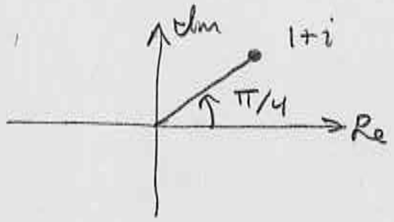
$\boxed{E1}$ $z_1 = 1 + i$, write in polar form via "cis" notation

$|z_1| = \sqrt{1+1} = \sqrt{2}$



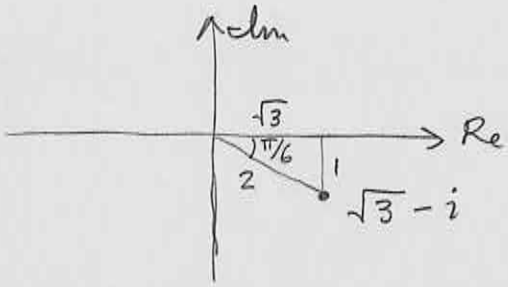
$$\Rightarrow \quad \underline{z_1 = \underbrace{\sqrt{2}\,\text{cis}\left(\tfrac{\pi}{4}\right)}_{\text{short for}}}.$$

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\boxed{E2}$ $z_2 = \sqrt{3} - i$, again express in polar form.

$$|z_2|^2 = z_2\bar{z}_2 = (\sqrt{3}-i)(\sqrt{3}+i) = 3 - i^2 = 4.$$

$\hookrightarrow |z_2| = 2.$



$$\underline{z_2 = 2\,\text{cis}\left(-\frac{\pi}{6}\right)}.$$

Properties of arg & Arg modulo 2π:

1.) $\arg(z_1 z_2) \equiv \arg(z_1) + \arg(z_2)$

2.) $\arg(z^{-1}) \equiv -\arg(z)$

3.) $\arg(z_1/z_2) \equiv \arg(z_1) - \arg(z_2)$

4.) $\arg(z) \equiv \text{Arg}(z)$

5.) properties 1, 2, 3 are <u>not</u> generally true for Arg

Def^{n}/ $A \equiv B$ iff $\exists k \in \mathbb{Z}$ such that $A = B + 2\pi k$

6.) $$\arg(x+iy) = \begin{cases} \tan^{-1}(y/x) + \frac{\pi}{2}(1-\text{sgn}(x)) & \text{if } x \neq 0 \\ (\pi/2)\,\text{sgn}(y) & \text{if } x = 0,\ y \neq 0 \\ \text{undefined} & \text{if } x = y = 0. \end{cases}$$

where "signum" fnct. $$\text{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

Complex Exponential:

Our text gives a different def$^{\underline{n}}$ which I'll discuss later. For now I offer this little argument from the text of Nagle and Saff. (not precisely, I changed it a bit)

<u>Discussion</u>: the def$^{\underline{n}}$ of e^z ought to have $e^{z_1+z_2} = e^{z_1}e^{z_2}$ thus $e^{x+iy} = e^x e^{iy}$. Moreover, e^x ought to match its usual meaning. <u>Assume</u> the chain rule $\frac{d}{dy}(e^{iy}) = ie^{iy}$ then $\frac{d}{dy}\frac{d}{dy}(e^{iy}) = i\frac{d}{dy}(e^{iy}) = i^2 e^{iy}$

hence e^{iy} solves $z'' = -z$ or $z'' + z = 0$.

It's well-known from DEq$^{\underline{n}}$'s $z = A\cos y + B\sin y$

To find A & B note $z(0) = e^{i(0)} = e^0 = 1 = A\cos 0 = A$

and $z'(0) = ie^{i(0)} = i = -A\sin(0) + B\cos(0) = B$

Hence $\underline{z(y) = e^{iy} = \cos y + i\sin y}$.

