(I.) If M and N are manifolds then so is $M \times N$ where charts on $M \times N$ are of the form

$$(U \times V,\ x \times y)$$

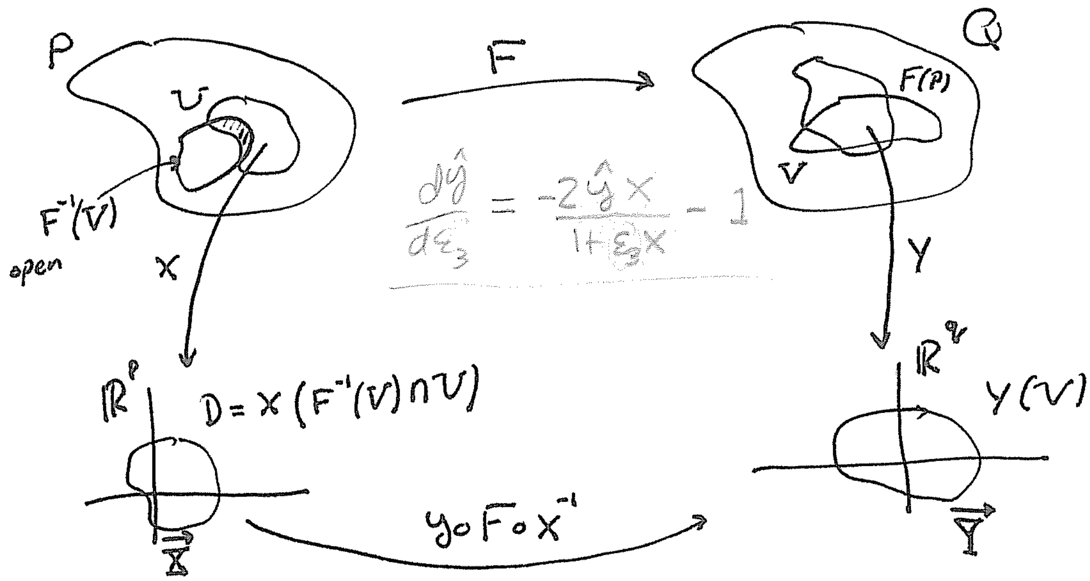
Where $(U, x) \in \mathcal{A}_M$ and $(V, y) \in \mathcal{A}_N$ and $U \times V$ is the cartesian product of U and V; $U \times V = \{(u,v) \mid u \in U \subseteq M,\ v \in V \subseteq N\}$ and $(x \times y)(u,v) = (x(u), y(v))$. The collection of all such charts forms the atlas for $M \times N$; $\mathcal{A}_{M \times N}$.

<u>Example:</u> $S^1 \times S^1 =$ Torus is a Lie Group actually
$\mathbb{R} \times S^1$ also a Lie group.

(II.) If P, Q are manifolds then $F : P \to Q$ is smooth iff

$$y \circ F \circ x^{-1} : D \to y(V)$$

is smooth for <u>every</u> $(U, x) \in \mathcal{A}_P$ and $(V, y) \in \mathcal{A}_Q$



$$d_p F(\mathbb{X}) = \mathbb{Y}$$

$$\mathbb{X} = \mathbb{X}^i \, \partial/\partial x^i$$

$$D_{x(p)}(y \circ F \circ x^{-1}) : \mathbb{R}^p \longrightarrow \mathbb{R}^q \quad \text{defined via}$$

$$D_{x(p)}(y \circ F \circ x^{-1})(\mathbb{X}^1, \mathbb{X}^2, \ldots, \mathbb{X}^p) = (\mathbb{Y}^1, \mathbb{Y}^2, \ldots, \mathbb{Y}^q) \quad \text{this defines } \mathbb{Y}$$

$$D_{x(p)}(y \circ F \circ x^{-1})(\vec{\mathbb{X}}) = \vec{\mathbb{Y}}$$

Frechet Derivative of $y \circ F \circ x^{-1}$ at $x(p)$
(the linear transformation for which the matrix is the Jacobean)

Frechet Derivative:

$f: W_1 \subseteq \mathbb{R}^p \longrightarrow W_2 \subseteq \mathbb{R}^q$

$$D_u f(h) = h J_f(u)^t$$

$$D_u f(h^t) = J_f(u) h^t$$

Def^n/ If P, Q are manifolds then $F: P \longrightarrow Q$ is smooth iff $y \circ F \circ x^{-1} : D \longrightarrow y(V)$ is smooth for all appropriate charts in P & Q. Where $y \circ F \circ x^{-1} : D \rightarrow y(V)$ is smooth if

$$y^j \circ F \circ x^{-1} : D \longrightarrow \mathbb{R} \subset y(V)$$

has partial derivatives of all orders.

Let $X \in T_p P$ then $\exists$ a curve $\alpha : (-\epsilon, \epsilon) \longrightarrow P$ such that $\alpha(0) = p$ and $\alpha'(0) = X$ then:

$$d_p F(X) = dF_{\alpha(0)}(\alpha'(0)) = (F \circ \alpha)'(0)$$

Differential of F via curves.

Def$^{\underline{n}}$ (1.1): G is a Lie Group means that G is a group and G is a manifold such that the group functions are smooth.

$$\mu_G : G \times G \longrightarrow G \qquad \mu_G(g_1, g_2) \equiv g_1 g_2$$

$$\text{inv} : G \longrightarrow G \qquad \text{inv}(g) = g^{-1}$$

These fncts. are smooth $\forall\, g_1, g_2, g \in G$. We give $G \times G$ the usual product manifold structure.

Def$^{\underline{n}}$/ (1.2) We say that $H \subseteq G$ is a subliegroup of G if G is a Lie group and H is a submanifold of G and the group operations restrict to H and the group operations are smooth on H (meaning H is a subgroup)

Th$^{\underline{m}}$/ If G is a Lie group and H is a subgroup which is also a submanifold of G then H is a sublie group.

<u>Remark</u>: $i : H \hookrightarrow G$ with $i(h) = h$ the inclusion map

$$\underbrace{\mu_G \circ (i \times i)}_{f} : H \times H \longrightarrow G$$

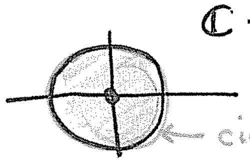
but $f(H \times H) \subset H$ Ex. 1.1 says $f : H \times H \to H$ is smooth.

$\boxed{\text{Th}^{\underline{m}}\text{(Proof in Warner): If } G \text{ is a Lie group and } H \subset G \text{ is a closed subgroup then } H \text{ is a subliegroup. (closed in the topological sense)}}$

Pf:/ Actually a bit long, worth looking over. this is a very non-trivial statement w/o Lie Structure we don't get such consequences

Application

$\mathbb{C}$-plane



circle is closed subset of $|z| \leq 1$ complex #'s.

$\boxed{\cancel{\text{Th}^{\underline{m}}} \text{ If } G \text{ and } H \text{ are Lie groups then } G \times H \text{ is a Lie Group}}$

Examples of Lie Groups

- $(\mathbb{R}, +)$
- $(\mathbb{R} \setminus \{0\}, \cdot)$
- $(\mathbb{R}^n, +)$
- $(S^1, \cdot)$: real - Lie group
- $S^1 \times S^1$: torus (geometrical $r=1$ for both is questionable)
- $\mathbb{R} \times S^1$: cylinder

  $$(a, e^{i\theta})(b, e^{i\varphi}) = (a+b, e^{i(\theta+\varphi)})$$ group operation on $\mathbb{R} \times S^1$

- $gl(n) = \mathbb{R}^{n \times n}$ $n \times n$ matrices over $\mathbb{R}$

  Remark: $\exists$ a bijection from $gl(n) \longrightarrow \mathbb{R}^{n^2}$ via the rule

  $$gl(n) \ni A \xmapsto{\Phi} (A_{11} A_{12} \dots A_{1n} A_{21} \dots A_{2n} \dots A_{n1} A_{n2} \dots A_{nn}) \in \mathbb{R}^{n^2}$$

  Now $gl(n)$ is the manifold with the differentiable structure that contains ~~[illegible]~~ Φ as a chart with range $\mathbb{R}^{n^2}$ aka

  $$x_{ij}(A) = A_{ij}$$

This $gl(n)$ is a vector space:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

$$(\lambda A)_{ij} = \lambda A_{ij}$$

These hold $\forall\, A,B \in gl(n)$ & $\lambda \in \mathbb{R}$. Clearly $gl(n)$ is a vectorspace. Moreover the chart $x = \Phi = (x_{ij})$ is also linear

$$x : gl(n) \longrightarrow \mathbb{R}^{n^2}$$

Its easy to see that

$$x(A+B) = x(A) + x(B)$$

$$x(\lambda A) = \lambda x(A)$$

Hence $gl(n) \cong \mathbb{R}^{n^2}$. Lets steal from $\mathbb{R}^{n^2}$ to see things for $gl(n)$ a metric for starters.

$$\langle A, B\rangle = \sum_i \sum_j A_{ij} B_{ij} = \text{trace}(AB^t)$$

So $gl(n)$ is an inner-product space. Likewise

$$\|A\|^2 = \langle A, A\rangle = \text{trace}(AA^t)$$

Hence $gl(n)$ is a normed-space. In fact its a finite dim'l Hilbert space.

<u>Remark</u>: $(gl(n), +)$ does form a Lie group but typically we care about $GL(n)$ the group of non-singular matrices.

All things $\mathcal{Gl}(n) = GL(n)$ the General Linear Group 8/20/04 ③

Def^n/ $\mathcal{Gl}(n) = \{ A \in gl(n) \mid A \text{ is non-singular} \}$

Notice that $\det : gl(n) \longrightarrow \mathbb{R}$ is a polynomial in the entries & so it's certainly smooth, all the partials exist and are continous.

$$\det(A) = \sum_{\sigma}{}' (-1)^{\sigma} A_{1\sigma_1} A_{2\sigma_2} \cdots A_{n\sigma_n}$$

Notice that $\mathcal{Gl}(n) = gl(n) \setminus \det^{-1}(0)$

now since $\{0\}$ is closed & det is smooth $\therefore$ continous we know inverse image of closed under cont. map is closed then $gl(n) \setminus$ closed set is itself an open set in $gl(n)$

$\mathcal{Gl}(n)$ is open in $gl(n)$

$\mathcal{Gl}(n)$ is a manifold ( )

Lie group?

$\mu : \mathcal{Gl}(n) \times \mathcal{Gl}(n) \longrightarrow \mathcal{Gl}(n)$

$\mu(A,B) = AB$ ← polynomial in the entries of A & B thus under the entry valued chart $x = (x_{ij})$ this is smooth.

$x_{ij}(\mu(A,B)) = \sum_{l} A_{il} B_{lj}$ can take $\frac{\partial}{\partial A_{pq}}$ & $\frac{\partial}{\partial B_{rs}}$ clearly enuf.

$Inv(A) = A^{-1} = \frac{1}{\det(A)} \operatorname{Cof}(A^t)$ ← whatever this means its just a polynomial, which is smooth.

Th$\underline{m}$/ Assume $U \subseteq \mathbb{R}^n$ is open and that $f : U \longrightarrow \mathbb{R}^m$ ~~[illegible]~~ $n > m$, is smooth. Then if $d_pf : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ has rank m at each point of $S = f^{-1}(0)$ then S is a sub manifold of U.

Remark :

$$d_pf(h) = h\, J_f(p)^t$$

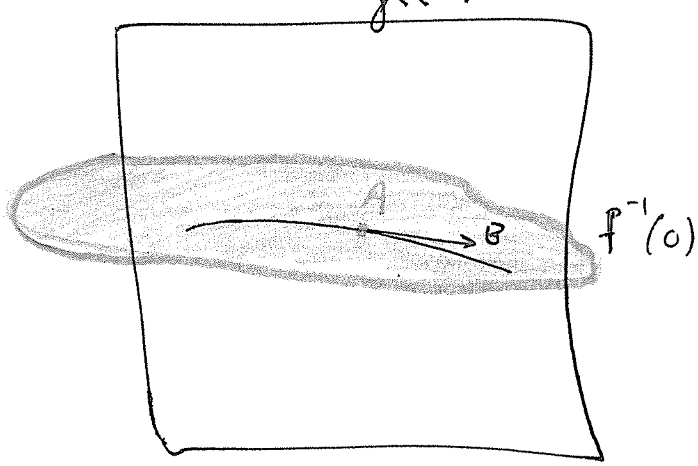
$$h^t \longmapsto J_f(p)\, h^t$$

$J_f(P)$ = the matrix of d_pf

$$\boxed{rk(d_pf) = rk(J_f(P))}$$

Exercise 1.2 : We can use the Th^m to see that the set of Isometries of $G = I$, $G = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \ddots \ -1 \end{pmatrix}$ exc... are in fact Lie Groups, by identifying them with the level set of f.

$gl(n)$



Def^n (1.3) If G is a Lie group and $a \in G$ then left (right) translation by a is the map

$$\ell_a : G \longrightarrow G \qquad (R_a : G \to G)$$
$$\ell_a(x) = ax \qquad (R_a(x) = xa)$$

The map $\sigma_a : G \longrightarrow G$ defined by $\sigma_a(x) = axa^{-1}$ is called the inner automorphism defined by a. ($\sigma_a = \mathrm{adj}_a$ sometimes)

<u>Remark</u>: $\sigma_a = R_{a^{-1}} \circ \ell_a = \ell_a \circ R_{a^{-1}}$

$L_a \equiv \ell_a$
$R_a \equiv r_a$

Def^n (1.4) A vector field X is left (right) invariant iff

$$d_x\ell_a(X(x)) = X(\ell_a(x)) = X(ax)$$
$$d_x R_a(X(x)) = X(R_a(x)) = X(xa)$$

<u>Existence</u>: look at the identity vect. field, but need smoothness.

<u>Recall</u>: If X is a vector field, x is a coordinate, f is a function on the domain (x)

$$X(f)(p) = X_p(f)$$
$$= d_pf(X)$$
$$= \sum_{i=1}^{n} X^i(p) \frac{\partial f}{\partial x^i}$$

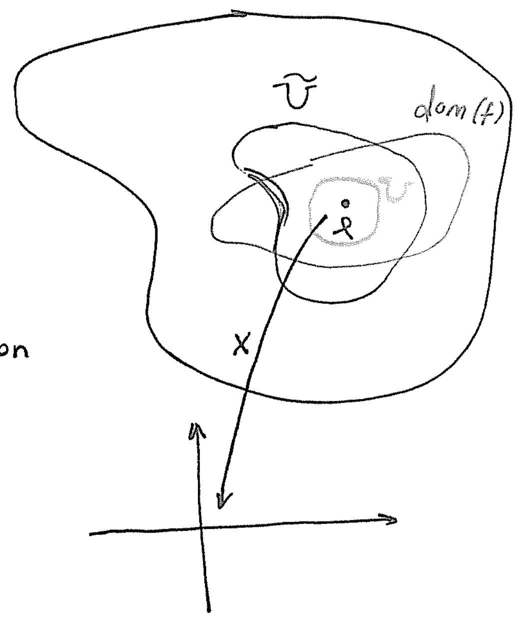
Where we have $d_pf = \sum_1 \frac{\partial f}{\partial x^i} d_px^i$

Thm/ Let G be a Lie group and $v \in T_e G$.
Define $X^v : G \longrightarrow TG$ by

$$X^v(x) = d_e l_x(v)$$

Then X^v is a smooth-left-invariant vector field on G

Proof: First proove X^v is smooth near $e \in G$. We show $X^v(f)$ is smooth $\forall f \in C^\infty_{loc}(e) \equiv \{f \mid f \in C^\infty(U),\ U \subseteq M \text{ is open}\}$.
Let $(\tilde{U}, \tilde{x})$ be a chart ~~domain~~ at e, and let $U \subseteq G$ be an open set such that $U^2 \subseteq \tilde{U} \cap \mathrm{dom}(f)$
Consider $X^v(f) \circ x^{-1}$ where $x = \tilde{x}|_U$;



$$
\begin{aligned}
\left[X^v(f) \circ x^{-1}\right](r) &= X^v(f)\left(x^{-1}(r)\right) \\
&= X^v_{x^{-1}(r)}(f) \\
&= d_e l_{x^{-1}(r)}(v)(f) \quad \text{by assumption} \\
&= df_{x^{-1}(r)}\left(d_e l_{x^{-1}(r)}(v)\right) \\
&= d_e\left(f \circ l_{x^{-1}(r)}\right)(v) \\
&= v\left(f \circ l_{x^{-1}(r)}\right) \\
&= \left(\sum_i v^i \frac{\partial}{\partial x^i}\bigg|_e\right)\left(f \circ l_{x^{-1}(r)}\right) \\
&= v^i \frac{\partial}{\partial u^i}\left(f \circ l_{x^{-1}(r)} \circ x^{-1}\right)(x(e))
\end{aligned}
$$

Can arrange x so that $x(e) = 0 \in \mathbb{R}^n$
$x = \tilde{x} - \tilde{x}(e)$ has
$x(e) = \tilde{x}(e) - \tilde{x}(e) = 0$.

