C = C_1 \cup C_2 \cup \dots \cup C_n$ and $\gamma_j(t_j) = \gamma_{j+1}(t_j)$ for $j = 1, 2, \dots, n-1$ then

$$\int_C f(z)dz = \sum_{j=1}^{n} \int_{C_j} f(z)dz$$

- In our 1st run through, I only defined $\int_C f(z)dz$ over a single arc. When we study a contour formed from pasting together finitely many arcs we simply add the integrals from each arc.

Remark: this integral has much in common with the line-integral from calculus III; $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\,dt$. In particular, (see pgs. 88 $\to$ 90 for explicit connection of contour & line integrals.)

Properties of the contour integral

1.) the definition, while given in terms of a particular parametrization, has no dependence on the choice of path to cover the given curve.

2.) $\int_C f\,dz = -\int_{-C} f\,dz$: reversing orientation of curve changes sign of the integral.

3.) $\int_{C_1 \cup C_2} f\,dz = \int_{C_1} f\,dz + \int_{C_2} f\,dz$: essentially part of the definition.

4.) $\int_C (c_1 f + c_2 g)dz = c_1 \int_C f\,dz + c_2 \int_C g\,dz$: the usual linearity

5.) $\left| \int_C f(z)dz \right| \leq M\ell(C)$ given $|f(z)| \leq M \ \forall z \in C$ and $\ell(C) = \int_C |dz|$. ⇐ arclength of C.

Proof of Properties of $\int_C f(z)dz$ as stated on (84)

1.) parameter independence. I'll prove for an arc. Suppose $\alpha_1 : [a_1, b_1] \longrightarrow \mathbb{C}$ and $\alpha_2 : [a_2, b_2] \longrightarrow \mathbb{C}$ parametrize a curve $C = \alpha_1[a_1, b_1] = \alpha_2[a_2, b_2]$. We assume α_1, α_2 are at least once differentiable. Notice there exists $\sigma : [a_1, b_1] \longrightarrow [a_2, b_2]$ such that $\alpha_1(t) = \alpha_2(\sigma(t))$. Consider,

$$\begin{aligned}
\int_{a_1}^{b_1} f(\alpha_1(t)) \frac{d\alpha_1}{dt} dt &= \int_{a_1}^{b_1} f(\alpha_2(\sigma(t))) \frac{d}{dt}\Big(\alpha_2(\sigma(t))\Big) dt \\
&= \int_{a_1}^{b_1} f(\alpha_2(\sigma(t))) \frac{d\alpha_2}{dt}(\sigma(t)) \frac{d\sigma}{dt} dt \\
&= \int_{a_2}^{b_2} f(\alpha_2(u)) \frac{d\alpha_2}{du} du.
\end{aligned}$$

To obtain result for a contour we apply this result to each arc.

2.) If $\alpha : [a, b] \longrightarrow \mathbb{C}$ parametrizes C then $\alpha_- : [a, b] \longrightarrow \mathbb{C}$ parametrizes $-C$ where $\alpha_-(t) = \alpha(a + b - t)$ (you can check: $\alpha_-(a) = \alpha(b)$, $\alpha_-(b) = \alpha(a)$)

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b f(\alpha(t)) \frac{d\alpha}{dt} dt \\
&= \int_b^a f(\alpha_-(u))\left(-\frac{d\alpha_-}{du}\right)(-du) \\
&= -\int_a^b f(\alpha_-(u)) \frac{d\alpha_-}{du} du \\
&= -\int_{-C} f(z)dz
\end{aligned}$$

$$\left[\begin{aligned}
& t = a + b - u \\
& u = a + b - t, \quad du = -dt \\
& u(a) = b \text{ and } u(b) = a \\
& \alpha(t) = \alpha(a + b - u) = \alpha_-(u) \\
& \frac{d\alpha}{dt} = \alpha'(a+b-u)\frac{d(a+b-u)}{dt} \\
& \Rightarrow \alpha'(t) = -\alpha_-'(u)
\end{aligned}\right.$$

Remark: there are three minus signs which collaborate to yield the result above. Mostly both 1.) and 2.) are the chain rule together with $\mathbb{C}$-valued FTC for a $\mathbb{R}$-variable.

3.) $\int_{C_1 \cup C_2} f dz = \int_{C_1} f dz + \int_{C_2} f dz$. If C_1 & C_2 are contours then $C_1 \cup C_2$ is also a contour and this follows by properties of finite sums.

continuing proofs:

4.) to prove linearity of the contour integral we rely on the linearity of the finite sum and the integral given on 79 (in particular part 1. of theorem on 79). Let C be a contour parametrized by $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ with $\text{dom}(\gamma_j) = [t_{j-1}, t_j]$ $\forall j$,

$$\begin{aligned}
\int_C (c_1 f + c_2 g)\,dz &= \sum_{j=1}^{n} \int_{\gamma_j} (c_1 f + c_2 g)\,dz \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[c_1 f(\gamma_j(t)) + c_2\, g(\gamma_j(t))\right] \frac{d\gamma_j}{dt}\,dt \\
&= c_1 \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(\gamma_j(t)) \frac{d\gamma_j}{dt}\,dt + c_2 \sum_{j=1}^{n} g(\gamma_j(t)) \frac{d\gamma_j}{dt}\,dt \\
&= c_1 \sum_{j=1}^{n} \int_{\gamma_j} f\,dz + c_2 \sum_{j=1}^{n} \int_{\gamma_j} g\,dz \\
&= c_1 \int_C f\,dz + c_2 \int_C g\,dz.
\end{aligned}$$

5.) Suppose $|f(z)| \leq M$ $\forall z \in C$ and let $\underbrace{\ell(C) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left|\frac{d\gamma_j}{dt}\right| dt}_{\text{arclength of the contour.}}$

$$\begin{aligned}
\left|\int_C f\,dz\right| &= \left|\sum_{j=1}^{n} \int_{\gamma_j} f\,dz\right| \\
&\leq \sum_{j=1}^{n} \left|\int_{\gamma_j} f\,dz\right| \\
&= \sum_{j=1}^{n} \left|\int_{t_{j-1}}^{t_j} f(\gamma_j(t)) \frac{d\gamma_j}{dt}\,dt\right| \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left|f(\gamma_j(t)) \frac{d\gamma_j}{dt}\right| dt \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} M \left|\frac{d\gamma_j}{dt}\right| dt \\
&= M \underbrace{\sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left|\frac{d\gamma_j}{dt}\right| dt}_{\ell(C)} = M\,\ell(C).
\end{aligned}$$

Lemma

property of Riemann integral

Lemma: $\left|\int_a^b (u(t) + i v(t))\,dt\right| \leq \int_a^b |u(t) + i v(t)|\,dt$

Proof: left to reader, possibly in hwk.

Th^{m} 1 (II.1.5.5 pg. 73 Freitag)
If $f: D \longrightarrow \mathbb{C}$ is continuous, D open has <u>primitive</u> F on D (meaning $F'(z) = f(z)\ \forall z \in D$) then for any contour α in D $\int_\alpha f(z)dz = F(\alpha(b)) - F(\alpha(a))$

<u>Proof</u>: mainly from FTC for $\mathbb{R} \xrightarrow{g} \mathbb{C}$ which was given on 79 and was just the standard FTC part II twice applied. Consider $\alpha : [a,b] \longrightarrow \mathbb{C}$ a smooth arc,

$$\begin{aligned}
\int_\alpha f(z)dz &= \int_a^b f(\alpha(t)) \frac{d\alpha}{dt}\, dt \\
&= \int_a^b \frac{dF}{dz}(\alpha(t)) \frac{d\alpha}{dt}\, dt \\
&= \int_a^b \frac{d}{dt}\Big(F(\alpha(t))\Big)\, dt \\
&= F(\alpha(b)) - F(\alpha(a)).
\end{aligned}$$

Now a contour is formed by joining arcs continuously. Suppose $\gamma : [a,b] \longrightarrow \mathbb{C}$ where $\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ and $a = t_0 < t_1 < \cdots < t_n < b$ where $\mathrm{dom}(\gamma_j) = [t_{j-1}, t_j]$. By definition,

$$\begin{aligned}
\int_\gamma f(z)dz &= \sum_{j=1}^n \int_{\gamma_j} f(z)dz \\
&= \sum_{j=1}^n \Big[F(\gamma_j(t_j)) - F(\gamma_j(t_{j-1}))\Big] \\
&= \Big(F(\gamma_1(t_1)) - F(\gamma_1(t_0))\Big) + \Big(F(\gamma_2(t_2)) - F(\gamma_2(t_1))\Big) + \\
&\quad + \cdots + F(\gamma_n(t_n)) - F(\gamma_n(t_{n-1}))
\end{aligned}$$

using def^{n} of contour which glues its arcs continuously.

$$\begin{aligned}
&= \cancel{F(\gamma(t_1))} - F(\gamma(a)) + \cancel{F(\gamma(t_2))} - \cancel{F(\gamma(t_1))} + \\
&\quad + \cdots + F(\gamma(b)) - \cancel{F(\gamma(t_{n-1}))} \\
&= F(\gamma(b)) - F(\gamma(a)). \quad //
\end{aligned}$$

<u>Remark</u>: to be pickier we ought to provide inductive proof on n.