Remark: we defer the proper def$^{\underline{n}}$ for next week. This is cheating a bit although the results are spot on.

$$\boxed{\text{Def}^{\underline{n}}/\quad \text{cis}(\theta) = e^{i\theta} = \cos\theta + i\sin\theta}$$

Clearly $\underline{e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)}$.

Many beautiful results flow from this profound notation. For example:

1.) $e^{z_1+z_2} = e^{z_1}e^{z_2}$ — laws of $\mathbb{R}$-exponents, add $\measuredangle$ f-la's

2.) $(e^z)^n = e^{nz}$ — de Moivre's f-la.

3.) $\underbrace{\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \text{ \& } \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})}_{\text{derive all manner of trig. f-la's.}}$

Trigonometry:

$$\cos(\theta+\beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$$
$$\sin(\theta+\beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$$

adding ∠'s f-la's. Can derive from basic geometry.

Notice that

$$\begin{aligned}\cos(\theta+\beta)+i\sin(\theta+\beta) &= \cos\theta\cos\beta - \sin\theta\sin\beta + i(\sin\theta\cos\beta+\cos\theta\sin\beta)\\ &= (\cos\theta+i\sin\theta)\cos\beta + \sin\beta(-\sin\theta+i\cos\theta)\\ &= (\cos\theta+i\sin\theta)\cos\beta + (\cos\theta+i\sin\theta)\, i\sin\beta\\ &= (\cos\theta+i\sin\theta)(\cos\beta+i\sin\beta).\end{aligned}$$

Therefore,

$$\boxed{\operatorname{cis}(\theta+\beta) = \operatorname{cis}(\theta)\operatorname{cis}(\beta)} \quad \text{— } \star$$

Now, in the imaginary exponential notation the identity ($\star$) simply reads:

$$\boxed{e^{i(\theta+\beta)} = e^{i\theta}e^{i\beta}} \quad \text{— } \tilde{\star}$$

We assumed this on 8 but the calculation above shows that if we define $e^z = e^x e^{iy}$ then $e^{z_1+z_2} = e^{z_1}e^{z_2}$,

$$\begin{aligned}e^{z_1+z_2} &= e^{x_1+iy_1+x_2+iy_2}\\ &= e^{x_1+x_2+i(y_1+y_2)}\\ &= e^{x_1+x_2}e^{i(y_1+y_2)}\\ &= e^{x_1}e^{x_2}e^{iy_1}e^{iy_2} \qquad \text{by Laws of exponents in } \mathbb{R} \text{ and } \tilde{\star}\\ &= e^{x_1+iy_1}e^{x_2+iy_2}\\ &= e^{z_1}e^{z_2}.\end{aligned}$$

Properties of The Complex Exponential

1.) $e^{x+iy} = e^x e^{iy}$ where $e^{iy} = \text{cis}(y) = \cos(y) + i\sin(y)$

2.) $e^{z+\pi i} = -e^z$

3.) $e^{z+2\pi k i} = e^z$

4.) $e^{z+w} = e^z e^w$

5.) $e^{-z} = (e^z)^{-1} = \dfrac{1}{e^z}$ $\left(\text{Def}^{\underline{n}} / \; w^{-1} = \dfrac{1}{w}\right)$

6.) $e^{0+0i} = 1$

7.) $|e^{x+iy}| = |e^x|$

8.) If $z = re^{i\theta}$ for $r \in (0, \infty)$ then $\exp(\ln(r) + i\theta) = z$

9.) $z(t) = z_o + Re^{it}$ for $0 \leq t \leq 2\pi$, $R > 0$ parametrizes a circle centered at z_o with radius R.

10.) $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ & $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

11.) $(e^z)^n = e^{nz}$ for $n \in \mathbb{Z}$.

12.) De Moivre's $\text{Th}^{\underline{m}}$/ $\text{cis}(\theta)^n = \text{cis}(n\theta)$.

Proof: we've shown 4 on 9, 1 is the current $\text{def}^{\underline{n}}$ of e^z. item 2 is hwk.