Since the mapping $(r, u) \longrightarrow f(\mu_G(x^{-1}(r), x^{-1}(u)))$ is smooth on $x(U) \times x(U) \subset \mathbb{R}^{2n}$, so all the partials exist & are continous, hence

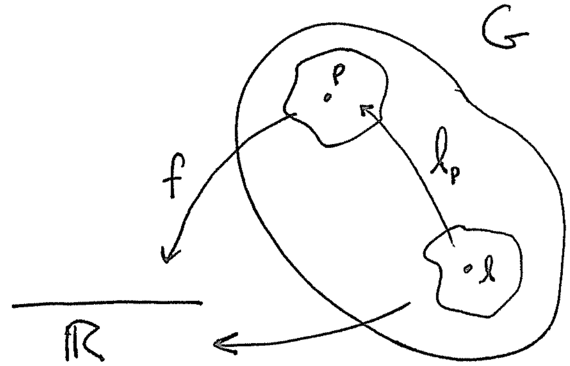
$$(r, u) \longmapsto \frac{\partial}{\partial u^i}\left(f(\mu_G(x^{-1}(r), x^{-1}(u)))\right) \quad \text{are smooth}$$

$$\therefore\ r \longrightarrow \frac{\partial}{\partial u^i} f(\mu_G(x^{-1}(r), x^{-1}(u)))\bigg|_{u=0} \quad \text{is smooth.}$$

Pf: (Continued).

Now let $p \in G$. We show that $\mathfrak{X}^v(f)$ is smooth $\forall f \in C^\infty_{loc}(P)$. We want to move it over to the identity e, but remember we need to work around p. Let q be any element of G near e

$$\mathfrak{X}^v(f) = \mathfrak{X}^v(f \circ l_p) \circ l_p^{-1}$$



Notice that $f \circ l_p$ is smooth from points near to e to $\mathbb{R}$.

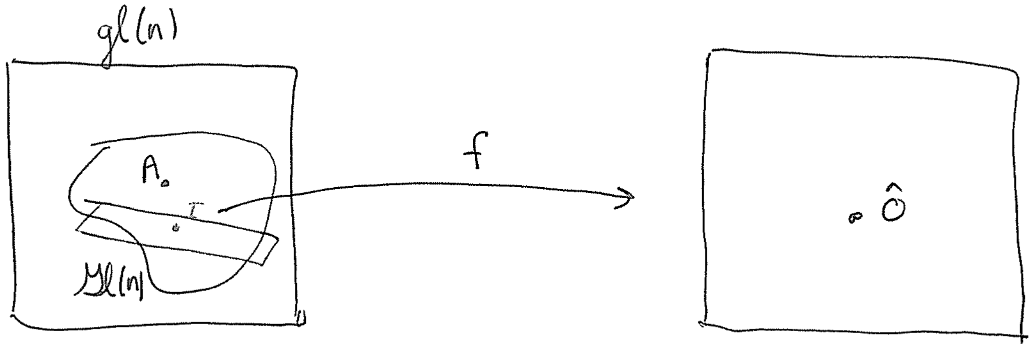
$f \circ l_p : \{\cdot e\} \longrightarrow \mathbb{R}$

Note $l_p(e) = p$.

$$\begin{aligned}
\mathfrak{X}^v(f \circ l_p)(q) &= \mathfrak{X}^v_q(f \circ l_p) \\
&= (d_e l_q)(v)(f \circ l_p) \\
&= d(f \circ l_p)(d_e l_q(v)) \\
&= d_e(f \circ l_p \circ l_q)(v) \\
&= d_e(f \circ l_{pq})(v) \\
&= \mathfrak{X}^v_{pq}(f) \\
&= \mathfrak{X}^v(f)(pq) \\
&= \mathfrak{X}^v(f)(l_p(q))
\end{aligned}$$

Hence $\mathfrak{X}^v(f)$ is smooth.

Exercise 2 Correction



$$d_A f : T_A Gl(n) \longrightarrow T_{f(A)}\, gl(n)$$

$$d_A f : gl(n) \longrightarrow gl(n)$$

Exercise 1

$\mathrm{dom}(z) = U \cap S'$ the picture is correct

$z = (\pi_1 \circ y)\big|_{U\cap S'}$ restriction of $\pi_1 \circ y$ onto $U \cap S'$

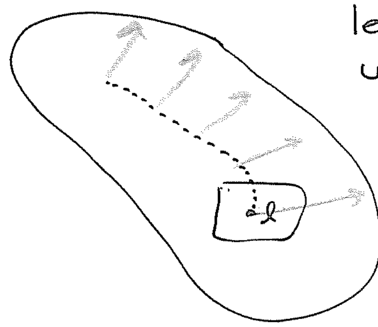
all need to do

$$y \circ f \circ x^{-1}\big|_{U\cap S} = z \circ f \circ x^{-1}$$

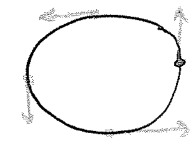
$v \in T_e G$

$\Xi^v(x) = d_e l_x(v)$ Left-invariant vector field

Ξ^v is a vector field on G



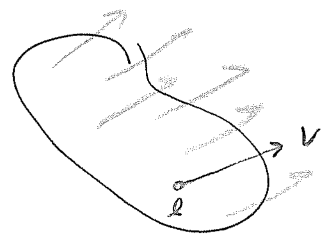
$$d_x l_a(\Xi^v(x)) = d_x l_a(d_e l_x(v))$$
$$= d_e(l_a \circ l_x)(v)$$
$$= d_e l_{ax}(v)$$
$$= \Xi^v(ax)$$



We know it's a vector field and we know it's left invariant

Remark: if Ξ is any left-invariant vector field on G then $\Xi = \Xi^v$ for some $v \in T_e G$. That is the only LIVF are those we've constructed.

Pf:/ Let $\Xi \in LIVF(G)$ and let $v = \Xi(e)$
~~let~~ $\Xi^v = \Xi$ is seen from



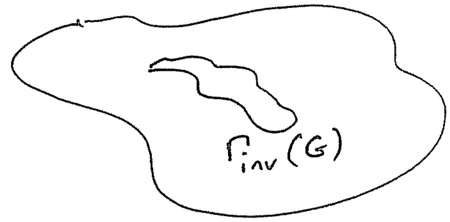
$$\Xi^v(x) = d_e l_x(v)$$
$$= d_e l_x(\Xi(e))$$
$$= \Xi(xe)$$
$$= \Xi(x) \quad \therefore \quad \Xi^v = \Xi.$$

<u>Remark</u>: all of the following can be def^{n} to be the Lie Alg $\mathfrak{g} \leftrightarrow G$

- Lie algebra = $T_e G$ with suitably defined brackets
- Lie algebra = 1-parameter groups thru identity
- LIVF

$\boxed{\text{Theorem: Let } G \text{ be a Lie Group then the vector space of all LIVF's}(G) \text{ is a vector space} \cong \text{ to } T_e G. \text{ Note } \Gamma_{inv}(G) = LIVF(G),\ \Gamma(G) \text{ all vector fields.}}$

Proof: Note that if X and Y are LIVF then so is λX & $X + Y$



$\Gamma(G)$

$\Gamma(G)$ is ∞'dim'l all vector fields on G

$\Gamma_{inv}(G)$ has finitely many

$$X = X^i \frac{\partial}{\partial x^i}$$

finitely generated module

X^i's are functions

$(f X)(x) = f(x)\, X(x)$

multiplying by $f(x)$ a $X \in \Gamma_{inv}(G)$

$fX \notin \Gamma_{inv}(G)$ necessarily

Let $\Phi : T_e G \longrightarrow \Gamma_{inv}(G)$ be defined by

$$\Phi(v) = X^v$$

We know $\Phi(Y(e)) = Y$ that is Φ is onto. (☆) should have shown linearity before checking kernel.

Φ is 1-1

$$\Phi(v) = 0 \Rightarrow X^v = 0 \Rightarrow X^v(x) = 0 \quad \forall x$$

So choose $x = e$ then

$$\begin{aligned} X^v(x) &= d_e l_x(v) \\ &= d_e l_e(v) \\ &= d_e\, id_G(v) \\ &= id_{T_e G}(v) \\ &= v = 0 \quad \Rightarrow \quad Ker(\Phi) = 0 \end{aligned}$$

(☆) Φ is linear $\Phi(av+bw)(x) = X^{av+bw}(x)$

$$\begin{aligned} &= d_e l_x(av+bw) \\ &= a\, d_e l_x(v) + b\, d_e l_x(w) \\ &= (a X^v + b X^w)(x) = (a\Phi(v) + b\Phi(w))(x) \end{aligned}$$

$\Phi : T_e G \longrightarrow \Gamma_{inv}(G)$ is interesting because it just so happens that $\Gamma_{inv}(G)$ has a natural bracket structure while $T_e G$ does not. So to remedy this we merely steal the bracket from $\Gamma_{inv}(G)$ via the bijection Φ.

$Th^{\underline{m}}$: If G and H are Lie Groups and $\varphi : G \longrightarrow H$ is a Lie group homomorphism (ie φ smooth with $\varphi(ab) = \varphi(a)\varphi(b)$) then $\ell(\varphi) : \ell(G) \to \ell(H)$ (where $\ell(G) = T_e G$ or $\ell(G) = \Gamma_{inv}(G)$) is a Lie Algebra homomorphism

$$\varphi : G \longrightarrow H \quad \text{and} \quad \psi : H \longrightarrow K$$

$$\ell(\psi \circ \varphi) = \ell(\psi) \circ \ell(\varphi)$$

Where $\ell(\varphi) \equiv d_{\ell}\varphi : T_{\ell_G} G \longrightarrow T_{\ell_H} H$.

Remark: For $v, w \in T_e G$ we defined $[v,w] \in T_e G$ such that $X^{[v,w]} = [X^v, X^w]$ then notice $T_e G \longrightarrow \Gamma_{inv}(G)$ proved that $\Phi(v) = X^v$

$$[v,w] = \Phi^{-1}([\Phi(v), \Phi(w)]) \quad \text{defines bracket}$$

We can move from $T_e G \underset{\Phi^{-1}}{\overset{\Phi}{\longleftrightarrow}} \Gamma_{inv}(G)$ hence its easy to calculate the tangent space at the identity & left-translate or just guess what $[v,w]$ should be ...

Pf// First note that

$$d_{\ell}\varphi : T_e G \longrightarrow T_e H$$

Is linear & so is the induced map

$$\tilde{\varphi} : \Gamma_{inv}(G) \longrightarrow \Gamma_{inv}(H)$$

defined by:

$$\tilde{\varphi} X_G^v = X_H^{d_e\varphi(v)}$$

Since we know $X^v(x) = d_e \ell_x(v)$ & $\exists$ only on v such that this works for for each thing in Γ_{inv}. We could write

$$\tilde{\varphi}(\Phi_G(v)) = \Phi_H(d_e\varphi(v))$$

So clearly $\tilde{\varphi} = \Phi_H \circ d_e\varphi \circ \Phi_G^{-1}$ so $\tilde{\varphi}$ and $d_e\varphi$ have same properties.

We show $\tilde{\varphi}$ preserves the bracket of $\Gamma_{inv}(G)$.

If $a, x \in G$ then consider

$$\begin{aligned}(\varphi \circ l_a^G)(x) &= \varphi(ax) \\ &= \varphi(a)\varphi(x) \qquad \varphi \text{ a homomorphism} \\ &= (l^H_{\varphi(a)} \circ \varphi)(x)\end{aligned}$$

Hence $\boxed{\varphi \circ l_a^G = l^H_{\varphi(a)} \circ \varphi}$ now take differential to see what results. The above is equivariance or perhaps covariance it says φ commutes with translations. If have vect. fields which are φ related its similar, (Point level $\to$ Vector Level)

$$\begin{aligned}d_x\varphi(\mathbb{X}_G^V(x)) &= d_x\varphi(d_e l_x^G(V)) \\ &= d_e(\varphi \circ l_x^G)(V) \\ &= d_e(l^H_{\varphi(x)} \circ \varphi)(V) \\ &= d_e l^H_{\varphi(x)}(d_e\varphi(V)) \\ &= \mathbb{X}_H^{d_e\varphi(V)}(\varphi(x))\end{aligned}$$

We have then in last Friday's notation that

we defined $[V,W]$ by the eq$^{\underline{n}}$ $[\mathbb{X}^V, \mathbb{X}^W] = \mathbb{X}^{[V,W]}$

$$\mathbb{X}_G^V \overset{\varphi}{\sim} \mathbb{X}_H^{d_e\varphi(V)}$$

$$\mathbb{X}_G^W \overset{\varphi}{\sim} \mathbb{X}_H^{d_e\varphi(W)}$$

$$[\mathbb{X}_G^V, \mathbb{X}_G^W] \overset{\varphi}{\sim} [\mathbb{X}_H^{d_e\varphi(V)}, \mathbb{X}_H^{d_e\varphi(W)}] = \mathbb{X}_H^{[d_e\varphi(V), d_e\varphi(W)]}$$

$$\Downarrow$$

$$\begin{aligned}d_e\varphi([\mathbb{X}_G^V, \mathbb{X}_G^W]) &= \mathbb{X}_H^{[d_e\varphi(V), d_e\varphi(W)]}(\varphi(e)) \\ &= \mathbb{X}_H^{[d_e\varphi(V), d_e\varphi(W)]}(e) \qquad \text{recalling } \mathbb{X}^V(e) = V \\ &= [d_e\varphi(V), d_e\varphi(W)]\end{aligned}$$

On the LHS we find ~~since~~

$$d_e\varphi\left([X_G^v, X_G^w](e)\right) = d_e\varphi\left(X_G^{[v,w]}(e)\right) = d_e\varphi([v,w])$$

$$\therefore \left[d_e\varphi(v), d_e\varphi(w)\right] = d_e\varphi([v,w])$$

Hence $d_e\varphi$ is a Lie Algebra Homomorphism.