Connecting with Calculus III (I actually did this 2/7/13)

Let us begin by recalling the theory of conservative vector fields. The end of the conversation should land on the theorem below:

Thm/ TFAE for connected $D \subseteq \mathbb{R}^n$, and $\vec{F}$ continuously differentiable,

1.) $\exists \varphi$ such that $\vec{F} = \nabla \varphi$ on D.

2.) For all paths C_1, C_2 in D connecting a given pair of pts, $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$.

3.) For all closed paths C in D, $\oint_C \vec{F} \cdot d\vec{r} = 0$

If $D \subseteq \mathbb{R}^3$ is simply-connected then we obtain an additional equivalence,

4.) $\nabla \times \vec{F} = 0$ on D.

For a vector field $\vec{F} = \langle P, Q, R \rangle$ we defined

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\,dt \quad : \text{for } \underbrace{\vec{r} : [a,b] \longrightarrow C \subset \mathbb{R}^3}_{\text{path covering } C.}$$

$$= \underbrace{\int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt} \right) dt}_{\text{tempting to think of } dt\text{'s formally cancelling}} \quad : \vec{r} = \langle x, y, z \rangle \text{ so } \frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

hence the notation: $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$

Just a reminder about how we defined the line-integral. In two-dimensional case, $\vec{F} = \langle P, Q \rangle$ hence,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

In this case the condition $\nabla \times \vec{F} = 0$ applied to $\vec{F} = \langle P, Q, 0 \rangle$ yields the exactness condition

$$\boxed{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0} \quad (\star)$$

The following formal calculation can be easily verified from the def$^{\underline{n}}$ of $\int_C \vec{F}\cdot d\vec{r}$ in $\mathbb{R}^2$ and $\int_C f dz$ in $\mathbb{C}$. We'll attend to the details, but first consider, (see 90 for details)

$$\begin{aligned}
\int f dz &= \int_C (u+iv)(dx+idy) \\
&= \int_C (u\,dx - v\,dy) + i\int(v\,dx + u\,dy) \\
&= \int_C \langle u,-v\rangle \cdot d\vec{r} + i\int \langle v,u\rangle \cdot d\vec{r}
\end{aligned}$$

Thus $\int_C f(z)dz$ is naturally identified as a complex linear combination of real-line-integrals on $\mathbb{R}^2$. If we suppose <u>both</u> $\langle u,-v\rangle$ and $\langle v,u\rangle$ are conservative then we need $\underbrace{u_y = -v_x \text{ and } v_y = u_x}_{\text{CAUCHY RIEMANN!}}$ (by $*$ on 88)

Apparently the CR-eq$^{\underline{n}}$'s $\Rightarrow \exists\, \varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{R}$ such that:

$$\begin{aligned}
\int_C f(z)dz &= \int_C (\nabla\varphi_1)\cdot d\vec{r} + i\int_C (\nabla\varphi_2)\cdot d\vec{r} \\
&= \varphi_1(z_2) - \varphi_1(z_1) + i(\varphi_2(z_2) - \varphi_2(z_1)) \\
&= \underbrace{(\varphi_1 + i\varphi_2)}_{\substack{\text{complex potential for } f(z) \\ \text{aka. the "antiderivative" or "primitive"}}}(z)\Big|_{z_1}^{z_2}
\end{aligned}$$

FTC for line-integrals assuming $C: z_1 \to z_2$

$$\boxed{\begin{array}{l}\text{Claim: If } f \text{ is complex-diff. on a simply connected} \\ \text{domain } D \text{ and } u_x, u_y, v_x, v_y \text{ are continuous then } \exists\, \varphi : \mathbb{C} \to \mathbb{C} \\ \text{with } \varphi'(z) = f(z) \text{ on for all } z \in D.\end{array}}$$

<u>Proof</u>: Let $\varphi = \varphi_1 + i\varphi_2$ constructed as indicated in discussion above the claim. We have $\nabla\varphi_1 = \langle u,-v\rangle$ and $\nabla\varphi_2 = \langle v,u\rangle$

$$\Rightarrow \partial_x\varphi_1 = u,\ \partial_y\varphi_1 = -v,\ \partial_x\varphi_2 = v,\ \partial_y\varphi_2 = u.$$

Hence $\partial_x\varphi_1 = \partial_y\varphi_2$ and $\partial_y\varphi_1 = -\partial_x\varphi_2$ hence $\varphi'(z)$ exists. Moreover, $\underbrace{\text{by CR-Th}^{\underline{m}}\text{ on 54}}$, $\varphi'(z) = \partial_x\varphi_1 + i\partial_x\varphi_2 = u + iv$.

(here φ_1 plays role of "u" and φ_2 plays role of "v" on 54) on 89 I began with a formal calculation. Here's the proof the conclusion is valid, let $C = t \longmapsto (x, y)$ for $a \leq t \leq b$,

$$\int_C \langle u, -v\rangle \cdot d\vec{r} = \int_a^b \left(u \frac{dx}{dt} - v \frac{dy}{dt} \right) dt$$

$$\int_C \langle v, u\rangle \cdot d\vec{r} = \int_a^b \left(v \frac{dx}{dt} + u \frac{dy}{dt} \right) dt$$

Thus,

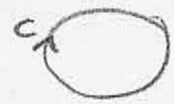
$$\begin{aligned} \int_C \langle u,-v\rangle \cdot d\vec{r} + i\int_C \langle v,u\rangle \cdot d\vec{r} &= \int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt} + iv\frac{dx}{dt} + iu\frac{dy}{dt}\right) dt \\ &= \int_a^b (u+iv)\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt \\ &= \int_C (u+iv)\, dz \end{aligned}$$

where $t \mapsto x(t) + iy(t)$ for $a \leq t \leq b$ parametrizes the curve $C \subset \mathbb{C}$.

This justifies the formal nonsense on the previous page ☺.

Continuing our thoughts from the Claim on 89 we just learned complex-diff and $f'(z)$ continuous on a simply connected domain $D \Rightarrow f(z) = \varphi'(z)$.
We can also borrow from calculus III to argue if $f'(z)$ continuous

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz \quad \& \quad \oint_C f(z)dz = 0$$



Since $f'(z)$ continuous on simply-connected domain
$\Rightarrow \langle u, -v\rangle, \langle v, u\rangle$ conservative and

$$\int_C f\,dz = \int_C \langle u, -v\rangle \cdot d\vec{r} + i\int_C \langle v, u\rangle \cdot dr$$

hence the path-independence & loop-trivialization for $\mathrm{Re}(\int f dz)$ and $\mathrm{Im}(\int f dz)$ are guaranteed from the equivalence Th^{m} on 88. All this said, we'll examine complete, independent from calc. III proofs on the pages to follow. 91 ⟶ 93

(I give proper proof of this etc.. 91 → 96)

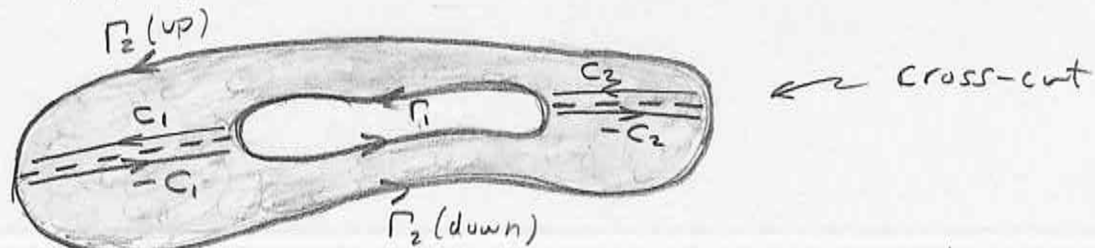
$\text{Th}^{\underline{m}}$/ If $D \subseteq \mathbb{C}$ is simply-connected and $f'(z)$ exists for each $z \in D$ then $\exists\, \varphi : U \to \mathbb{C}$ such that $\varphi'(z) = f(z)\ \forall z \in D$. Moreover, if C goes from z_1 to z_2 inside D then $\int_C f\,dz = \varphi(z_2) - \varphi(z_1)$