//

Now show $l(\varphi)([v,w]) = [l(\varphi)(v), l(\varphi)(w)]$

Also $l(\varphi) = d_e\varphi$ and $l(\varphi) = \tilde{\varphi}$

$$\begin{aligned}
\tilde{\varphi}\left([X_G^v, X_G^w]\right) &= \tilde{\varphi}\left(X_G^{[v,w]}\right) \\
&= X_H^{d_e\varphi([v,w])} \\
&= X_H^{[d_e\varphi(v),\, d_e\varphi(w)]} && \text{just proved} \\
&= \left[X_H^{d_e\varphi(v)}, X_H^{d_e\varphi(w)}\right] \\
&= \left[\tilde{\varphi}(X_G^v), \tilde{\varphi}(X_G^w)\right]
\end{aligned}$$

$\therefore \tilde{\varphi}$ a Lie Alg. Homomorphism.

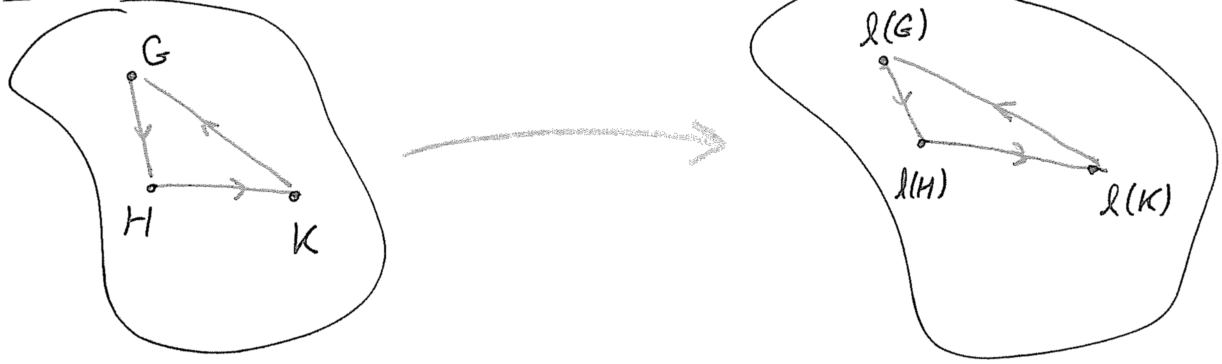
//

Now ~~β~~ we have two maps $\varphi : G \to H$ & $\psi : H \to K$

$$\begin{aligned}
l(\psi \circ \varphi) &= d_e(\psi \circ \varphi) \\
&= d_e\psi \circ d_e\varphi \\
&= l(\psi) \circ l(\varphi)
\end{aligned}$$

As an excercise try it for the ~ notation.

Lie Groups — Lie Algebra



Diagrams commute.

Remark: If $\varphi = id_G$ then $d_e\varphi = id_{T_eG}$

So if $\psi = \varphi^{-1}$ then we know $\psi \circ \varphi = id_G$, $\varphi \circ \psi = id_H$

$$l(\psi) \circ l(\varphi) = l(id_G) = id_{l(G)}$$

$$l(\varphi) \circ l(\psi) = id_{l(H)}$$

$$l(\varphi^{-1}) = l(\varphi)^{-1}$$

$\boxed{\text{Thm/ If } G \,\&\, H \text{ are simply connected and } l(G) \cong l(H) \text{ as Lie Algebras then } G \cong H \text{ as Lie Groups.}}$

Question: If G and H are not both simply connected but $l(G) \cong l(H) \overset{?}{\Rightarrow} G \cong H$?

Definition 1.5 A *Lie algebra* is a vector space $(L, +, \cdot)$ on which there is defined a bilinear mapping from $L \times L$ to L such that for $x, y \in L$ the value of the mapping at (x, y) is denoted $[x, y]$ and in addition to the bilinearity in x and y one also has the properties:

(i) $[x, y] = -[y, x]$

(ii) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

for all $x, y, z \in L$. When (i) is satisfied we say that the operation $[\cdot,\cdot]$ is *skew-symmetric* and we refer to (ii) as the *Jacobi identity*.

Remark: Note that generally a Lie algebra operation $[\cdot,\cdot]$ is not associative and in fact the Jacobi identity is a "replacement" for the associative property.

Example 1 Let $L = \mathbb{R}^3$ with the usual operations $+$ and $\cdot$. Define $[a, b] = a \times b$ for $a, b \in \mathbb{R}^3$. Then L is a Lie algebra with this definition of the Lie operation $[\cdot,\cdot]$.

Example 2 Let $L = gl(n)$ with the usual operations $+$ and $\cdot$. Let $[\cdot,\cdot]$ be defined on $gl(n)$ by $[A, B] = AB - BA$ for $A, B \in gl(n)$. Then L is a Lie algebra relative to these operations. Notice that if $\mathfrak{g}$ is any subspace of $gl(n)$ such that $A, B \in \mathfrak{g}$ implies $AB - BA \in \mathfrak{g}$ then $\mathfrak{g}$ will also be a Lie algebra and $\mathfrak{g}$ is

Before getting into the proof of the theorem we need a definition and a lemma.

Definition 1.7 Let M and N be manifolds and $\varphi : M \to N$ a smooth mapping. If $X \in \Gamma(M)$ and $Y \in \Gamma(N)$ then we say X and Y are <u>φ-related</u> iff for each $x \in X$

$d_x\varphi(X_x)(g) = Y_{\varphi(x)}(g)$

$$d_x\varphi(X_x) = Y_{\varphi(x)}.$$

$X_x(g\circ\varphi) = Y_{\varphi(x)}(g)$

Notice that we may also write this condition as $d\varphi \circ X = Y \circ \varphi$.

<u>Lemma</u> Let M and N be manifolds and $\varphi : M \to N$ a smooth mapping. Assume that $X_1, X_2 \in \Gamma(M)$ and $Y_1, Y_2 \in \Gamma(N)$. If X_i is φ-related to Y_i for $i = 1,2$ then $[X_1, X_2]$ is φ-related to $[Y_1, Y_2]$.

Proof of the Lemma. Since X_i is φ-related to Y_i for $i = 1,2$ we see that

$$[Y_i(g)\circ\varphi](x) = (Y_i)_{\varphi(x)}(g) = d_x\varphi(X_{ix})(g) = (X_i)_x(g\circ\varphi) = X_i(g\circ\varphi)(x)$$

for $x \in M$, $g \in C^\infty_{loc}(\varphi(x))$. It follows that

$$d\varphi([X_1, X_2]_x)(g) = [X_1, X_2]_x(g\circ\varphi) = (X_1)_x(X_2(g\circ\varphi)) - (X_2)_x(X_1(g\circ\varphi))$$

$h = Y_2(g)$

$(X_1)_x(h\circ\varphi) = (Y_1)_{\varphi(x)}(h)$

$$= (X_1)_x(Y_2(g)\circ\varphi) - (X_2)_x(Y_1(g)\circ\varphi)$$

$$= (Y_1)_{\varphi(x)}(Y_2(g)) - (Y_2)_{\varphi(x)}(Y_1(g)).$$

Notice that $\Gamma(M)$ is generally a vector space, a Lie-algebra and a $C^\infty(M)$ module where the module operation is defined by $(f\cdot X)_x = f(x) X_x$ for $f \in C^\infty(M)$, $X \in \Gamma(M)$. Moreover, locally

$$X = a^i \frac{\partial}{\partial x^i}$$

in terms of a chart (x^i). Since the a^i are smooth functions defined on the domain U of (x^i) and since $\partial/\partial x^i$ are also defined on U we see that, as a module, $\Gamma(U)$ is finitely generated. Thus $\Gamma(M)$ is locally finitely generated but as a vector space is infinite dimensional.

On the other hand $\Gamma_{inv}(G)$ is a finite dimensional vector space but is not a submodule of $\Gamma(G)$. Generally if X is left-invariant then fX will not be left-invariant.

Recall that $gl(n)$ is a vector space and so is a manifold with global chart the identity map. Moreover the identity on $Gl(n)$ maps $Gl(n)$ onto an open subset of the vector space $gl(n)$ as $Gl(n)$ is open in $gl(n)$. Define real-valued functions $x_{ij} : Gl(n) \to \mathbb{R}$ by $x_{ij}(A)$ is the entry in ~~the~~ i-th row and j-th column of A. Clearly x_{ij} is smooth for all i,j and are ~~the~~ components of an admissible chart of $Gl(n)$.

$$d_e\varphi([\mathfrak{X}_G^v, \mathfrak{X}_G^w](e)) = [\mathfrak{X}_H^{d_e\varphi(v)}, \mathfrak{X}_H^{d_e\varphi(w)}](e)$$

or

$$d_e\varphi\left(\mathfrak{X}_G^{[v,w]}(e)\right) = \mathfrak{X}_H^{[d_e\varphi(v),\, d_e\varphi(w)]}(e)$$

or

$$d_e\varphi([v,w]) = [d_e\varphi(v), d_e\varphi(w)].$$

Thus

$$l(\varphi)([v,w]) = [l(\varphi)(v), l(\varphi)(w)]$$

for $v, w \in T_eG$. For the sake of clarity also observe that

$$\tilde{\varphi}([\mathfrak{X}_G^v, \mathfrak{X}_G^w]) = \tilde{\varphi}(\mathfrak{X}_G^{[v,w]})$$

$$= \mathfrak{X}_H^{d_e\varphi([v,w])}$$

$$= \mathfrak{X}_H^{[d_e\varphi(v),\, d_e\varphi(w)]}$$

$$= [\mathfrak{X}_H^{d_e\varphi(v)}, \mathfrak{X}_H^{d_e\varphi(w)}] = [\tilde{\varphi}(\mathfrak{X}_G^v), \tilde{\varphi}(\mathfrak{X}_G^w)]$$

and so $\tilde{\varphi}$ is indeed a Lie-algebra homomorphism as it should be if our identifications are to be consistent. Finally observe that $d_e(\psi\circ\varphi) = d_e\psi \circ d_e\varphi$ and also

$$\widetilde{\psi\circ\varphi}(\mathfrak{X}_G^v) = \mathfrak{X}_K^{d_e(\psi\circ\varphi)(v)} = \mathfrak{X}_K^{d_e\psi(d_e\varphi(v))}$$

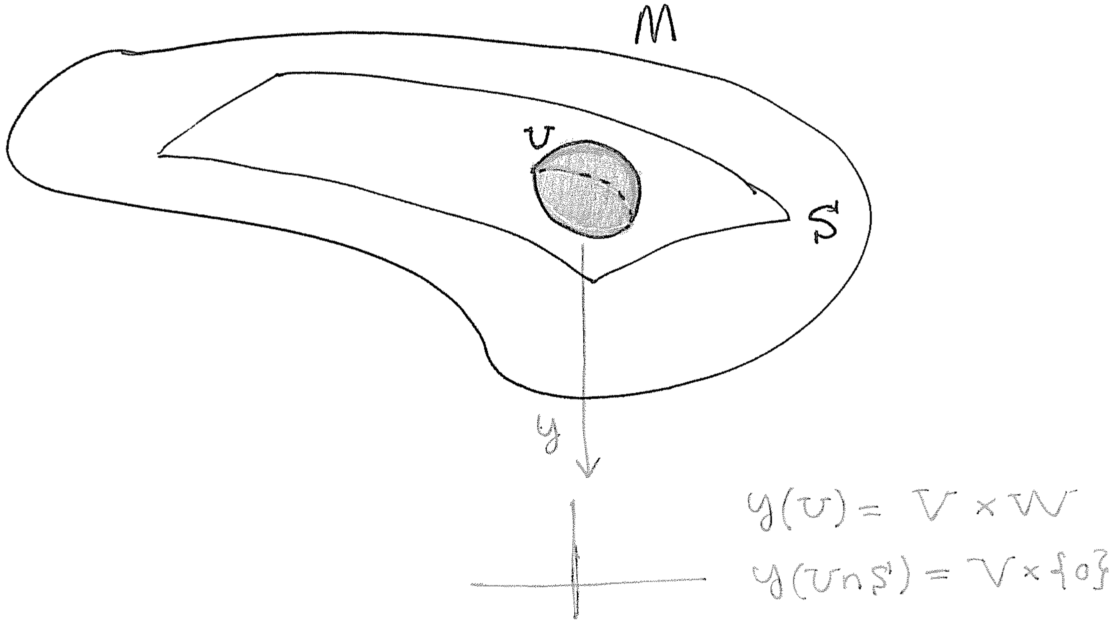
$$= \tilde{\psi}\left(\mathfrak{X}_H^{d_e\varphi(v)}\right) = (\tilde{\psi}\circ\tilde{\varphi})(\mathfrak{X}_G^v)$$

for all $v \in T_eG$. Thus we have both $d_e(\psi\circ\varphi) = d_e\psi \circ d_e\varphi$ and $\widetilde{\psi\circ\varphi} = \tilde{\psi}\circ\tilde{\varphi}$ and consequently $l(\psi\circ\varphi) = l(\psi)\circ l(\varphi)$

Remark Observe that if $\varphi: G \to G$ is the identity then $l(\varphi): l(G) \to l(G)$ is the identity on $l(G)$. Moreover, if $\varphi: G \to H$ and $\psi: H \to G$

- Actually what we discussed last time ~~th~~ was a functor from categorie of Lie Algebras to Lie groups.

- We define Submanifolds so that the charts are easily excessable



$(\pi_1 \circ y)|_{U\cap S}$

<u>Digression</u>: Submanifold's how to define

- Def^n (SUBMANIFOLD, almost the same) $\varphi : S \longrightarrow M$
  1-1 smooth and a homeomorphism onto it's image.

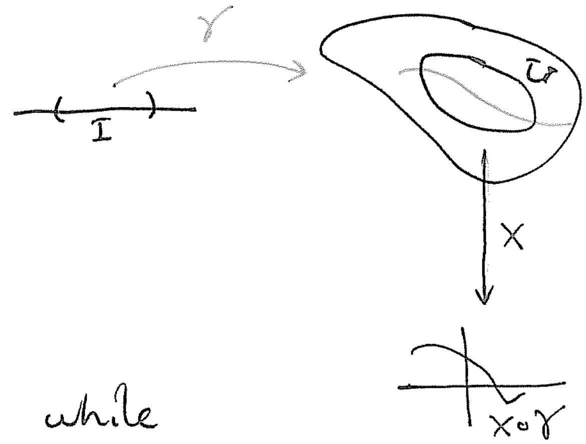
- In their case $\varphi(S)$ is the submanifold, but saying φ^{-1} is smooth doesn't necessarily make sense till we give S a diff. structure.

- In our case $S \subset M$ so $i : S \hookrightarrow M$ with $i(x) = x$ is smooth.
  Additionally i becomes lie group homorphism and in fact $\ell(i) : \ell(S) \longrightarrow \ell(M)$ can identify $\ell(S)$ as sublie algebra of $\ell(M)$

$$\{A \mid A^T G A = G\} \subseteq Gl(n)$$

So the lie algebra of $\{A \mid A^T G A = G\}$ of $gl(n)$. Anyway all the matrix groups are subsets of $Gl(n)$.

Def^n/ If $\mathbb{X}$ is a vector field on M then an integral curve of $\mathbb{X}$ is a function $\gamma : I \to M$ such that $\gamma'(t) = \mathbb{X}(\gamma(t)) \quad \forall t \in I$ where I interval $\subseteq \mathbb{R}$.