Proof in limited case: adding assumption of continuity of $f'(z)$ we can follow calculation of 89 & 90 to construct $\varphi = \varphi_1 + i\varphi_2$ and the $\text{Th}^{\underline{m}}$ follows.//

Deformation Th^{m}

If $f'(z)$ exists for each z in a domain D and if Γ_1 and Γ_2 are two consistently oriented loops then $\int_{\Gamma_1} f\,dz = \int_{\Gamma_2} f\,dz$

Proof: again, the $\text{Th}^{\underline{m}}$ at top is not yet proved, but taking a logical loan for a moment we proceed, Assume Γ_1, Γ_2 are CCW oriented,



Observe the cross-cuts and half-loops form loops whose interior forms a simply connected set on which $f'(z)$ exists.

$$\int_{C_1 \cup (-\Gamma_2(up)) \cup C_2 \cup (\Gamma_1(up))} f(z)\,dz = 0$$

$$\int_{-C_1 \cup \Gamma_1(down) \cup (-C_2) \cup (-\Gamma_2(down))} f(z)\,dz = 0$$

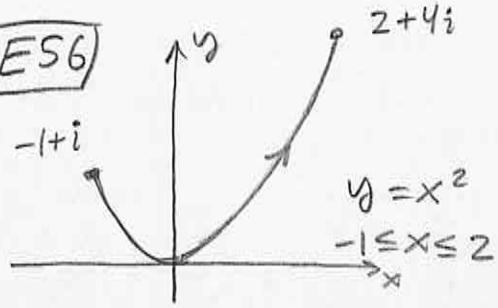
Adding these $eq^{\underline{n}}$'s and using props 2., 3.) of $\int_C f(z)\,dz$ on 84 yields

$$\int_{-\Gamma_2(up) \cup \Gamma_1(up) \cup \Gamma_1(down) \cup (-\Gamma_2(down))} f(z)\,dz = 0$$

$$\Rightarrow \int_{\Gamma_1} f(z)\,dz + \int_{-\Gamma_2} f(z)\,dz = 0 \Rightarrow \int_{\Gamma_1} f\,dz = \int_{\Gamma_2} f\,dz. //$$

Remark: the examples that follow were offered 2/11/13 initially.

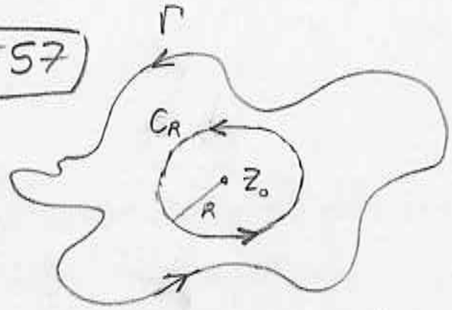
$\boxed{E56}$



$\left.\begin{array}{l} y = x^2 \\ -1 \leq x \leq 2 \end{array}\right\}$ curve C

$$\int_C z\,dz = \frac{1}{2}z^2\Big|_{-1+i}^{2+4i} = \frac{1}{2}(2+4i)^2 - \frac{1}{2}(-1+i)^2$$
$$= \frac{1}{2}\left[4+8i-16-(1-2i-1)\right]$$
$$= \underline{-6+5i}.$$

$\boxed{E57}$



Calculate $\int_\Gamma \frac{dz}{z-z_o}$ by using the deformation Th$^{\underline{m}}$ to replace Γ with C_R : $z(t) = z_o + Re^{2\pi i t}$ for $0 \leq t \leq 1$

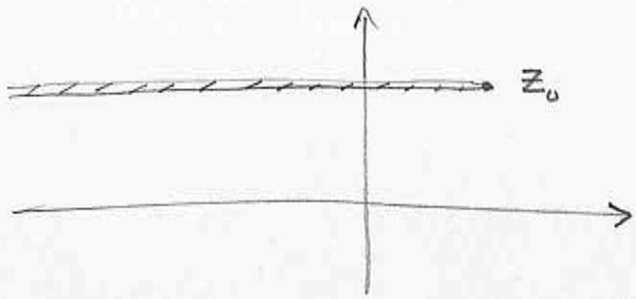
$$\int_{C_R} \frac{dz}{z-z_o} = \int_0^1 \frac{1}{Re^{2\pi i t}} \frac{d}{dt}\left(z_o + Re^{2\pi i t}\right) dt$$
$$= \int_0^1 \frac{R(2\pi i)}{Re^{2\pi i t}} e^{2\pi i t} dt$$
$$= 2\pi i \int_0^1 dt$$
$$= \underline{2\pi i}.$$

remark: if we wound k-times around Γ then we'd also need to wind k-times around C_R and we would obtain $2\pi i k$.

Alternatively, we can derive by using antiderivative of $\frac{1}{z-z_o}$ on a slit-plane. In particular $\text{Log}(z-z_o)$ has

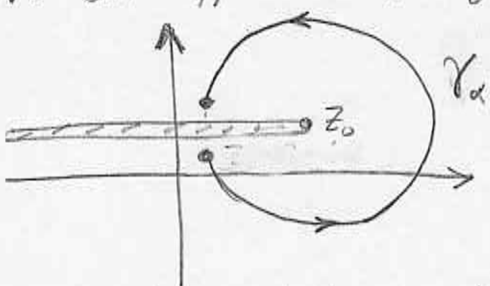
$$\frac{d}{dz}\left[\text{Log}(z-z_o)\right] = \frac{1}{z-z_o}$$

continued



$\mathbb{C}_-(z_0)$ = set on which $\text{Log}(z-z_0)$ is continuous.

We can approach C_R by allowing $\alpha \to 0^+$



$\gamma_\alpha \leftarrow$ arc on circle $|z-z_0| = R$ which goes CCW from $z_0 - R - i\alpha$ to $z_0 - R + i\alpha$

with this notation settled,

$$\int_{\gamma_\alpha} \frac{dz}{z-z_0} = \text{Log}(z-z_0)\Big|_{z_0-R-i\alpha}^{z_0-R+i\alpha}$$

$$= \text{Log}(-R+i\alpha) - \text{Log}(-R-i\alpha)$$

$$= \cancel{\ln\sqrt{R^2+\alpha^2}} + i\,\text{Arg}(-R+i\alpha) - \cancel{\ln\sqrt{R^2+\alpha^2}} - i\,\text{Arg}(-R-i\alpha)$$

$$= i\left(\pi - \tan^{-1}\left(\tfrac{\alpha}{R}\right)\right) - i\left(-\pi + \tan^{-1}\left(\tfrac{\alpha}{R}\right)\right)$$

$$= 2\pi i - 2i\tan^{-1}(\alpha/R)$$

Thus $\displaystyle\int_{C_R} \frac{dz}{z-z_0} = \lim_{\alpha\to 0^+}\left(2\pi i - 2i\underbrace{\tan^{-1}(\alpha/R)}_{0}\right) = 2\pi i.$

After we've settled the basics on contour integrals we'll prove Cauchy's Integral Th$^{\underline{m}}$ which states $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0}$ for $f(z)$ which is complex-diff. on and inside C

$\boxed{E58}$ $\displaystyle\frac{1}{2\pi i}\int_C \frac{dz}{z-z_0} = 1$ (identify $f(z)=1$ and apply Cauchy's Integral Th$^{\underline{m}}$)

$$\Rightarrow \int \frac{dz}{z-z_0} = 2\pi i$$

(we'll return to Cauchy's $\int$ Th$^{\underline{m}}$ later on, I included this here to give yet another view on the integral of $\frac{1}{z-z_0}$, in some sense this integral & the deformation theorem yield Cauchy's Th$^{\underline{m}}$...)