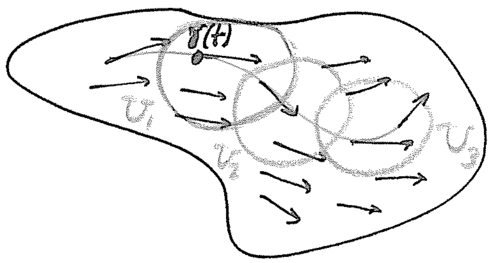
__Remark:__ $$\mathbb{X}(p) = \sum_i \mathbb{X}^i(p)\left(\frac{\partial}{\partial x^i}\Big|_p\right)$$

$$\gamma'(t) \in T_{\gamma(t)} M$$

$$\gamma'(t) = \sum_i \frac{d(x^i \circ \gamma)}{dt}(t) \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

$x \circ \gamma$ is a curve in Euclidean space while
$x^i \circ \gamma$ is a curve into $\mathbb{R}$ itself.

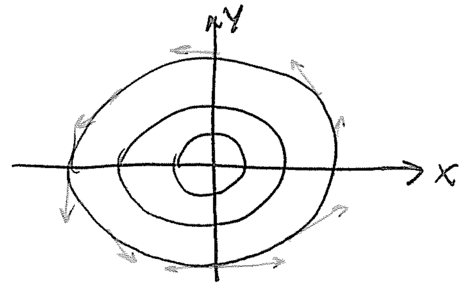
M



$$\gamma'(t) = \mathbb{X}(\gamma(t)) \iff \frac{d(x^i \circ \gamma)}{dt}(t) = \mathbb{X}^i(\gamma(t))$$

Theory of DEq^{n}'s says we can pick a compact nbhd and sol^{n} is gauranteed. Sometimes we can continue the curve all over M, but not always.

eg/ $$\mathbb{X}(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$



$$\gamma(t) = (x(t), y(t))$$

$$\mathbb{X}(\gamma(t)) = -y(t)\frac{\partial}{\partial x} + x(t)\frac{\partial}{\partial y}$$

$$\gamma'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$$

Thus equating yields

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= +x(t) \end{aligned} \Rightarrow x'' + x = 0 \Rightarrow \underline{\text{Circles}}$$

Thm/ Let $\mathfrak{X}$ be a left-invariant vector field on a Lie group G then

(1) If γ is an integral curve of $\mathfrak{X}$ through $e^2=e$ then γ is a 1-parameter group in G. ($\gamma: I \to G$ with $0 \in I^\circ$ and $\gamma(0)=e$)

(2) If γ is a 1-parameter group in G then γ is an integral curve of $\mathfrak{X}^v$ where $v=\gamma'(0) \in T_eG$

~~Remark~~: If $\gamma: I \to G$ is a curve then it is a (local) 1-parameter group if for $s,t \in I$ such that $s+t \in I$ then $\gamma(s+t) = \gamma(s)\gamma(t)$

When $I = \mathbb{R}$ then $\gamma: \mathbb{R} \to G$ is a Lie Homomorphism we say that γ is a global 1-parameter group.

Remark: ppl. did similar things in DEq^n's with flowcharts.

$$\varphi : \mathbb{R} \times M \longrightarrow M$$

(1) $\varphi(0,x) = x$

(2) $\frac{d}{dt}(\varphi(t,x)) = \mathfrak{X}(\varphi(t,x))$

$$t \longrightarrow \varphi(t,x) = \varphi_t(x)$$

$\varphi_t : M \longrightarrow M$ diffeomorphism

becomes 1-parameter group of diffeomorphisms

But usually this works only locally.

$$\varphi(t, \varphi(s,x)) = \varphi(t+s, x)$$ flow-box

Arnold cleaned up fluid-mechanics via such formalism.

Read Abraham & Marsden's Foundation of Mechanics

Proof: Assume on a ~~compact manifold~~ Lie group
(1) we can make any ~~vecto~~ integral curve complete, that is $I = \mathbb{R}$.
(Actually for any compact manifold the integral curves are complete)

Assume X is a left-invariant & $\gamma : \mathbb{R} \longrightarrow G$ is an integral curve such that $\gamma(0) = e$, meaning $X(\gamma(t)) = \gamma'(t)$

$$\frac{d}{dt}\left[\gamma(s+t)\right] = \gamma'(s+t) = X(\gamma(s+t))$$

$$\begin{aligned}
\frac{d}{dt}\left[\gamma(s)\gamma(t)\right] &= \frac{d}{dt}\left[l_{\gamma(s)}(\gamma(t))\right] \\
&= d_{\gamma(t)} l_{\gamma(s)}(\gamma'(t)) \qquad \text{Chain-rule.} \\
&= d_{\gamma(t)} l_{\gamma(s)}\left(X(\gamma(t))\right) \\
&= X\left(\gamma(s)\gamma(t)\right) \qquad \text{since } X \text{ is left-invariant}
\end{aligned}$$

Thus $t \longmapsto \gamma(s+t)$ and $t \longmapsto \gamma(s)\gamma(t)$ are both integral curves of X. At $t=0$ both curves yield $\gamma(s)$ now then the uniqueness thm of DEq^n's for a manifold $\Rightarrow$ $\underline{\gamma(s+t) = \gamma(s)\gamma(t)}$

Hence its a 1-parameter group.

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(2) Assume $\gamma : \mathbb{R} \longrightarrow G$ and $\gamma(s+t) = \gamma(s)\gamma(t)$

$$\frac{d}{dt}\left(\gamma(s+t)\right) = \frac{d}{dt}\left(\gamma(s)\gamma(t)\right)$$

$$\gamma'(s+t) = d_{\gamma(t)} l_{\gamma(s)}(\gamma'(t)) \qquad \text{true } \forall s,t \in \mathbb{R} \text{ then let } t=0,$$

$$\gamma'(s) = d_e\, l_{\gamma(s)}(\gamma'(0)) = X^v(\gamma(s)) \qquad (v = \gamma'(0))$$

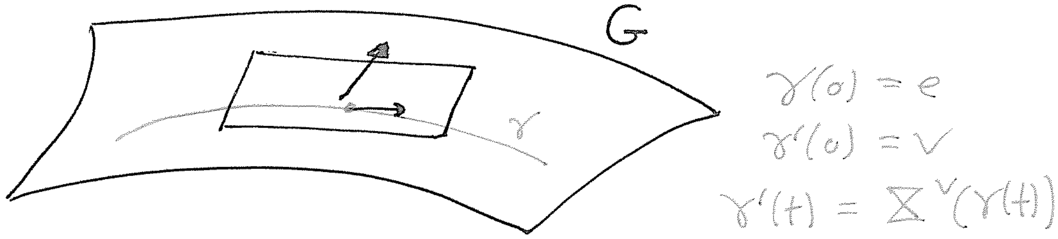
//

Def^n/ The exponential mapping is the function from $T_e G$ into G defined by

$$\exp(v) = \gamma_v(1)$$

where γ_v is the integral curve of X^v thru the identity.

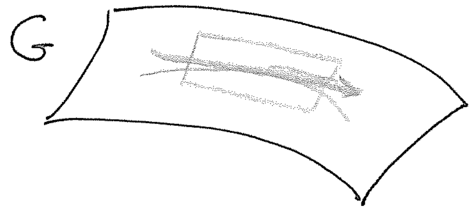
Remark : $\exp : T_e G \longrightarrow G$

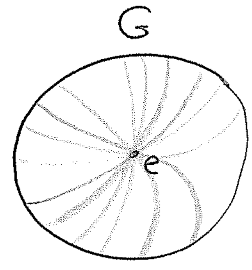


the intial conditions are at zero but exp is defined at 1 but every int. curve in Lie group can be extended to $I = \mathbb{R}$ so no problem.

- Same exp happens in Riem. Geom pushes straight lines on Tangent Space to Geodesics in Manifold
- Here exp pushes straight lines to 1-parameter groups $\gamma(t) = \exp(sv)$

$\boxed{\text{Th}^{\underline{m}}\text{: If } \gamma \text{ is a 1-parameter subgroup of a Lie group } G \text{ then } \gamma(s) = \exp(sv) \text{ where } v = \gamma'(0)}$





the whole nbhd of e is ruled with 1-param. groups.

Analogously: The exp map for a Rimannian Manifold that maps from tangent space of manifold onto the manifold sending straight lines down to geodesics.
Choice MTW + SPIVAK or Sachs & Wu for general relativity. Berkeley. (General Relativity for Mathematicians)
(Difference is manifolds have tangent spaces everywhere!)

Proof:
Since γ is a 1-parameter subgroup it is a sol$^{\underline{n}}$ of the "diff. eq^{n}" or vect. field.

$$\gamma'(t) = X^v(\gamma(t)) \quad \text{where } v = \gamma'(0)$$

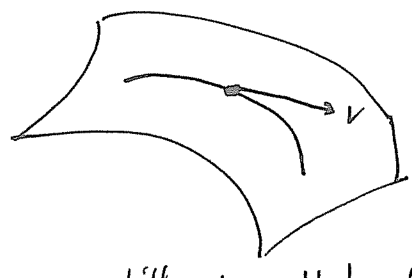
Where $X^v(\gamma(t)) = d_e l_{\gamma(t)}(v)$ which says $X^v \in \Gamma_{inv}(M)$. (oops $M = G$)

Consider the curve $t \longrightarrow \gamma(st)$ where $s \in \mathbb{R}$, notice that

$$\gamma(s(t_1+t_2)) = \gamma(st_1 + st_2)$$
$$= \gamma(st_1)\gamma(st_2)$$

Take derivative then and use chain rule

$$\frac{d}{dt}\left[\gamma(st)\right] = \gamma'(st)s$$
$$= s X^v(\gamma(st))$$
$$= s\, d_e l_{\gamma(st)}(v)$$
$$= d_e l_{\gamma(st)}(sv)$$
$$= X^{sv}(\gamma(st))$$



different multiples of v push you further out the 1-parameter group

Now s is fixed here so
$t \longmapsto \gamma(st)$
is a sol$^{\underline{n}}$ of X^{sv} whose value at zero is e.

$$\exp(sv) = \gamma(s(1)) = \gamma(s) \quad //$$

(Only gap is we take domain to be $\mathbb{R}$, okay as previously explained)

Remark: The mapping $\exp : T_e G \longrightarrow G$ is smooth

$$\exp(v) = \gamma_v(1)$$

$$\gamma_v'(t) = X^v(\gamma_v(t))$$

You write

$$\frac{\partial}{\partial t}\left[\gamma(t,v)\right] = X(\gamma(t,v), v)$$

this X is a function of two-variables. Have a parametrized family of diff. eqⁿ's. This is a sol^Δ which is smooth.... a non-trivial Thᵐ in Dif. Eqⁿ's.

Thᵐ/ The $\exp : T_e G \longrightarrow G$ ~~has~~ is a local diffeomorphism meaning that locally it is smooth with smooth inverse.

Proof: Let $\gamma(s) = \exp(sv)$. Now we differentiate the exponential & use the Chain Rule. We know $\gamma(s)$ is 1-parameter group & $\gamma'(0) = v$

$$\gamma'(s) = d_{sv}(\exp)(v)$$

$$\gamma'(0) = d_0(\exp)(v) = v \quad (\text{since } \gamma'(0) = v)$$

$$\therefore \quad d_0(\exp) = id_{T_e G}$$

$$\begin{tikzcd}
T_e G \arrow[r] \arrow[d, "id"'] & G \arrow[d, "x"] \\
\mathbb{R}^n \arrow[r, "x^{-1}\circ\exp", "f"'] & \mathbb{R}^n
\end{tikzcd}$$

apply inverse function thᵐ to $x^{-1}\circ\exp$ is local diffeomorphism

$$\exp = x^{-1}\circ f$$

{diffeomorphism ⇓ Compatibility}

$\therefore$ $\exists$ an open set (U) about $0 \in T_e G$ such that $\exp|_U : U \to \exp(U)$ where $\exp(U)$ is a open nbhd of the identity in G and $\exp|_U$ is a diffeomorphism.

So $\log_U$ is a chart at e!

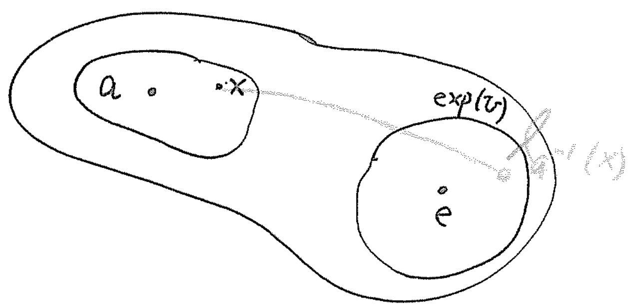
$$\log_U \equiv (\exp|_U)^{-1} \qquad \log_U : \exp(U) \longrightarrow U$$

$\exp(U)$: open in G ; U: open in a vector space. $(\mathbb{R}^n)$

9/4/04 3

How to construct charts at the identity

$\ell_a \circ \log_v$ is a chart at $a \in G$



take points near a then $\ell_{a^{-1}}$ transports those points close to e

$$\log_v \circ \ell_{a^{-1}} : \ell_a(\exp(v)) \longrightarrow \mathbb{R}^n$$

is a diffeomorphism $\Rightarrow$ compatible with other charts.
we can hence construct charts at a via log.

//

$A \in gl(n)$ the matrix exponential is

$$e^A = I + A + \frac{1}{2!}A^2 + \cdots$$

$$S_p = \sum_{n=0}^{p} \frac{1}{n!}A^n = I + A + \frac{1}{2}A^2 + \cdots + \frac{1}{p!}A^p$$

This is a Cauchy sequence. Take $p > q$

$$S_p - S_q = \sum_{n=0}^{p} \frac{A^n}{n!} - \sum_{m=0}^{q} \frac{A^m}{m!} = \sum_{n=q+1}^{p} \frac{A^n}{n!}$$

$$\|S_p - S_q\| = \left\| \sum_{n=q+1}^{p} \frac{1}{n!}A^n \right\| \le \sum_{n=q+1}^{p} \frac{1}{n!}\|A^n\| \quad \text{Triangle Inequality.}$$

Because $gl(n) \cong \mathbb{R}^{n^2}$ it is a Banach Algebra meaning $\|AB\| \le \|A\|\|B\|$ and $\|I\| = 1$. Apply this to the above

$$\|S_p - S_q\| \le \sum_{n=q+1}^{p} \frac{1}{n!}\|A\|^n$$

Note that $\sum_{n=0}^{\infty} \frac{1}{n!}\|A\|^n = e^{\|A\|}$ convergence from ordinary calculus

If $\mathcal{S}_p = \sum_{n=0}^{p} \frac{1}{n!} \|A\|^n$ then $\mathcal{S}_p \longrightarrow e^{\|A\|}$ and $\{\mathcal{S}_p\}$ is a Cauchy sequence, $\forall\, \varepsilon > 0\ \exists\, n \in \mathbb{N}$ such that $p, q \geq N$

$$\Rightarrow |\mathcal{S}_p - \mathcal{S}_q| < \varepsilon$$

Therefore since

$$\| S_p - S_q \| = |\mathcal{S}_p - \mathcal{S}_q| < \varepsilon$$

There fore $\{S_p\}$ is Cauchy & since $gl(n)$ is complete (it is the same as $\mathbb{R}^{n^2}$). So $S_p \longrightarrow S$ //

$$\boxed{\text{Th}^{m}/\ \text{If } [A,B] = 0 \text{ then } e^{A+B} = e^A e^B = e^B e^A}$$

Th^{m}/ If $A \in gl(n)$ then $\gamma(t) = e^{tA}$ is a 1-parameter group in $Gl(n)$.

Proof Sketch:

tA & sA commute

$$e^A e^{-A} = e^0 = I$$

almost trivial

Next: How to calculate Γ_{inv} on $Gl(n)$

Vector fields on $\mathcal{GL}(n)$

$$e^A = I + A + \frac{1}{2!}A^2 + \cdots$$

$$e^A e^B = e^{A+B} = e^B e^A \quad \text{provided } [A,B]=0$$

$$\gamma(t) = e^{tA}$$

$$\gamma(t+s) = e^{(t+s)A} = e^{tA}e^{sA} = \gamma(t)\gamma(s)$$

$$e^A e^{-1(A)} = e^0 = I \quad \therefore \; e^A \in \mathcal{GL}(n)$$

<u>Remark</u>:

$$\gamma(0) = I$$

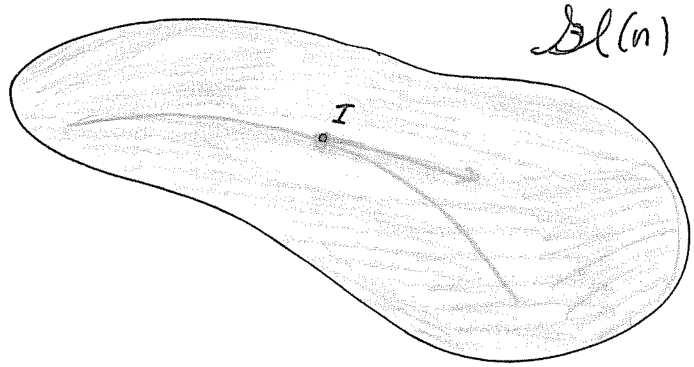
$$\gamma'(0) = ?$$

$$e^{tA} = I + tA + \tfrac{1}{2}(tA)^2 + \cdots$$

$$\begin{aligned}\frac{d}{dt}e^{tA} &= A + tA^2 + \tfrac{1}{2}t^2A^3 \\ &= A(I + tA + \tfrac{1}{2}t^2A^2 + \cdots) \\ &= Ae^{tA} = \frac{d}{dt}(e^{tA})\end{aligned}$$

$$\left(\text{Since } \frac{d}{dt}\left(\frac{1}{n!}t^n\right) = \frac{n-1}{n!}t^{n-1} = \frac{1}{(n-1)!}t^{n-1}\right)$$

Anyway $\underline{\gamma'(0) = A} \in \mathfrak{gl}(n) = T_I\mathcal{GL}(n)$, $\gamma'(0) = \sum_{i,j} A_{ij}\frac{\partial}{\partial x_{ij}}$



have a 1-parameter group through the identity.

Think of B as a matrix or as a tang. vector at identity of $Gl(n)$

$$\underline{X}^B(x) = d_I l_x(B)$$

and $\gamma : \mathbb{R} \to Gl(n)$
with $\gamma'(0) = B$ & $\gamma(t) = e^{tB}$

$$B = \sum B_{ij} \frac{\partial}{\partial x_{ij}}$$

$$\frac{d}{dt}\left[l_x(\gamma(t)) \right] = d_{\gamma(t)} l_x(\gamma'(t))$$

$$\|$$

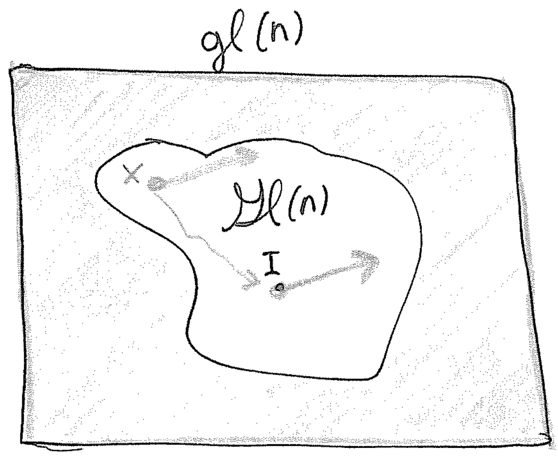
$$\frac{d}{dt}\left[x\gamma(t) \right] = d_{\gamma(t)} l_x(\gamma'(t))$$

$$\therefore \quad xB = d_I l_x(B)$$

Or in more manifestly manifold language

$$d_I l_x\left(\sum_{ij} B_{ij} \frac{\partial}{\partial x_{ij}} \right) = \sum (xB)_{ij} \frac{\partial}{\partial x_{ij}}$$

Since $T_x Gl(n) = gl(n)$ is the identification.



$Gl(n)$ is open & dense in $gl(n)$.

$$T_I Gl(n) = gl(n) \quad \text{but} \quad T_x Gl(n) = gl(n)$$

allways move vector to I & use same $gl(n)$
just like vectors in $\mathbb{R}^3$.

$$x_{ij}(M) = M_{ij} \qquad \underline{X}^B(x) = \sum_{ij} \sum_k x_{ik} B_{kj} \frac{\partial}{\partial x_{ij}}$$

What are the brackets of 2 $Gl(n)$ vector fields? 9/8/04 ③

$$[\mathbb{X}^B, \mathbb{X}^C] = \left[\mathbb{X}^B((\mathbb{X}^C)^{pq}) - \mathbb{X}^C((\mathbb{X}^B)^{pq})\right]\frac{\partial}{\partial X_{pq}}$$

$$= \left\{\mathbb{X}^B\left(\sum_k X_{pk}C_{kq}\right) - \mathbb{X}^C\left(\sum_k X_{pk}B_{kq}\right)\right\}\frac{\partial}{\partial X_{pq}} \quad \text{summed over } p, q$$

$$= \sum (BC - CB)_{pq}\frac{\partial}{\partial X_{pq}} \quad \text{(exercise to reader)}$$

$$= \mathbb{X}^{[B,C]}$$

Hence when you think of Γ_{inv} you can think in two ways

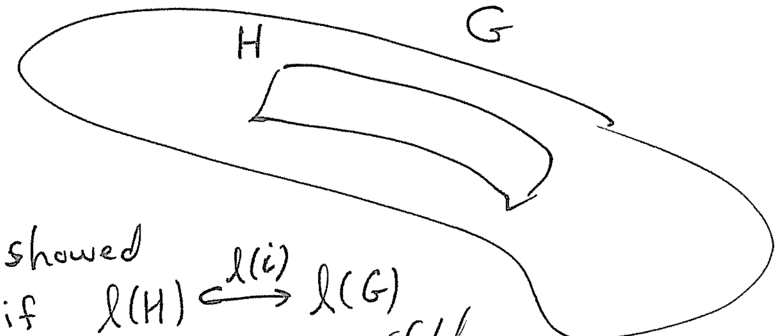
$\mathbb{X}^B$ where $B \in T_e Gl(n)$

$\mathbb{X}^{[v,w]} = [\mathbb{X}^v, \mathbb{X}^w]$ abstractly defined

is actually the specific $\mathbb{X}^d$ gotten from $d = [B, C]$

where $[B, C]$ is the plain-old commutator.

//



What we showed is that if $l(H) \xrightarrow{l(i)} l(G)$

provided H is a submanifold

$i : H \hookrightarrow G$

take any $v \in T_I H$ that gives you $\mathbb{X}_H^v \xrightarrow{l} \mathbb{X}_G^v$
in H in G

Take any subliegroup of $Gl(n)$ and apply this trickery.

Example

$$G^2 = I$$
$$G^t = G$$

$\mathcal{O}(n) = \{ A \in gl(n) \mid A^t A = I \}$ take $G = I$

Let $A(\lambda)^t A(\lambda) = I$ with $A(0) = 0$ & $A'(0) = B$

$$B^t + B = 0$$

$o(n) = \{ B \mid B^t + B = 0 \}$ Lie Algebra of $\mathcal{O}(n)$

$$o(n) = T_I \mathcal{O}(n)$$

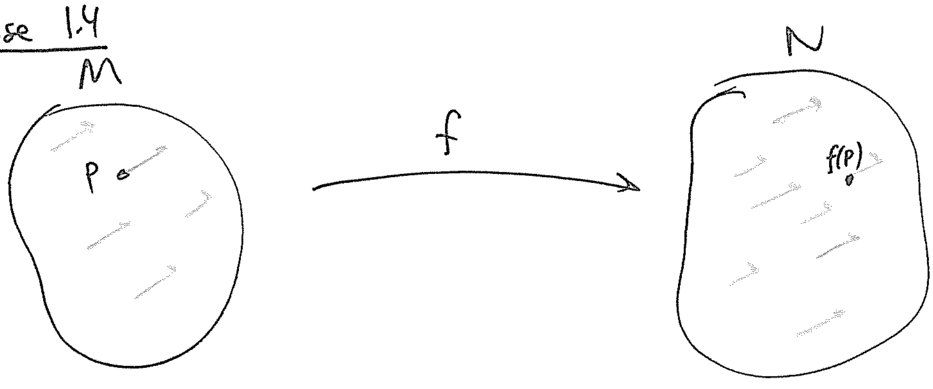
Take brackets on $Gl(n)$ just commutator brackets
Now abstractly we can already conclude $o(n)$ is Lie Alg.

$$X^B(x) = d_I l_x^{\mathcal{O}(n)}(B)$$

$$[X^{B_1}, X^{B_2}]_{o(n)} = X^{[B_1, B_2]}$$

//

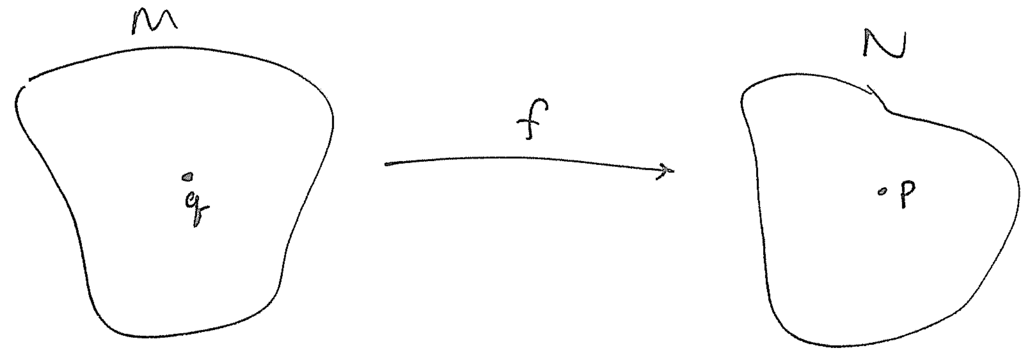
Exercise 1.4



Φ related is to get away from problem below

can push forward provided f is nice enuf.

$$d_p f(X_p) = Y_{f(p)}$$



$\beta_p(V_1, V_2, \ldots, V_k)$, $\beta_p : \underbrace{T_pN \times T_pN \times \cdots \times T_pN}_{k \text{ of these}} \longrightarrow \mathbb{R}$

$$\beta_p = \frac{1}{k!}\beta_{i_1 i_2 \cdots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

not using a basis if we were take increasing indices & lose the $\frac{1}{n!}$

$$\beta_p = \beta_I\, dx^I$$

$$\beta_p \in \Omega^k_p(N)$$

Pull back

$$(f^*\beta)_q(W_1, W_2, \ldots, W_k) = \beta_{f(q)}(d_qf(W_1), d_qf(W_2), \ldots, d_qf(W_k))$$

$$(f^*\beta)_q \in \Omega^k_q(M)$$

$$f^*\beta = f^*\left[\frac{1}{k!}\beta_{i_1 i_2 \cdots i_k}(dx^{i_1} \wedge \cdots \wedge dx^{i_k})\right]$$

We know its linear & $f^*(\alpha \wedge \gamma) = f^*\alpha \wedge f^*\gamma$

$$f^*\beta = \frac{1}{k!}(f^*\beta_{i_1 \cdots i_k})\left[d(f^*x^{i_1}) \wedge \cdots \wedge d(f^*x^{i_k})\right]$$

Now then pull-back of function is easy, $f^*(g) = g \circ f$

$$f^*\beta = \frac{1}{k!}(\beta_{i_1 i_2 \cdots i_k} \circ f)\, d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_k} \circ f)$$

$$= \frac{1}{k!}(\beta_{i_1 i_2 \cdots i_k} \circ f)\, \frac{\partial(x^{i_1} \circ f)}{\partial y^{j_1}} dy^{j_1} \wedge \cdots \wedge \frac{\partial(x^{i_k} \circ f)}{\partial y^{j_k}} dy^{j_k}$$

Now we're just doing 1-forms for now but...

Exercise 1.4

$\alpha \in T_e^* G$

left translate all over group. Need to do it via the pull-back.

$\ell\alpha$ – left-invariant form

$(\ell\alpha)_g = L_{g^{-1}}^* \alpha = \alpha \circ d_g L_{g^{-1}}$ acts on $V_g \in T_g G$

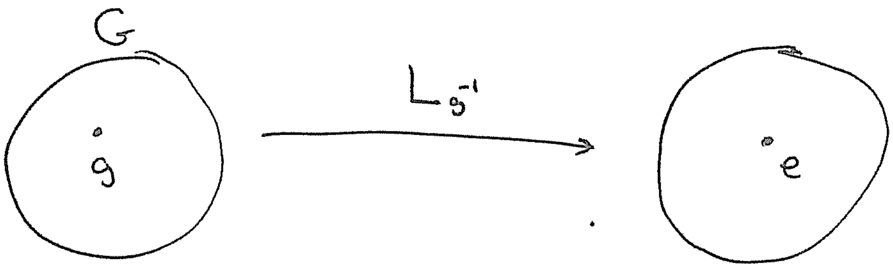
$(\ell\alpha)_g (v) = \alpha \circ d_g L_{g^{-1}} (V_g)$

In part 1) $L_h^* (\ell\alpha) = L_h^* (\alpha \circ d_g L_{g^{-1}})$

use chain rule on differentials

& $L_a L_b = L_{ab}$ it'll pop right on out. //

$\alpha \in T_e^* G$



$L_{g^{-1}} = (L_g)^{-1}$

$l : T_e^* G \longrightarrow \Omega_{inv}(G)$

$(L_{g^{-1}})^* (\alpha) = \alpha \circ dL_{g^{-1}}$

$(l\alpha)_g \in T_g^* G$

$l\alpha$ is a 1-form

$(l\alpha)_g : T_g G \longrightarrow \mathbb{R}$

---

② Show that for every left 1-form β $\exists\, \alpha$ with $\beta = l\alpha$
that is show l is onto

$\alpha = \beta_e$

$l\alpha$ & use left-invariance of β to finish

$l\beta_e$

Then show $\alpha \longmapsto l\alpha$ is vect. space

$l(\alpha_1 + \alpha_2) = l\alpha_1 + l\alpha_2$

1-1 show $\mathrm{Ker}(l) = 0$

$l\alpha = 0 \Rightarrow \alpha \circ L_{g^{-1}} = 0 \quad \forall g$

compose with $dL_g \Rightarrow \alpha \circ L_{g^{-1}} \circ dL_g = \alpha = 0.$

3 $\beta \in \Omega_{inv}$ by part 2 $\Rightarrow \beta = l\alpha$ for $\alpha = \beta_e$

$X \in \Gamma_{inv}$ means $X = X^v$

Use left invariance of both use

$$\beta_g = \alpha \circ d_g L_{g^{-1}} \quad \& \quad X_g = d_e L_g(v)$$

Some cancellation

4 $\beta_g(X_g) = (l\alpha)_g(X_g^v) = \alpha(v)$ in #3. then

apply this to special cases

T_eG has basis $\{e_i\}$

T_e^*G has dual basis $\{e^j\}$ with $e_i(e^j) = \delta_i^j$

Take any $v \in T_eG$ we get X^v but we write for $v = e_i$

$$X_i = X^{e_i} \equiv l(e_i)_g = d_e L_g(e_i)$$

Likewise take any covector $\alpha \in T_e$ but we write for $\alpha = e^i$

$$(\beta^j)_g = l(e^j)_g \equiv e^j \circ d_g L_{g^{-1}}$$

Then $X_i \in \Gamma_{inv}(G)$ have n- of those, they're L.I & span T_gG

Similarly $\beta \in \Omega^1_{inv}(G)$

$\beta_g \in T_g^*G$ and $\{\beta^j(g)\}$ basis of T_g^*G

Digression If γ is any differential form on G then γ can be written as

$$\gamma = \sum f_j \beta^j \quad \text{for } f_j \in C^\infty(G)$$

Can prove smooth-ness from contracting over Γ_{inv}. But if γ is left invariant then.

$$\gamma = \sum f_j \beta^j \quad \text{for } f_j \in \mathbb{R}.$$

Just hit on Γ_{inv} & use problem. The space of diff. forms on G is ∞-dim'l, but the $\Gamma_{inv}(G)$ is finite dim'l, its the same as the $\dim(T_e^*G)$ aka $\dim(G)$.