One last comment before we go on,

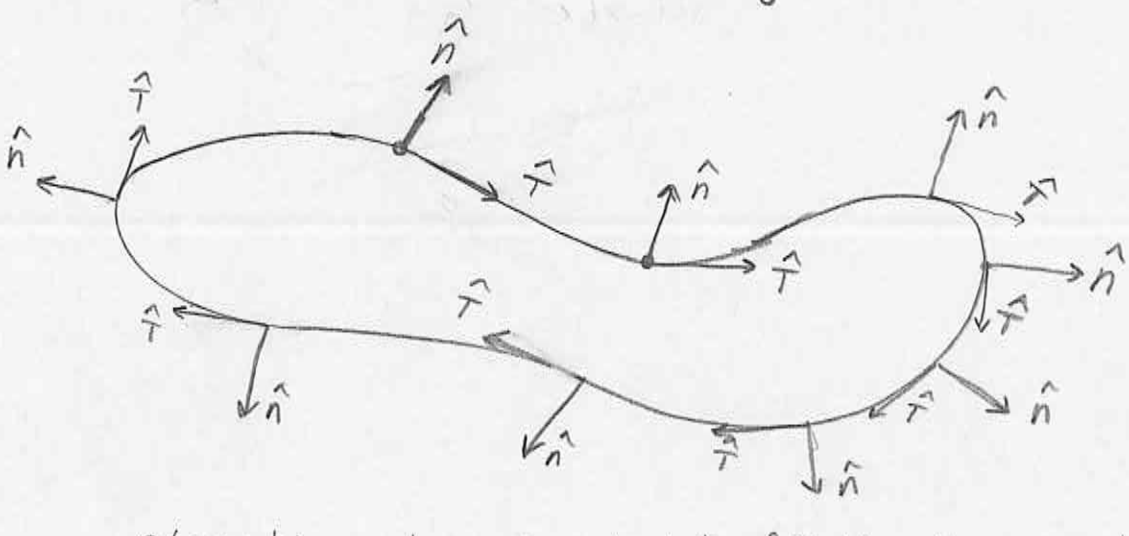
Let C be a simple, closed, contour, we seek to comment on the geometric meaning of $\int_C f\,dz$ where $f = u+iv$

$$\int_C f\,dz = \int_C (u+iv)(dx+idy)$$

$$= \underbrace{\int_C (u\,dx - v\,dy)}_{\substack{\text{this calculates}\\ \text{the flux of the}\\ \text{vector field}\\ \vec{F} = \langle v,u\rangle\\ \text{through } C}} + i\underbrace{\int_C (v\,dx + u\,dy)}_{\substack{\text{this calculates}\\ \text{the circulation}\\ \text{of the vector field}\\ \vec{F} = \langle v,u\rangle\\ \text{along } C \ (\text{aka work of } \langle v,u\rangle \text{ on } C)}}$$

$\Phi_C = \int_C (\vec{F}\cdot\hat{n})\,ds$: adds-up parts of $\vec{F}$ which cut through C ($\hat{n}$-component)

$W = \int_C (\vec{F}\cdot\hat{T})\,ds$: adds-up parts of $\vec{F}$ which align with C ($\hat{T}$-component)



Observation: when we calculate $\int_C f(z)dz$ it amounts to measuring flux and work of $\langle \mathrm{Im}(f), \mathrm{Re}(f)\rangle$ w.r.t C.

- Note: I discuss flux of two-dim'l vector fields in my calculus III notes if you'd like to see more on why the integral $\int_C (u\,dx - v\,dy)$ gives us flux.

Th^m (2) (initially covered 2/12/13)

If D is a <u>domain</u> and $f : D \longrightarrow \mathbb{C}$ is continuous then TFAE,

a.) f has a primitive (aka antiderivative) on D

b.) $\int_C f(z)dz = 0$ for any closed curve in D.

c.) $\int_C f(z)dz$ depends only on the endpoint and starting point of C for each contour in D

<u>Proof Outline</u>: we show a.) $\Rightarrow$ b.) $\Rightarrow$ c.) $\Rightarrow$ a.) to establish that a.) $\Leftrightarrow$ b.) $\Leftrightarrow$ c.)

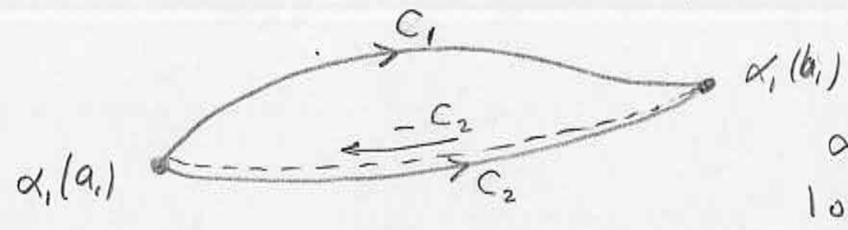
<u>a.) $\Rightarrow$ b.)</u> Assume $\exists F : D \longrightarrow \mathbb{C}$ s.t. $F'(z) = f(z)$. Let $\alpha : [a,b] \longrightarrow \mathbb{C}$ be a closed curve $\alpha(a) = \alpha(b)$. Consider

$$\int_\alpha f(z)\,dz = F(\alpha(b)) - F(\alpha(a)) = F(\alpha(b)) - F(\alpha(b)) = 0.$$

where we have used Th^m (1) from (87).

//

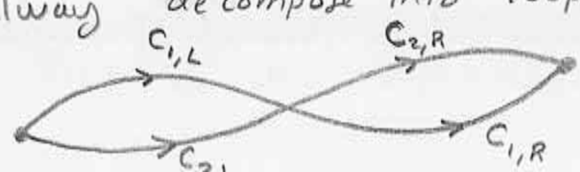
<u>b.) $\Rightarrow$ c.)</u> Suppose C_1 and C_2 share terminal points then $C_1 = \alpha_1([a_1, b_1])$ and $C_2 = \alpha_2([a_2, b_2])$ with $\alpha_1(a_1) = \alpha_2(a_2)$ and $\alpha_1(b_1) = \alpha_2(b_2)$. Consider geometrically:



$\alpha_1 \cup (-\alpha_2)$ is a loop here.

$$\text{Hence, } \int_{C_1 \cup (-C_2)} f(z)dz = \int_{C_1} f(z)dz + \int_{-C_2} f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz = 0.$$

Technically, C_1 and C_2 could have many intersections but we can always decompose into loops which yield the desired result:



$C_{1,L} \cup (-C_{2,L})$ loop

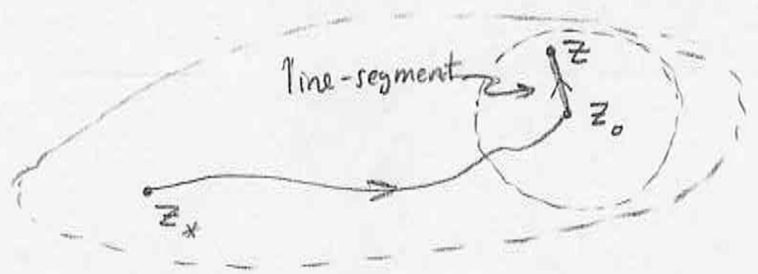
$C_{2,R} \cup C_{1,R}$ loop.

I leave the case of infinitely many intersections to the reader ☺.

$c \Rightarrow a$ | Suppose $\int_C f(z)dz$ depends only on the terminal pts. of C. (for each contour C inside some open, connected set D). Let z_* be a fixed point in D and define (for continuous f)

$$F(z) = \int_{z_*}^{z} f(w)dw$$

where this notation means $F(z) = \int_C f(w)dw$ for C a contour from $z_* \to z$. This is single-valued by assumption. An open, connected subset of $\mathbb{C}$ is path-connected. Consider,



if $z \in D$ then z is interior. Thus, $\exists z_0 \in D$ and $\delta_1 > 0$ where $z \in D(z_0, \delta_1)$.