④ (a) $\beta^j(X_i) = \delta^j_i \Rightarrow$ differentiating yields zero.

(b) We've got LIVF & $[X_i, X_j]$ is LIVF & we have Γ_{inv} forms a basis. thus

$$[X_i, X_j] = \sum_k f_{ij}^k X_k \qquad f_{ij}^k \in \mathbb{R}$$

(Notice f_{ij}^k are basis dependent, see what happens if you change basis. (Digression))

Show that then

$$d\beta^j = -\frac{1}{2} f_{ij}^k \beta^i \wedge \beta^j$$

($d\beta^j$ is LIV 2-form & $\beta^i \wedge \beta^j$ is LIV 2-form ...)

Just take X_p, X_q & evaluate LHS & RHS

using $d\beta(X, Y) = \cancel{X(\beta(Y))}^{0} - \cancel{Y(\beta(X))}^{0} - \underbrace{\beta([X, Y])}_{\text{all here}}$

falls out of identity: $X_i(\gamma(\cdots))$

$$d\gamma(X_0, X_1, \dots, X_p) = \sum_{i=0}^{p} (-1)^i \cancel{\gamma_i} X_i (X_0, \dots, \hat{X}_i, \dots, X_p)$$
$$+ \sum_{i<j} (-1)^{i+j} \gamma(X_0, \dots, [X_i, X_j], \dots, X_p)$$

(delete X_i & X_j) replacing X_i with $[X_i, X_j]$

Mauer-Cartan Formula:

$$\frac{\mathbb{R}^n \times SO(n)}{SO(n)} \cong \mathbb{R}^n \qquad \frac{\mathbb{R}^4 \times \text{Lorentz}}{\text{Lorentz}} \cong \mathbb{R}^4 = \mathcal{M}$$

take Mauer Cartan up here...

An important idea

$$\det(e^A) = e^{\text{Trace}(A)}$$

Why well $O(n) = \{A \mid A^T A = I\}$ and $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$

$o(n) = \{B \mid B^t = -B\}$ what about $so(n)$? let $A \in so(n)$

$$1 = \det(e^A) = e^{\text{Trace}(A)} \Rightarrow \underline{A \text{ is traceless.}}$$

$$A^t \eta A = \eta \qquad A \text{ is Lorentz Matrix.}$$

$$\det(A^t)\,\cancel{\det(\eta)}\,\det(A) = \cancel{\det(\eta)}$$

$$(\det(A))^2 = 1$$

$$\det(A) = \pm 1. \Rightarrow \text{Lorentz group disconnected.}$$

Actually $\det(A) = 1$ has future & past components

Lie algebra: $B^t \eta = -\eta B$ and $tr(B) = 0$ to get $\det(e^B) = 1$.

$Gl(n, \mathbb{C})$ let $f(A) = (A^t)^* G A - G$

↑ real $2n^2$ dim'l Lie Algebra.

$U(n)$ & $SU(n)$ exc....

Proof of $\det(e^A) = e^{\text{trace}(A)}$

$$f(t) = \det(e^{tB})$$

$$\frac{d}{dh}\left[f(t+h)\right] = \frac{d}{dh}\left(\det(e^{tB})\det(e^{hB})\right)$$
$$= \det(e^{tB})\frac{d}{dh}\left(\det(e^{hB})\right)$$

Call this $g(h)$

Well det is multilinear mapping, concatenate $g(h)$

$$g(h) = \left[g^{(1)}(h) \mid g^{(2)}(h) \mid \cdots \mid g^{(n)}(h)\right]$$

$$\frac{d}{dh}\left(\det\left[g^{(1)}(h) \mid g^{(2)}(h) \mid \cdots \mid g^{(n)}(h)\right]\right)$$

Th$^{\underline{m}}$ say diff. multilinear map work's like product.

$$\det\left[\frac{d}{dh}\left(g^{(1)}(h)\right) \middle| g^{(2)}(h) \middle| \cdots \middle| g^{(n)}(h)\right] + \det\left[g^{(1)}(h) \middle| \frac{d}{dh} g^{(2)}(h) \middle| \cdots \middle| g^{(n)}(h)\right]$$

$$+ \cdots + \det\left[g^{(1)}(h) \middle| \cdots \middle| \frac{d}{dh}\left(g^{(n)}(h)\right)\right]$$

Substitute $h=0$ on both sides.

$$f'(t) = \det(e^{tB})\left[\det\left[B^{(1)}I^{(2)}\cdots I^{(n)}\right] + \cdots + \det\left[I^{(1)}\cdots B^{(n)}\right]\right]$$
$$= \det(e^{tB})\left[B^{11} + B^{22} + \cdots + B^{nn}\right]$$
$$\frac{df}{dt} = \det(e^{tB})\ \text{trace}(B) = f(t)\,\text{trace}(B)$$

$$\Rightarrow f(t) = e^{\text{Tr}(B)\,t} = \det(e^{tB}) \qquad \underline{t=1}$$

$$\therefore \boxed{e^{\text{Trace}(B)} = \det(e^{B})}$$

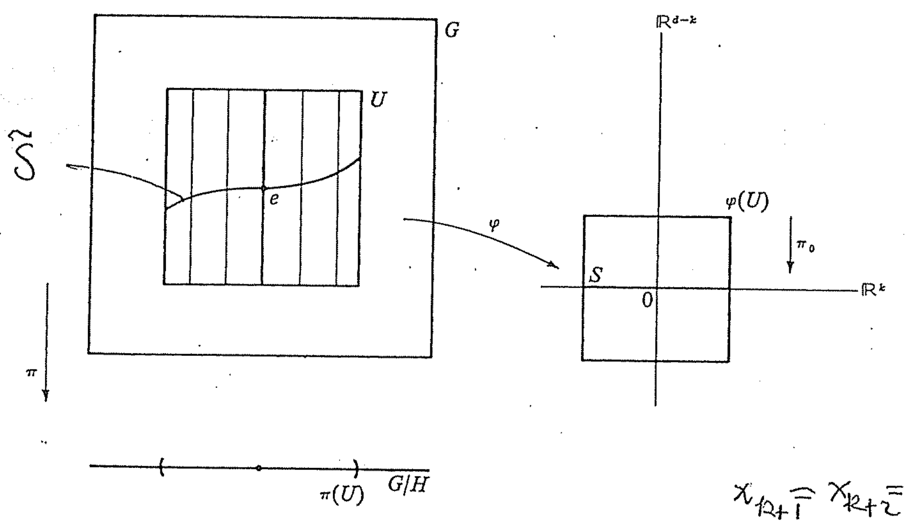
Choose neighborhoods U and V_1 of e, cubic relative to the coordinate system (V,φ), such that

(3) $$V_1V_1 \subset V \quad \text{and} \quad U^{-1}U \subset V_1.$$

Now suppose that σ and τ are points of U which lie in the same coset modulo H, so $\sigma \in \tau H$. Then

(4) $$\tau^{-1}\sigma \in V_1 \cap H = V_1 \cap S_0.$$

So $\sigma \in \tau(V_1 \cap S_0)$. Now $\tau(V_1 \cap S_0)$ is an integral manifold of $\mathscr{D}$ which lies in V by the choice of V_1 in (3), and $\tau(V_1 \cap S_0)$ is connected. Therefore $\tau(V_1 \cap S_0)$ lies in a single slice of V. So σ and τ lie in the same slice. Conversely, it is easily seen that a single slice lies on a single coset. Thus (U,φ) is the desired coordinate system.



$x_{k+1} = x_{k+2} = \cdots = x_d = 0$ defines a submanifold $\tilde{S}$ of U and $\varphi|_{\tilde{S}}$ is a diffeomorphism from $\tilde{S}$ onto S.

Let S be the slice of $\varphi(U)$ on which $x_{k+1}, \ldots, x_d$ vanish. Let $\tilde{\varphi}^{-1}$ be the map defined by setting

(5) $$\tilde{\varphi}^{-1} = \pi \circ \varphi^{-1} \big| S \colon S \to \pi(U).$$

Then $\tilde{\varphi}^{-1}$ is one-to-one by the choice of the coordinate system (U,φ), and is also continuous and an open map; hence it is a homeomorphism. Let $\tilde{\varphi}$ be the inverse,

(6) $$\tilde{\varphi} \colon \pi(U) \to S \subset \mathbb{R}^k.$$

Then $(\pi(U),\tilde{\varphi})$ is a coordinate system about the identity coset in G/H. We obtain coordinate systems about other points of G/H by left translations. Indeed, if $\sigma \in G$, we let $\tilde{l}_\sigma$ be the homeomorphism of G/H induced by left translation l_σ on G; that is,

(7) $$\tilde{l}_\sigma(\tau H) = \sigma\tau H.$$

$\pi|\tilde{S}$ is a diffeom from $\tilde{S}$ onto $\pi(U)$. Its inverse is the section at e.

$m \in M$. A smooth distribution $\mathscr{D}$ is called *involutive* (or *completely integrable*) if $[X, Y] \in \mathscr{D}$ whenever X and Y are smooth vector fields lying in $\mathscr{D}$.

1.57 Definition A submanifold (N,ψ) of M is an *integral manifold* of a distribution $\mathscr{D}$ on M if

$$(1) \qquad d\psi(N_n) = \mathscr{D}(\psi(n)) \quad \text{for each } n \in N.$$

1.58 Remarks Our object in this section is to prove that a necessary and sufficient condition for there to exist integral manifolds of $\mathscr{D}$ through each point of M is that $\mathscr{D}$ be involutive. Perhaps a word of explanation is in order about the expression "*completely integrable*" sometimes used in place of "involutive." We have required integral manifolds to be submanifolds whose tangent spaces coincide with the subspaces determined by the distribution. One could define a weaker notion of integral manifold by requiring only that the tangent spaces of the submanifold be contained in but not necessarily equal to the distribution at each point. It is possible for a distribution $\mathscr{D}$ to be "integrable" in the sense that it has low-dimensional "integral manifolds," but not completely integrable in the sense that $\mathscr{D}$ does not have integral manifolds of the maximal dimension. For us, unless specified otherwise, integral manifolds of distributions will always be taken to mean integral manifolds of maximal dimension, that is, as defined in 1.57.

1.59 Proposition *Let $\mathscr{D}$ be a smooth distribution on M such that through each point of M there passes an integral manifold of $\mathscr{D}$. Then $\mathscr{D}$ is involutive.*

PROOF Let X and Y be smooth vector fields lying in $\mathscr{D}$, and let $m \in M$. We must prove that $[X, Y]_m \in \mathscr{D}(m)$. Let (N,ψ) be an integral manifold of $\mathscr{D}$ through m, and suppose that $\psi(n_0) = m$. Since $d\psi\colon N_n \to \mathscr{D}(\psi(n))$ is an isomorphism at each n in N, there exist vector fields $\tilde{X}$, $\tilde{Y}$ on N such that

$$(1) \qquad \begin{aligned} d\psi \circ \tilde{X} &= X \circ \psi, \\ d\psi \circ \tilde{Y} &= Y \circ \psi. \end{aligned}$$

It is easily checked that $\tilde{X}$ and $\tilde{Y}$ are smooth. By 1.55, $[\tilde{X},\tilde{Y}]$ and $[X, Y]$ are ψ-related. Hence $[X, Y]_m = d\psi([\tilde{X},\tilde{Y}]_{n_0}) \in \mathscr{D}(m)$.

1.60 Theorem (Frobenius) *Let $\mathscr{D}$ be a c-dimensional, involutive, C^∞ distribution on M^d. Let $m \in M$. Then there exists an integral manifold of $\mathscr{D}$ passing through m. Indeed, there exists a cubic coordinate system (U,φ) which is centered at m, with coordinate functions $x_1, \ldots, x_d$ such that the slices*

$$(1) \qquad x_i = \text{constant} \qquad \text{for all } i \in \{c+1, \ldots, d\}$$

are integral manifolds of $\mathscr{D}$; and if (N,ψ) is a connected integral manifold of $\mathscr{D}$ such that $\psi(N) \subset U$, then $\psi(N)$ lies in one of these slices.

Th$^{\underline{m}}$

Let H be a closed subgroup of a Lie group G and let

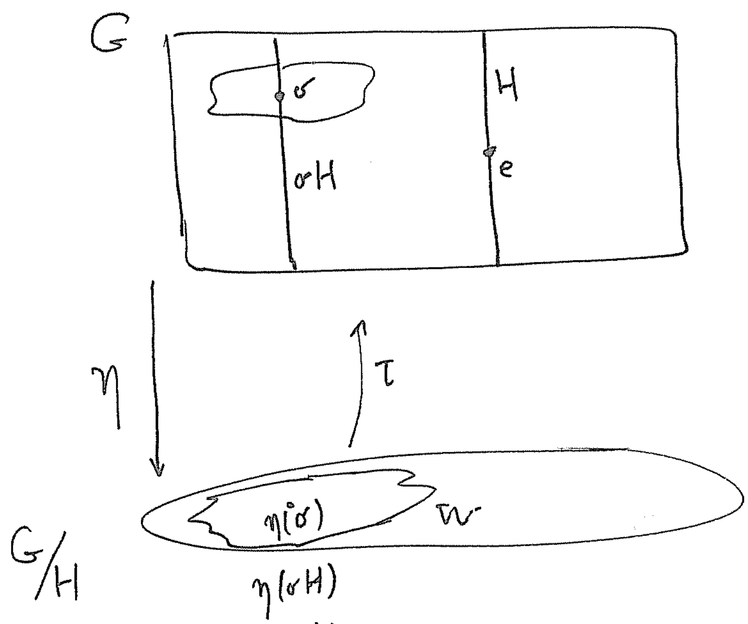
$$G/H = \{ \sigma H \mid \sigma \in G \} \quad \text{(left-cosets)}$$

Let $\eta : G \to G/H$ denote the map $\eta(\sigma) = \sigma H$. Then $\exists!$ manifold structure on G/H such that

(1) η is smooth

(2) at each point $y = \sigma H \in G/H$ there exists an open set W about y and a map $\tau : W \to G$ such that

$$\eta \circ \tau = id \quad (\tau\text{'s a local section} \ldots \text{gauge choice.})$$



τ - section takes G/H & pushes it up locally.

$\eta(\sigma H)$

$\eta(\sigma h)$

$\eta(\sigma H) \in G/H$ but $\sigma H \subseteq G$

Given any

Proof on Handout discussion

9/13/04 (1)

Field has a monopole number which is a topological invariant catalogued by homotopy classes of maps. Fields are :'ed into homotopy classes & this handles charged magnetic monopole. Physicsists integrate some diff. form & find it must be an integer anyway mathematicians use Principal fiber bundles.

Def^n/ A distribution in a Manifold M is a function $\mathcal{D}$ from M to TM such that $\mathcal{D}(P)$ is a subspace of T_pM $\forall p \in M$. Moreover one requires

$$P \longmapsto \dim \mathcal{D}(P)$$

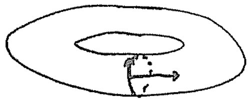
is constant. We've got a fixed dimesion. Another langauge is a distribution of k-planes in TM with $\dim \mathcal{D}(P) = k$ $\forall p \in M$.

Def^n/ A Distribution $\mathcal{D}$ is smooth if there exist k linearly independent vector fields $X_1, X_2, \ldots, X_k$ on M such that

$$\mathcal{D}(P) = \left\{ \sum_{i=1}^{k} \lambda^i X_i(P) \mid \lambda^i \in \mathbb{R} \right\}$$

and such that $X_1(P), X_2(P), \ldots, X_k(P)$ are linearly independent $\forall p \in M$.

Remark: There is no smooth 1-dim'l distribution on the sphere, but a torus is okay



left-translate around the torus $\Rightarrow$ 1-dim'l distribution.

Take any vector $v \in T_eG$ and left translate over Lie group G we get a 1-dim'l distribution.

- If you have M with a non-zero vector field on M then take $\mathcal{D}(P)$ to be non-zero multiples of each point....

- We say a vector field (X) belongs to a distribution $\mathcal{D}$ on M iff $X(P) \in \mathcal{D}(P)$ $\forall p \in M$. (People write $X \in \mathcal{D}$ but it doesn't really make sense except with the understanding above).

- A distribution is involutive (completely integrable) iff $X \in \mathcal{D}$, $Y \in \mathcal{D} \Longrightarrow [X, Y] \in \mathcal{D}$

$\psi : N \longrightarrow M$

$\psi(N)$ is a submanifold of M

$N \longrightarrow \psi(N)$ assume $\psi(N)$ has subspace topology.

A submanifold $N \xrightarrow{\psi} M$ where ψ is diffeomorphism & $\psi(N)$ is the submanifold, is a ~~integrable~~ integral manifold of a distribution $\mathcal{D}$ on M iff $\forall q \in N \quad d_q\psi(T_qN) = \mathcal{D}(\psi(q))$

$$d_q\psi : T_qN \longrightarrow T_{\psi(q)}M$$

if N is k-dim'l and $d_q\psi(T_qN) \subseteq \mathcal{D}(\psi(q))$ then we would automatically have $\underbrace{d_q\psi(T_qN)}_{k\text{-dim'l}} = \underbrace{\mathcal{D}(\psi(q))}_{k\text{-dim'l}}$

- Maximal integral manifold
  could take 10-dim'l dist. in 25-dim'l space & take surface that lied in 10-dim'l dist. ...

<u>Proposition 1.59</u>: Let $\mathcal{D}$ be a smooth distribution on M such that through each point of M passes an integral manifold of $\mathcal{D}$. Then $\mathcal{D}$ is involutive.

<u>Thm / 1.60 Frobeninius</u>: If $X, Y \in \mathcal{D} \Rightarrow [X, Y] \in \mathcal{D}$ with $\dim(\mathcal{D}) = c$ on M^d. Let $m \in M$ then $\exists$ a integral manifold of $\mathcal{D}$ passing through m. Indeed $\exists$ cubic coordinate systems (U, φ)

$\varphi|_{\varphi^{-1}(\text{Rectangle})}$ translate so that $\varphi|_{\varphi^{-1}(\text{Rect.})}(m) = 0$

now we know U is connected ...

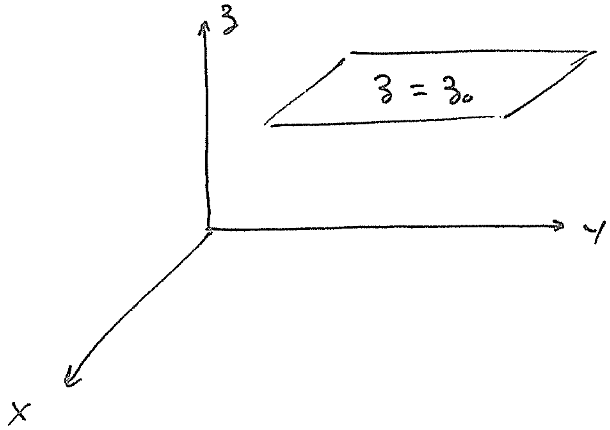
$x_i = \text{constant} \quad \forall i = c+1, \ldots, d$

<u>Warner</u>:

$$\mathbb{R}^n = (r_1, r_2, \ldots, r_n)$$

$$M^n = (x_1, x_2, \ldots, x_d) \quad \& \; x_i = r_i \circ \varphi$$

example $\mathcal{D}(x_0, y_0, z_0)$ = the plane $z - z_0 = 0$



$f(x, y, z) = z_0$

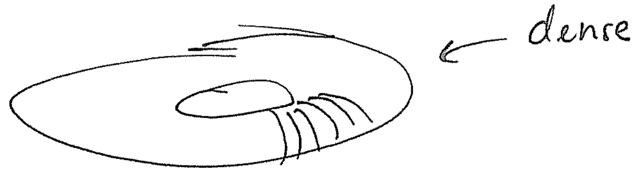
$f^{-1}(z_0)$ = the plane.

$$df_{(x_0,y_0,z_0)}\left(\frac{\partial}{\partial x}\right) = dz\left(\frac{\partial}{\partial x}\right) = 0$$

Same for $\frac{\partial}{\partial y} \Rightarrow \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ are tangent to distribution & its involutive & the planes are the maximal integral manifolds.

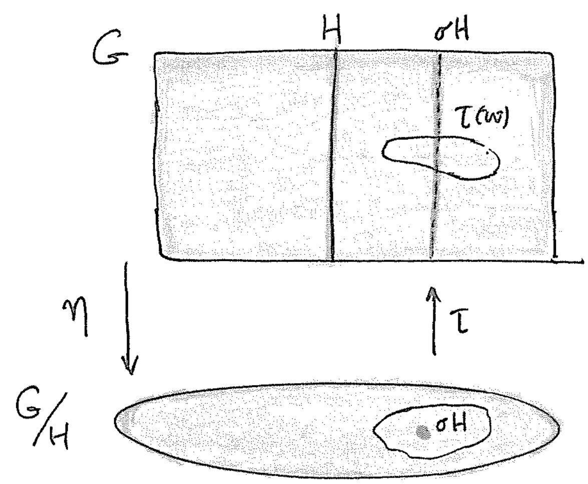
- Remark: Notice Lie Algebra Structure $\Rightarrow$ Involutive. The set of involutive vector fields $\Rightarrow \exists$ maximal integ. submanifold.
- Remark: $[X, X] = 0$ so a manifold has an 1-dim'l integral submanifold aka the existence of diffeq$^{\underline{s}}$!!!
- Frobenious Th$^{\underline{m}}$/ Generalizes the Existence of DE$_q$$^{\underline{s}}$ th$^{\underline{m}}$/.

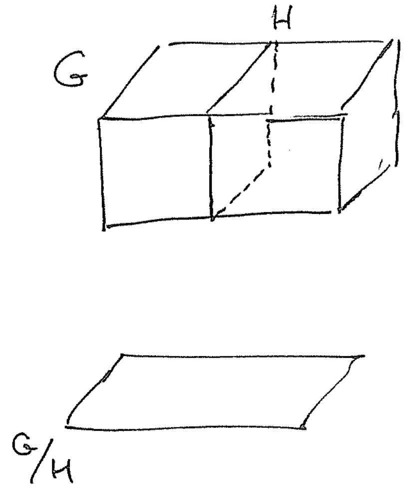
Remark $\lambda(x) = (e^{2\pi i x}, e^{ix})$



Last time we're in middle of Th^{m}.

Imagine have Lie group G & closed subgroup H





$G/H = \{\sigma H \mid \sigma \in G\}$ left cosets.

$\eta(g) = gH$ the natural quotient map.

Th^{m} says there's exactly one way of placing a maximal atlas on G/H so that η is smooth and each nbhd of G/H has a local section

① η is smooth

② $\forall\, y \in G/H$ $\exists$ open W about y and a smooth map $\tau : W \to G$ such that $\eta \circ \tau = id_W$

$\tau(W)$

Notice that $\tau(W)$ has same dimensionality as G/H.

Lemma 1

There is a topology on G/H such that

① G/H is Hausdorff

② η is a continous open map.

Pf/ Quotient spaces are notorious for being non-Hausdorff. But Warner gives this proof.

Let $\mathcal{O} = \{ U \subset G/H \mid \eta^{-1}(U) \text{ is open in } G \}$ so you look at union of cosets over U is open then U is open. Then you can check it's a Topology

① Let $\sigma_1 H \neq \sigma_2 H$ in G/H. Consider

$$R = \{ (\sigma, \tau) \in G \times G \mid \sigma = \tau h \text{ for some } h \in H \}$$

And consider (he calls it α)

$\mu(\sigma, \tau) = \tau^{-1}\sigma$ is a smooth hence continous from $G \times G \longmapsto G$

$$\mu^{-1}(H) = R$$

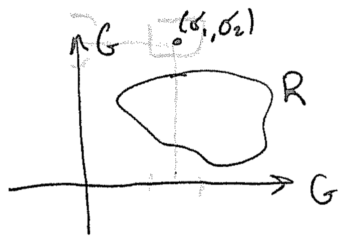
Let $(\sigma,\tau) \in \mu^{-1}(H) \iff \mu(\sigma,\tau) \in H \iff \tau^{-1}\sigma \in H$ ~~$\iff \sigma \in \tau H \iff$~~

$\iff \tau^{-1}\sigma = h \quad \exists h \in H$

$\iff \sigma = \tau h$

$\iff (\sigma, \tau) \in R \iff \sigma H = \tau H.$

Hence $\mu^{-1}(H) = R$ and since H is closed $\Rightarrow \mu^{-1}(H)$ is closed $\therefore R$ closed & $R \subset G \times G$

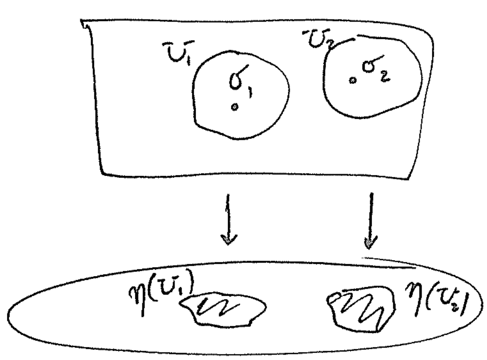


$(\sigma_1, \sigma_2) \in G \times G$

$(\sigma_1, \sigma_2) \notin R$ since $\sigma_1 H \neq \sigma_2 H$

Hence $\exists\ U_1, U_2$ open in G such that

$$(\sigma_1, \sigma_2) \in U_1 \times U_2 \subseteq (G \times G) - R$$



With ② we can conclude that $\eta(U_1)$ & $\eta(U_2)$ are open

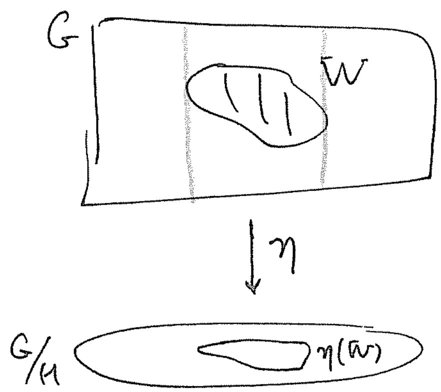
Claim $\eta(U_1) \cap \eta(U_2) = \phi$. Well if not $\exists\, y \in \eta(U_1) \cap \eta(U_2)$ with $y = \eta(\bar{\sigma}_1) = \eta(\bar{\sigma}_2)$ where $\bar{\sigma}_1 \in U_1$ & $\bar{\sigma}_2 \in U_2$

$$\eta(\bar{\sigma}_1) = \eta(\bar{\sigma}_2) \Rightarrow \bar{\sigma}_1 H = \bar{\sigma}_2 H \Rightarrow \bar{\sigma}_2 = \bar{\sigma}_1 h \text{ for some } h \in H$$
$$\Rightarrow (\bar{\sigma}_1, \bar{\sigma}_2) \in R$$

But then $R \cap U_1 \times U_2 = \phi$ so no such point exists.

$\therefore$ G/H is Hausdorff. (Although we ought to prove ② 1st.) //

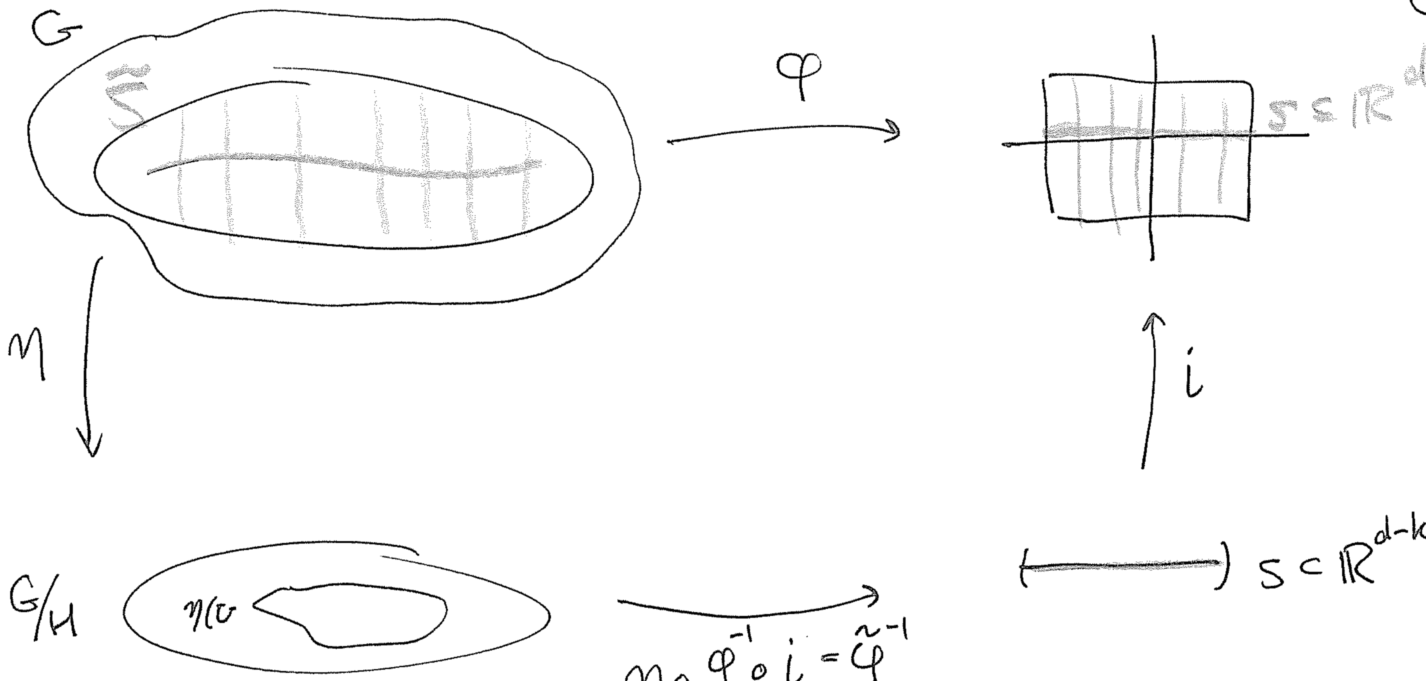
② to show η is open we must show that if $W \subseteq G$ is open then $\eta(W)$ is open in G/H