Let C be the contour joining $z_* \to z_0 \to z$.

$$F(z) = \int_C f(w)dw = \int_{z_*}^{z_0} f(w)dw + \int_{z_0}^{z} f(w)dw$$

$$= F(z_0) + \int_{z_0}^{z} f(w)\,dw$$

Let $\epsilon > 0$ and choose $\delta_2 > 0$ such that $0 < |z - z_0| < \delta_2$ $\Rightarrow |f(z) - f(z_0)| < \epsilon$. Let $\delta = \min(\delta_1, \delta_2)$ and consider $z \in \mathbb{C}$ such that $0 < |z - z_0| < \delta \le \delta_1, \delta_2$. (this means z is inside the disk so we can form the line-segment within D at the same time we control $f(z)$ to be ϵ-units close to $f(z_0)$)

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| = \left| \frac{1}{z - z_0}\int_{z_0}^{z} f(w)dw - \frac{1}{z - z_0}\int_{z_0}^{z} f(z_0)\,dw \right|$$

$$= \frac{1}{|z - z_0|} \left| \int_{z_0}^{z} (f(w) - f(z_0))\,dw \right|$$

$$< \frac{1}{|z - z_0|}\, \epsilon\, |z - z_0| = \epsilon.$$

Thus $F'(z_0) = f(z_0)$. It follows that $F'(z) = f(z)$ $\forall z \in D$. //

Remark: maybe you can find a cleaner choice of notation so we prove $F'(z) = f(z)$ in the end. Since any point in D is at the center of (many) disks in D the argument above can be given to show $F'(z) = f(z)$. (Just swap $z \longleftrightarrow z_0$)

sloppy rectangles triangles

Th^m 3 (Cauchy - Goursat - Pringsheim) Let f be analytic on D and let $z_1, z_2, z_3 \in D$ with $\langle z_1, z_2, z_3 \rangle \subseteq D$

$\left(\langle z_1, z_2, z_3 \rangle = \{ t_1 z_1 + t_2 z_2 + t_3 z_3 \mid 0 \le t_1, t_2, t_3 \;\&\; t_1 + t_2 + t_3 = 1 \}\right)$

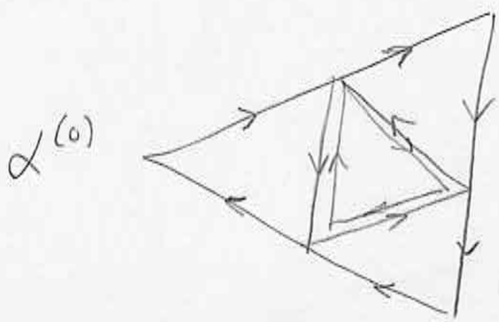
then $\displaystyle\int_{\langle z_1, z_2, z_3 \rangle} f(\zeta)\, d\zeta = 0$

Remark: see Frietag for carefully written proof. Here I just work through some details.

Proof: take the Δ and subdivide, one of the Δ has max-value of $|\int f|$

inner paths cancel

cont funct. on closed set.



$$\partial \Delta^{(n)} = \alpha^{(n)}.$$

- each time sub-divide so that $\alpha^{(n+1)}$ has

$$\left| \int_{\alpha^{(n)}} f \right| \le 4 \left| \int_{\alpha^{(n+1)}} f \right|$$

$$\Rightarrow \left| \int_\alpha f(\zeta)\, d\zeta \right| \le 4^n \left| \int_{\alpha^{(n)}} f(z)\, dz \right|$$

$$\Delta = \Delta^{(0)} \supset \Delta^{(1)} \supset \Delta^{(2)} \supset \cdots \supset \Delta^{(n)} \ni z_0$$

f complex-diff at $z_0 \in \Delta^{(j)} \quad \forall j = 0, 1, 2, \ldots$

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + r(z)$$

with $\displaystyle\lim_{z \to z_0} \left(\frac{r(z)}{z - z_0} \right) = 0$

have antiderivatives on $\Delta^{(n)}$

$$\left| \int_\alpha f(z)\, dz \right| \le 4^n \int_{\alpha^{(n)}} \Big[\underbrace{f(z_0)}_{0} + \underbrace{f'(z_0)(z - z_0)}_{0} \Big] + r(z) \Big] dz$$

$$\le 4^n \int_{\alpha^{(n)}} r(z)\, dz$$

continuing

$$\left| \int_\alpha f(z)\,dz \right| \le 4^n \left| \int_{\alpha^{(n)}} r(z)\,dz \right|$$

where $\frac{r(z)}{z-z_0} \to 0$ as $z \to z_0$. Chose n large enough such that $0<|z-z_0|<\delta \Rightarrow z \in \Delta^{(n)}$ and

$$\left| \frac{r(z)}{z-z_0} - 0 \right| < \epsilon \Rightarrow |r(z)| < \epsilon |z-z_0| \le \epsilon\delta$$

Hence $\left| \int_{\alpha^{(n)}} r(z)\,dz \right| \le \epsilon\delta\, l_{\alpha^{(n)}}$

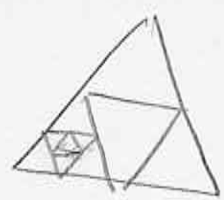
Let $\epsilon > 0$. There exists $\delta > 0$ s.t. $0<|z-z_0|<\delta$

$$\Rightarrow \left| \frac{r(z)}{z-z_0} - 0 \right| < \epsilon \Rightarrow |r(z)| < \epsilon |z-z_0|.$$

For n sufficiently large, $n \ge N$ we'll find $\Delta^{(n)} \subset \underbrace{U_\delta(z_0)}_{D(\delta, z_0)}$

hence, (think about the nesting)

$$\underbrace{|z-z_0| \le l(\alpha^{(n)}) = \frac{1}{2^n} l(\alpha)}_{\Rightarrow |r(z)| < \epsilon \frac{1}{2^n} l(\alpha)}$$



$l(\alpha)$ total l around big Δ.

so,

$$\begin{aligned} \left| \int_\alpha f(z)\,dz \right| &\le 4^n \left| \int_{\alpha^{(n)}} r(z)\,dz \right| \\ &\le 4^n \frac{\epsilon}{2^n} l(\alpha)\, l(\alpha^{(n)}) \\ &\le 4^n \frac{\epsilon}{2^n} l(\alpha) \frac{1}{2^n} l(\alpha) \\ &= \epsilon\, l(\alpha)^2 \end{aligned}$$

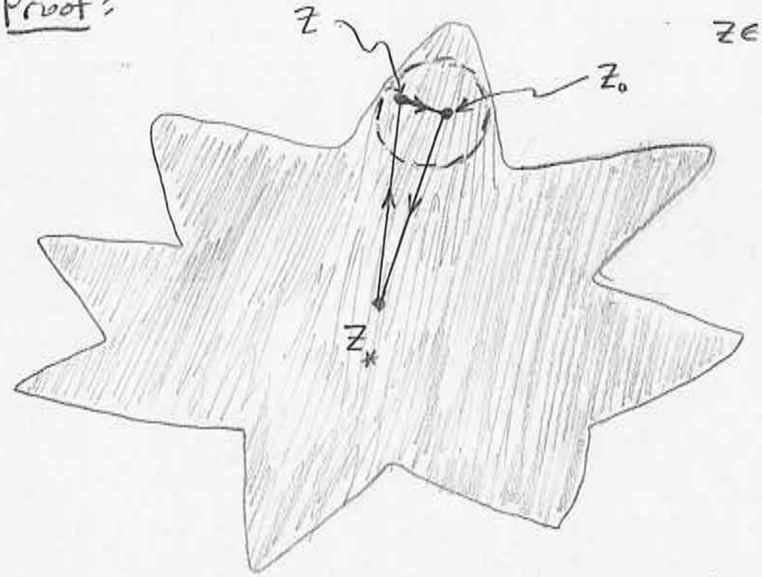
But, this holds for any $\epsilon > 0 \Rightarrow \left| \int_\alpha f(z)\,dz \right| = 0$

$$\Rightarrow \underline{\int_\alpha f(z)\,dz = 0}$$

//

$\boxed{\text{Th}^{m} \text{④ If } f \text{ is analytic (complex-differentiable) on a star-shaped domain } D \text{ then } f \text{ has an antiderivative (primitive) on } D}$

Proof:



$z \in D$ open $\Rightarrow z$ interior
$\Rightarrow \exists \delta > 0$ s.t. $z \in D(z_0, \delta)$.

Let $\Delta = \langle z_*, z, z_0 \rangle \subset D$
By Th^{m} ③, aka Cauchy Goursat for a triangle (Pringsheim 1901)

$$\int_\Delta f(z)\,dz = 0$$

As in our proof of $c \Rightarrow a$ of Th^{m} ② on pg. 92 (see 2/12/13 notes) define $F(z) = \int_{z_*}^{z} f(\zeta)\,d\zeta$. Observe,

$$\int_\Delta f(\zeta)\,d\zeta = \int_{z_*}^{z} f(\zeta)\,d\zeta + \int_{z}^{z_0} f(\zeta)\,d\zeta + \int_{z_0}^{z_*} f(\zeta)\,d\zeta$$

Hence, $F(z) = -\int_{z_0}^{z_0} f(\zeta)\,d\zeta - \int_{z_0}^{z_*} f(\zeta)\,d\zeta$. We claim, by proof similar to $c \Rightarrow a$ of pg. 92 we can show $F'(z) = f(z)$. //

Remark: this claim is likely bunk.

$\boxed{E59}$ Let $L(z) = \int_1^z \frac{d\zeta}{\zeta}$ for $z \in \mathbb{C}_-$. Observe that $f(\zeta) = \frac{1}{\zeta}$ has $f'(\zeta) = \frac{-1}{\zeta^2}$ for $\zeta \neq 0$ hence it is clear f is analytic (complex-diff.) on $\mathbb{C}_-$. By Th^{m} ④ there exists F such that $F'(z) = \frac{1}{z}$ for all $z \in \mathbb{C}_-$. Observe $L(z) = \int_1^{z} \frac{d\zeta}{\zeta}$ is an antiderivative for $\frac{1}{z}$ since 1 is a star-center for $\mathbb{C}_-$ (we can go through the proof of Th^{m} ④ to see that $L'(z) = \frac{1}{z}$ as desired.)

Th^{m} ⑤ If f is analytic on a simply-connected domain D then f has an antiderivative on D.

Proof: any simply connected domain can be chopped into star-shaped pieces then we apply Th^{m} ④ of p. 95 to obtain this result. Pages 233–249 make this precise. In those pages Freitag explains the equivalence of an elementary domain (every analytic fnct. on D has a primitive) see pg. 85–86 and simply-connected domains. //

Remark: $\mathbb{C} - \{0\} = \mathbb{C}^{\bullet}$ is not elementary domain since the function $f(z) = \frac{1}{z}$ is analytic on $\mathbb{C}^{\bullet}$ however, no antiderivative covers the whole of $\mathbb{C}^{\bullet}$. We always miss some ray due to the angle-jump problem. It's neat that this complex-analytic criteria of all analytic fncts having antiderivatives detects the topological property of simply connected.

Th^{m} (II.3.2, Cauchy Integral Formula)(1831)(pg. 93 Freitag)

Let $f : D \longrightarrow \mathbb{C}$ be analytic, D open. Assume $\overline{D(z_0, r)} \subseteq D$ then for each $z \in D(z_0, r)$ we find for $\alpha(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$,

$$f(z) = \frac{1}{2\pi i} \oint_{\alpha} \left(\frac{f(w)}{w - z} \right) dw$$

Proof: following Frietag pg. 94, let $g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & \text{for } w \neq z \\ f'(z) & \text{for } w = z \end{cases}$

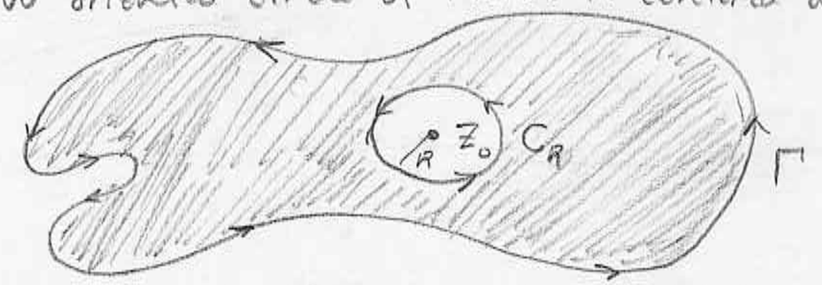
We can show (I leave details out since I'm about to give another proof on 97) that $g'(w)$ exists on D hence

$$\int_{\alpha} g(w)\, dw = 0 \Rightarrow \int_{\alpha} \frac{f(w)}{w - z}\, dw = \int_{\alpha} \left(\frac{f(z)}{w - z} \right) dw = 2\pi i\, f(z). //$$

Remark: Freitag's argument here is slick, but the one that follows is more relevant to how we tend to calculate contour integrals

$\text{Th}^{\underline{m}}$ (CAUCHY'S INTEGRAL THEOREM): If f is analytic on a simply connected domain D which contains a CCW-oriented simple, closed contour Γ containing z_0 then
$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}\,dz$$

"Proof": I use the Deformation $\text{Th}^{\underline{m}}$ proved on (90a). Let C_R be a CCW oriented circle of radius R centered at z_0



where C_R is inside Γ as pictured. Notice $g(z) = \dfrac{f(z)}{z - z_0}$ is clearly analytic on the region between C_R and Γ hence $\int_\Gamma g(z)\,dz = \int_{C_R} g(z)\,dz$. Consider

$$\begin{aligned}
\lim_{R\to 0}\left(\int_{C_R} g(z)\,dz\right) &= \lim_{R\to 0}\int_{C_R}\frac{f(z)\,dz}{z - z_0} && \text{since } z \approx z_0 \text{ as } R \to 0. \\
&= \lim_{R\to 0} f(z_0)\int_{C_R}\frac{dz}{z - z_0} && \text{By } \boxed{E57} \\
&= \lim_{R\to 0}\left(2\pi i f(z_0)\right) \\
&= 2\pi i f(z_0).
\end{aligned}$$

The theorem follows by solving for $f(z_0)$.

<u>Remark</u>: there's no need to work through cross-cuts here if we've already established the Deformation $\text{Th}^{\underline{m}}$.

- To remove the " " from "Proof" we need to add detail on pulling out $f(z_0)$ in place of $f(z)$. (details)

Details of Proof expanded

$$\int_{C_R} \frac{f(z)\,dz}{z-z_o} = \int_{C_R} \frac{f(z)-f(z_o)+f(z_o)}{z-z_o}\,dz$$

$$= \int_{C_R} \frac{f(z)-f(z_o)}{z-z_o}\,dz + f(z_o)\int_{C_R}\frac{dz}{z-z_o}$$

$$= 2\pi i f(z_o) + \int_{C_R}\frac{f(z)-f(z_o)}{z-z_o}\,dz$$

Let $M_R = \max\{\,|f(z)-f(z_o)| \;|\; z \in C_R\,\}$ then clearly, for $z \in C_R$

$$\left|\frac{f(z)-f(z_o)}{z-z_o}\right| \leq \frac{M_R}{R}$$

Hence,

$$\left|\int_{C_R}\frac{f(z)-f(z_o)}{z-z_o}\,dz\right| \leq \frac{M_R}{R}(2\pi R) = 2\pi M_R$$

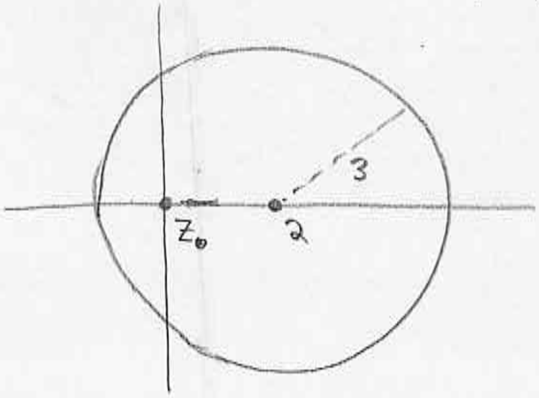
Notice, as $R \longrightarrow 0$ we find $z \longrightarrow z_o$ for $z \in C_R$ thus $M_R \longrightarrow 0$ by continuity of f at z_o. Cauchy's integral Th^{m} follows.

//

Remark: at this point I discussed Cauchy's Generalized $\int$-fla but, I postpone it here until I derive a new result

E60 Let Γ : circle $|z-2|=3$, CCW (once) calculate $\displaystyle\int_\Gamma \frac{e^z+\sin z}{z}\,dz$

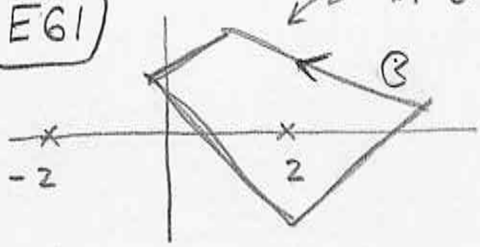
Observe $f(z) = e^z + \sin z$ is everywhere complex-diff., $f(z)$ is entire. Furthermore, by Cauchy's $\int$-fla,



$$\int_\Gamma \frac{e^z+\sin z}{z}\,dz = 2\pi i f(0)$$
$$= 2\pi i(e^0 + \sin(0))$$
$$= \boxed{2\pi i}$$

Remark: discussion of Γ vs. r evoked introduction of which is not intended to indicate a shape, rather it is merely to end discussion of inane nomenclature.

E61



of course, many other simple, closed, CCW curves enclosing only 2 would produce the same result.

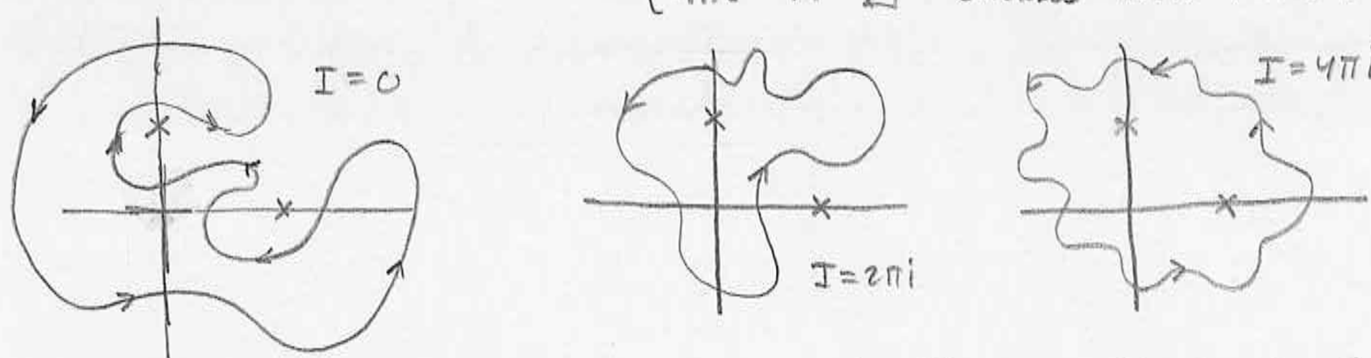
analytic inside and on , let's call it $g(z)$

$$\int \frac{\cos(z)}{z^2-4}\,dz = \int \frac{\overbrace{\cos(z)/z+2}^{g(z)}}{z-2}\,dz = 2\pi i\, g(z) = \boxed{\frac{\pi i \cos(2)}{2}}$$

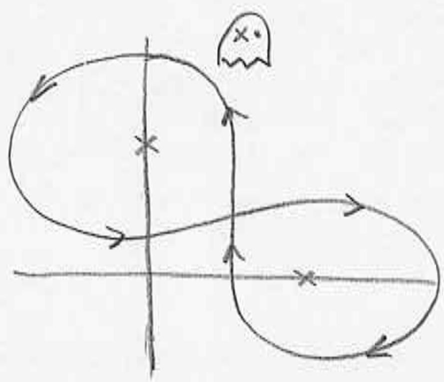
Cauchy's $\int$-fmla.

E62

$$I = \int \left(\frac{1}{z-1} + \frac{1}{z-i}\right)dz = \begin{cases} 0 & \text{if does not enclose } 1 \text{ or } i \\ 2\pi i & \text{if encloses just } 1 \text{ or } i \text{ (but not both)} \\ 4\pi i & \text{if encloses both } 1 \text{ and } i \end{cases}$$



Remark: to be clear I assumed was CCW, simple and closed. If we drop simple and CCW then you can get CCW around $z=i$ and CW around $z=1$ in which case



$$\int \left(\frac{1}{z-1} + \frac{1}{z-i}\right)dz = \underbrace{2\pi i}_{\text{from } z=i} - \underbrace{2\pi i}_{\text{from } z=1} = \boxed{0}.$$

(self-intersection $\Rightarrow$ not simple at non terminal point)

Th^m (15 of Saff & Snider): Let g be continuous on contour $\mathcal{C}$ and for each $z \notin \mathcal{C}$ set

$$G(z) = \int_{\mathcal{C}} \frac{g(\zeta)}{\zeta - z}\, d\zeta.$$

Then G is analytic at each $z \notin \mathcal{C}$ and $G'(z) = \int_{\mathcal{C}} \frac{g(\zeta)}{(\zeta - z)^2}\, d\zeta$

<u>Proof</u>: (mostly borrowed from Saff & Snider). Consider, set $\mathcal{C} = \Gamma$

$$\frac{G(z+\Delta z) - G(z)}{\Delta z} = \frac{1}{\Delta z}\int_\Gamma \left(\frac{g(w)}{w - z - \Delta z} - \frac{g(w)}{w-z}\right) dw$$

suppose Δz is small enough so that $z + \Delta z$ not on Γ

$$= \frac{1}{\Delta z}\int_\Gamma g(w) \left[\frac{w - z - (w - z - \Delta z)}{(w-z)(w-z-\Delta z)}\right] dw$$

$$= \frac{1}{\Delta z}\int_\Gamma g(w) \left[\frac{\Delta z}{(w-z)(w-z-\Delta z)}\right] dw$$

$$= \int_\Gamma \frac{g(w)\, dw}{(w-z)(w-z-\Delta z)}$$

Set $J = \dfrac{G(z+\Delta z) - G(z)}{\Delta z} - \displaystyle\int_\Gamma \frac{g(w)\,dw}{(w-z)^2}$ consequently,

$$J = \int_\Gamma g(w) \left[\frac{1}{(w-z)(w-z-\Delta z)} - \frac{1}{(w-z)^2}\right] dw$$

$$= \int_\Gamma g(w) \left[\frac{w - z - (w - z - \Delta z)}{(w-z)^2 (w - z - \Delta z)}\right] dw$$

as $\Delta z \to 0$

$$= \Delta z \int_\Gamma \frac{g(w)\, dw}{(w-z)^2 (w-z-\Delta z)} \underset{(\star)}{\leq} \left(\frac{2M\ell(\Gamma)}{d^3}\right)\Delta z \longrightarrow 0$$

Let $M = \max\{|g(w)| : w \in \Gamma\}$ and d = minimal distance from z to Γ so that $|w - z| \geq d > 0$ $\forall w \in \Gamma$. Further Saff/Snider suggest $\Delta z \to 0$ hence we can suppose $|\Delta z| < d/2$,

$$|w - z - \Delta z| \geq |w - z| - |\Delta z| \geq d - \frac{d}{2} = \frac{d}{2} \quad \text{for } w \in \Gamma.$$

Hence, for all $w \in \Gamma$,

$$\left| \frac{g(w)}{(w-z)^2 (w - z - \Delta z)} \right| \leq \frac{M}{d^2\left(\frac{d}{2}\right)} = \frac{2M}{d^3} \quad (\star)$$

Remark: the Th$^{\underline{m}}$ on 100 is justified formally by

$$\frac{dG}{dz} = \frac{d}{dz}\int_{\mathcal{C}} \frac{g(w)dw}{w-z} \underset{\substack{\uparrow \\ \text{Jump!}}}{=} \int_{\mathcal{C}} \frac{d}{dz}\left(\frac{g(w)}{w-z}\right) dw = \int \frac{g(w)dw}{(w-z)^2}.$$

Here the exchange of $\int_{\mathcal{C}}$ and $\frac{d}{dz}$ can be justified from the theory of uniform continuity. Or, as we just saw in 100 explicit, particular, calculation. In any event, the Th$^{\underline{m}}$ is easy to recall if you know this formal observation.

> Th$^{\underline{m}}$/ (Generalized Cauchy Integral Formula)
> Let Γ be a CCW, simple, closed contour and let f be analytic inside and on Γ then for $n = 1, 2, 3, \dots$
> $$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\Gamma} \frac{f(w)\,dw}{(w-z)^{n+1}}$$

Proof: we assume $\frac{d}{dz}$ and $\int_{\Gamma}$ can be exchanged. This can be proved by studying uniform continuity or giving an argument similar to that offered on 100. To begin the proof by induction on $n \in \mathbb{N}$ note that $n=1$ was proved already, see 98. Assume $f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\Gamma} \frac{f(w)}{(w-z)^{n+1}}$ for some $n \geq 1$ and consider,

$$\begin{aligned}
f^{(n+1)}(z) &= \frac{d}{dz}\left(f^{(n)}(z)\right) && : \text{def}^{\underline{n}} \text{ of } f^{(n)}(z). \\
&= \frac{d}{dz}\left[\frac{n!}{2\pi i}\int_{\Gamma} \frac{f(w)dw}{(w-z)^{n+1}}\right] && : \text{by induction hypothesis.} \\
&= \frac{n!}{2\pi i}\int_{\Gamma} \frac{d}{dz}\left[\frac{f(w)}{(w-z)^{n+1}}\right] dw && : \text{exchanging } \tfrac{d}{dz} \;\&\; \textstyle\int_{\Gamma}, \\
&= \frac{n!}{2\pi i}\int_{\Gamma} f(w)\,(w-z)^{-n-1-1}(-n-1)(-1)\,dw && : \text{chain \& power rule} \\
&= \frac{(n+1)\,n!}{2\pi i}\int_{\Gamma} \frac{f(w)dw}{(w-z)^{n+1+1}} && : \text{notation} \\
&= \frac{(n+1)!}{2\pi i}\int_{\Gamma} \frac{f(w)\,dw}{(w-z)^{n+1+1}} && \text{thus the Th}^{\underline{m}} \text{ is true } \forall n \in \mathbb{N} \text{ by induction.} //
\end{aligned}$$

$\boxed{\text{Th}^{m}/\ \text{If } f \text{ is analytic on a domain } D \text{ then } f', f'', \ldots, f^{(n)}, \ldots \text{ exist and are analytic on } D}$

Proof: Let $z_o \in D$ and assume $\Gamma \subset D$ hence f is analytic inside and on Γ thus by Cauchy's generalized-$\int$-formula,

$$f^{(n)}(z_o) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)\,dw}{(w-z)^{n+1}}$$

thus $f^{(n)}(z_o)$ exists as does $f^{(n+1)}(z_o)$ which proves the Th^m. //

Remark: another way of communicating this result is to say complex-differentiable $\Longrightarrow$ complex-smooth. This is not typical of real differentiation. For example, $f(x) = x|x|$ has $f'(x) = |x|$ but $f''(x) = \begin{cases} -1 & x<0 \\ 1 & x>0 \end{cases}$ and $f''(0)$ d.n.e. Once differentiable at $x=0$ does not imply infinitely many (smooth) diff of $f(x)$ at $x=0$.

E63) Calculate $\displaystyle\int_\Gamma \frac{\cosh(z)}{z^3}\,dz$ where $\Gamma: |z|=1$ traversed once CCW.

Notice it is convenient to express Cauchy's-Generalized-$\int$-fla as

$$\int_\Gamma \frac{f(z)\,dz}{(z-z_o)^n} = \frac{2\pi i\, f^{(n-1)}(z_o)}{(n-1)!}$$

Identify, $f(z) = \cosh(z)$ and $z_o = 0$, $n = 3$

note $f'(z) = \sinh(z)$ and $f''(z) = \cosh(z)$ hence

$$\int_\Gamma \frac{\cosh(z)}{z^3}\,dz = \frac{2\pi i \cosh(0)}{2!} = \boxed{\pi i}$$

$\boxed{\text{Th}^{\underline{m}}/ \text{ If } f \text{ is analytic inside and on } C_R = \{z \mid |z-z_o| = R\} \text{ and } |f(z)| \le M \ \forall z \in C_R \text{ then for } n = 1, 2, \dots \quad |f^{(n)}(z_o)| \le \dfrac{n!\,M}{R^n}.}$

Cauchy's Inequality

<u>Proof</u>: simply combine $\left|\int_C g(z)\,dz\right| \le M_g\, l(c)$ where $|g(z)| \le M_g$ and Cauchy's Gen. $\int$-formula, note $\left|\frac{f(w)}{(w-z)^{n+1}}\right| \le \frac{M}{R^{n+1}}$ for $w \in C_R$,

$$|f^{(n)}(z_o)| = \left| \frac{n!}{2\pi i} \int_{C_R} \frac{f(w)\,dw}{(w-z)^{n+1}} \right| \le \frac{n!}{2\pi}\left(\frac{M}{R^{n+1}}\right)(2\pi R) = \frac{Mn!}{R^n} \,//$$

(LIOUVILLE'S THEOREM)

$\boxed{\text{Th}^{\underline{m}}/ \text{ The only bounded entire functions are the constant fncts.}}$

<u>Proof</u>: Suppose $f(z) = z_o$ then clearly $M = |z_o|$ bounds $|f(z)|$ $\forall z \in \mathbb{C}$ and $f'(z) = 0$ hence f is entire and bounded. Conversely, suppose $f(z)$ is bounded and entire. We're given $|f(z)| \le M$ and $f'(z) \in \mathbb{C}$ for all $z \in \mathbb{C}$. Apply Cauchy's Inequality on a circle $C_R = \{z \mid |z| = R\}$

$$|f'(z)| \le \frac{M}{R}.$$

But, R is arbitrary! Hence $|f'(z)| = 0 \Rightarrow f'(z) = 0$ for all $z \in \mathbb{C}$ hence $f(z) = z_o$ $\forall z \in \mathbb{C}$. //

<u>Remark</u>: contrast with $f : C \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ where f is continuous and C is closed and bounded attaining <u>every</u> value between the minimum & maximum value of f on C. (In calculus I we call this the Intermediate Value Th$^{\underline{m}}$). Apparently bounding a complex-diff. fnct. is a <u>very</u> strict condition.

Th^{m}/ (FUNDAMENTAL THEOREM OF ALGEBRA)

Every nonconstant polynomial with complex coefficients has at <u>least</u> one zero.

<u>Proof</u>: (By contradiction)

Suppose $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$ with $a_n \neq 0$ and $n \geq 1$. Suppose, towards a $\rightarrow\leftarrow$, $P(z) \neq 0 \ \forall z \in \mathbb{C}$. Observe $f(z) = \dfrac{1}{P(z)}$ is entire since the quotient rule simply yields $f'(z) = \dfrac{-P'(z)}{(P(z))^2}$ and $P(z) \neq 0 \ \forall z \in \mathbb{C}$. The students will show in hwk that $f(z)$ is bounded on $\mathbb{C}$. By Liouville's Th^{m} on 103 $\Rightarrow f(z) = z_0 = \dfrac{1}{P(z)}$.

Since $P(z)$ nonconstant $\Rightarrow \exists z_1, z_2$ s.t. $P(z_1) \neq P(z_2)$ and clearly this gives $\rightarrow\leftarrow$ since $z_0 = \dfrac{1}{P(z_1)} \neq \dfrac{1}{P(z_2)} = z_0$ $\Rightarrow z_0 \neq z_0$. Therefore, by proof by contradiction we find $\exists$ at least one zero of $P(z)$. //

Th^{m}/ If $f(z) \in \mathbb{C}[z]$ and $\deg(f) = n \geq 1$ then $\exists r_1, r_2, \ldots, r_n \in \mathbb{C}$ and $A \in \mathbb{C}$ such that

$$f(z) = A(z - r_1)(z - r_2) \cdots (z - r_n)$$

<u>Proof</u>: repeated apply FTA and use (or better yet derive) the <u>factor</u> <u>theorem</u> $\left(f(r) = 0 \Rightarrow \exists g \text{ s.t. } f(z) = (z - r) g(z) \right)$. //

Cauchy's Formula on a Circle and Max. Modulus Th^{m}'s

Let f be analytic in and on C_R : $z = z_0 + Re^{it}$ $0 \le t \le 2\pi$, Cauchy's formula says:

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_R}\left(\frac{f(z)}{z-z_0}\right)dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+Re^{it})}{Re^{it}}\,Rie^{it}\,dt$$

plugging in the Parametrization of C_R

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{it})\,dt$$

Or, perhaps we should write

$$\underbrace{\boxed{f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\theta})\,d\theta}}_{\text{the average of } f(z) \text{ on a circle centered at } z_0 \text{ is given by the value of } f \text{ at the center of the circle.}} \quad -(*)$$

(assuming f analytic of course!)

Th^{m}/ If f analytic on dish $D = \overline{D(z_0,R)}$ and the max. of $|f(z)|$ $\forall z \in D$ is $|f(z_0)|$ then $|f(z)|$ is constant on D

Proof: left to reader. See Churchill Section 42, lot's of detail.

(maximum modulus principle)

Th^{m}/ If f is analytic in a domain D and $|f(z)|$ attains its maximum value at z_0 inside D then f is constant on D

Proof: left to reader. See Churchill Section 42.

(maximum modulus theorem)

Th^{m}/ A function analytic in a bounded region D and continuous up to and including its boundary attains its maximum modulus on the boundary

Proof: Since analytic fncts are continuous in the real-sense the theorem of real analysis $\Rightarrow \exists$ min/max hence $\exists$ pts. inside D or on ∂D for which $|f(z)|$ is extremal. But, the max. modulus principle $\Rightarrow$ any interesting extrema on ∂D. //