Need to show ~~$\eta^{-1}(W)$~~ is $\eta^{-1}(\eta(W))$ is open in G
for each point in $\eta(W)$ we get a fiber hence

$$\begin{aligned}
\eta^{-1}(\eta(W)) &= \{ g \mid \eta(g) \in \eta(W) \} \\
&= \{ g \mid \eta(g) = \eta(w) \ \exists\, w \in W \} \\
&= \{ g \mid gH = wH \ \exists\, w \in W \} \\
&= \{ g \mid g \in wH \text{ for some } w \in W \} \\
&= \bigcup_{w \in W} (wH) \\
&= \bigcup_{h \in H} (Wh) \\
&= \bigcup_{h \in H} R_h(W)
\end{aligned}$$

R_H are homeomorphisms & W is open hence $R_h(W)$ open
$\therefore$ $\eta^{-1}(\eta(W))$ is the union of open sets $\therefore$ its open
$\therefore$ $\eta(W)$ is open. //



$$\eta \circ \varphi^{-1} \circ i = \tilde{\varphi}^{-1}$$

$\tilde{\varphi}^{-1}$ is one-one and its continous, even open map. Then we finally have a chart on the quotient space

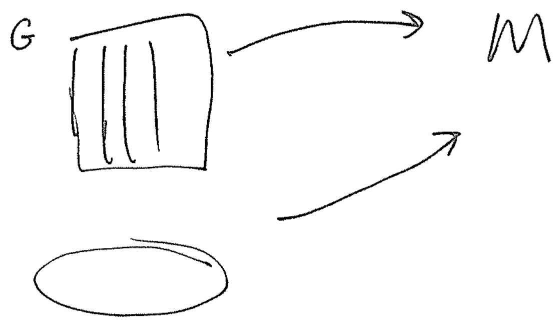
$$\tilde{l}_\sigma(aH) = (\sigma a)H$$

One pg. 122. He shows the overlap is smooth. Once we have this atlas we automatically have a local section.

Next

(1) $H \triangleleft G$ then G/H is Lie group

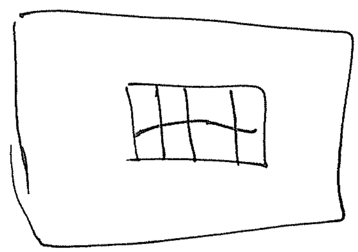
(2)

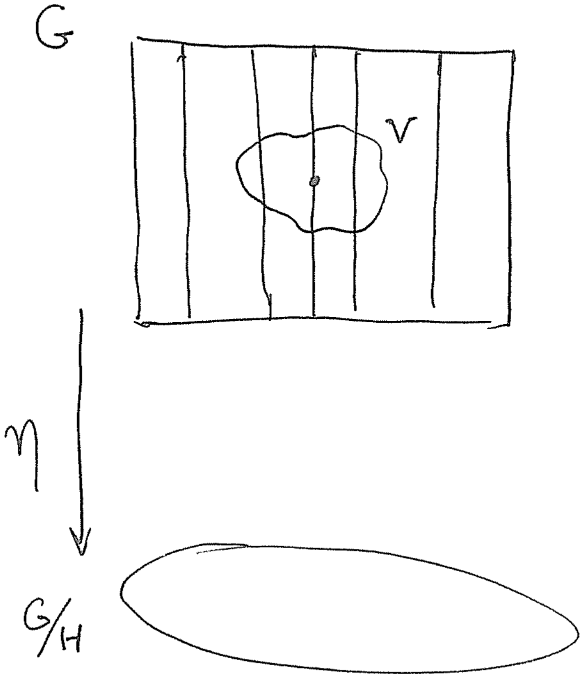


Lemma 2

Let $\mathcal{D}(g) = T_g(gH)$. Then $\mathcal{D}$ is an integrable distribution on G

Remark: Possible using Frobenious its possible to find chart whose image is rectangle such that the cosets become slices.





Lemma

If $\mathcal{D}(g) = T_g(gH)$ then $\mathcal{D}$ is an integrable distribution

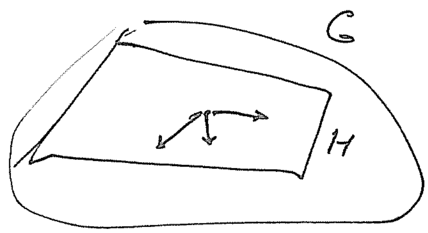
Thus, By Frobenious Thm $\forall$ points in G $\exists$ a chart (V, φ) at the point such that $\varphi(m) = 0$ and $\varphi(V)$ is a cube and the integral manifolds of $\mathcal{D}$ are slices $x_i = \text{constant}$ $1 \le i \le k$. Hence $\varphi = (x_1, x_2, \dots, x_d)$ $d > k$.

Proof of Lemma

He has a slick proof but here's an intuitive sketch

$$X(g) \in \mathcal{D}(g), \quad Y(g) \in \mathcal{D}(g)$$

$$X = \sum f_i X_i$$



$$\left[\sum_i f_i X_i, \sum_j g_j X_j\right]$$

there'll be linear comb. of X's

... its a proof of Lemma. //

Remark: Warner uses notation $r_i : \mathbb{R}^d \to \mathbb{R}$ with $r_i(x) = x_i$ & $x_i = r_i \circ \varphi$

- Now consider the point e with H around it then $V \cap H$ is a slice.

$\boxed{H \text{ is a submanifold if at each point } p \in H \ \exists \text{ a chart}}$

Enigmatic Remark: Since H is a closed subgroup of G, V can be chosen small enough such that $V \cap H$ = slice thru e.

Well we know H is a sub lie group by early this semester, that is H is a submanifold of G $\therefore$ $\exists$ a chart $(W; \Phi)$ of G such that $\Phi(W) = \mathcal{O}_1 \times \mathcal{O}_2 \subseteq \mathbb{R}^{d-k} \times \mathbb{R}^k$ where

$$\Phi(W \cap H) = \mathcal{O}_1 \times \{0\}$$

Choose $\mathcal{O}_1$ & $\mathcal{O}_2$ such that they're connected

$$W \cap H = \Phi^{-1}(\mathcal{O}_1 \times \{0\}) \quad \text{is connected.}$$

On pg. 42. If ... then $\Psi(N)$ lies on one of the slices. With original V didn't know $V \cap H$ was connected...

So V can be shrunk such that $\cancel{\exists}\, \varphi(V \cap H)$ is a single slice through e. (we call it S_0)

Then assume V is chosen that way

$$\begin{aligned} x_1 &= c_1 \\ x_2 &= c_2 \\ &\vdots \\ x_k &= c_k \end{aligned}$$

each set of constants gives you a slice

$c_1 = c_2 = \cdots = c_k = 0 \Rightarrow S_0$ slice through identity.

Choose U_1, V_1 open about e such that

$$V_1 V_1 \subseteq V \quad \text{and} \quad U^{-1} U \subseteq V_1$$

$$\mu_G(x,y) = xy \quad \& \quad \mu_G(e,e) = e$$

U_1, U_2 then $\mu_G(U_1 \times U_2) \subseteq V$

$U_1 \cap U_2 = V_1$ any pair of things in V_1 $\Rightarrow$ both in U_1 & U_2 & product of them ends up inside V.

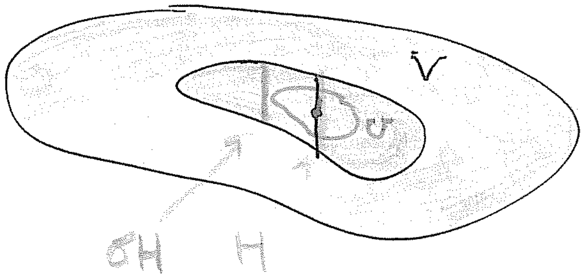
$\nu(x,y) = x^{-1}y$ is smooth

open around each, open

take intersection & then similar to above...

Again shrink those sets so that they are cubical.

Let $\sigma, \tau \in U$ then he wants $\sigma H = \tau H$, that is σ, τ are on the same coset



Modified V such that any 2 things in U lying on same coset lie on same slice (which is submanifold via frobenous)

Assume $\sigma, \tau \in U$ such that $\sigma H = \tau H$

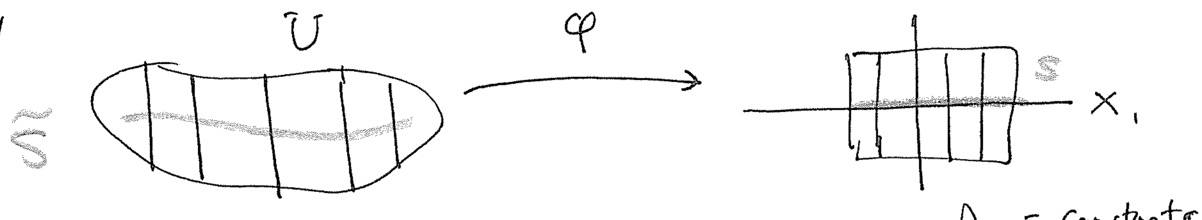
that $\tau^{-1}\sigma \in \tau^{-1}\sigma H = H \Rightarrow \tau^{-1}\sigma \in H$. But then by setup $\tau^{-1}\sigma \in V_1 \cap H \subseteq V_{\beta} \cap H = S_0$

$$\tau^{-1}\sigma \in V_1 \cap H \subseteq V_1 \cap (V_1 \cap H) \subseteq V_1 \cap S_0$$

Consequently $\tau^{-1}\sigma \in V_1 \cap S_0 \Rightarrow \sigma \in \tau(V_1 \cap S_0)$ and $V_1 \cap H$ being connected $\Rightarrow$ becomes a slice again... τ & σ are in the same slice. Hence

$$U \cap \sigma H = \text{single slice}.$$

Conversely a single slice goes on a single coset. Just above it says we proved chart system ... with x_i = constant whose image of chart is cube with identity $\to 0$ open sets and



$X_1 = c_1$
$x_2 = c_2$
$\vdots$
$x_n = c_n$

that's why he meant λ_i not X_i's.

λ_k = constants

the λ_i slices map exactly under φ^{-1} to X_i slices

Let S be slice of $\varphi(U)$ such that $\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_d = 0$

then $\varphi^{-1}(S) = \tilde{S}$

On G/H we constructed a topology and a single chart $(W, \tilde{\varphi})$ along with a section

$$s : W \longrightarrow \eta^{-1}(W)$$

In this chart η and s are smooth

For $\sigma \in G$ define $\tilde{l}_\sigma$ by $\tilde{l}_\sigma : G/H \to G/H$

$$\tilde{l}_\sigma(xH) = (\sigma x)H$$

Then $\tilde{l}_\sigma$ is a homeomorphism and the set of all pairs

$$\left(\tilde{l}_\sigma(W),\ (\tilde{\varphi}\circ\tilde{l}_\sigma)^{-1}\big|_{\tilde{l}_\sigma(W)}\right)$$

is an atlas on G/H relative to which

$$\eta : G \longrightarrow G/H$$

is smooth and such that these are smooth local sections

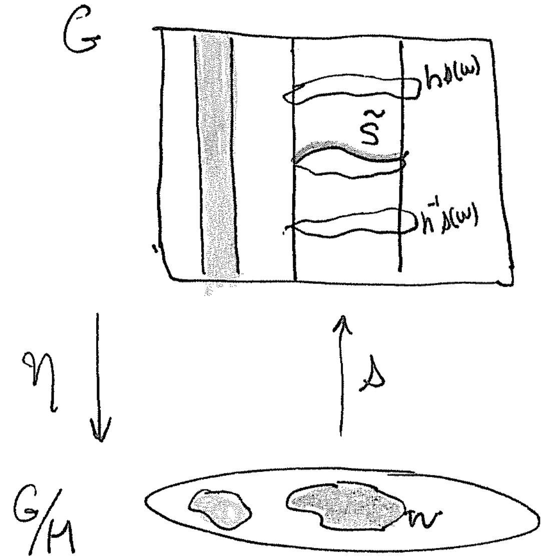
<u>Cor</u>: If H is a closed normal subgroup of a Lie group G then G/H is a Lie group with operations

$$\mu : G/H \times G/H \longrightarrow G/H$$

$$\mu(g_1H, g_2H) = g_1g_2H$$

$$i : G/H \longrightarrow G/H$$

$$i(gH) = g^{-1}H$$



$\tilde{s} = s(w) h$

$$s(w) \in \eta^{-1}(W) = gH \subseteq G$$

You can take the local section and multiply it by h you left translate the whole section up the fibre; ~~$h s(w)$~~ $s(w) h$ also ~~$h(w) s(w)$~~ $s(w)\, h(w)$ will be a section its just not vertical

$$\cancel{\eta(h(w) s(w))}$$

$$\eta(s(w) h(w)) = s(w) h(w) H = s(w) H = \eta(s(w)) = id_W$$

We think of s as a local gauge and h as a gauge transformation. We can use the local section to get

$$\eta^{-1}(W) = \text{local trivialization.}$$

Dirac Monopole

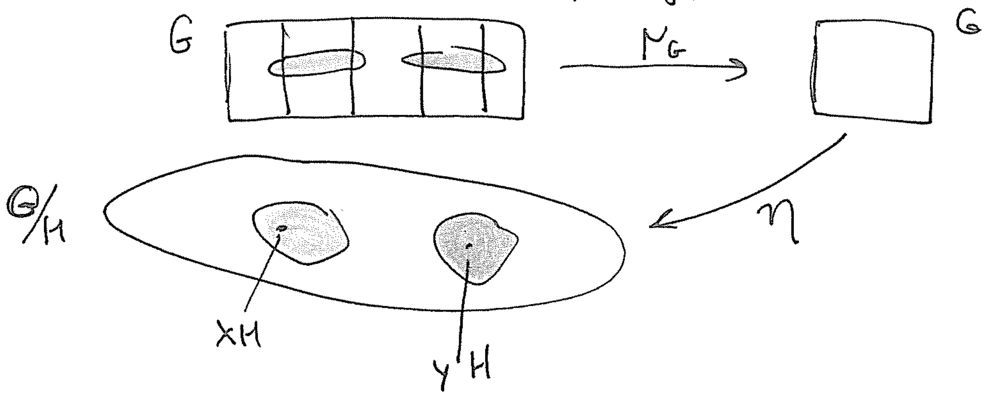
$$SU(2) \quad U(1)$$
$$\downarrow$$
$$\frac{SU(2)}{U(1)} \cong S^2$$

take Minkowski Space with parametrization,

$$\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^+ \times S^2$$
$$\downarrow$$
$$\mathbb{R} \times \mathbb{R}^* \times SU(2)$$

Proof of Corollary

$\mu : G/H \to G/H$ is well defined by the normality of $H \triangleleft G$, algebra result.
We must show μ is smooth near a point $(xH, yH) \in G/H \times G/H$
Choose local sections $s_x : W_{xH} \longrightarrow G$ and $s_y : W_{yH} \longrightarrow G$
defined on open sets $W_{xH} \subseteq G/H$, $W_{yH} \subseteq G/H$ about xH, yH



Notice that on $W_{xH} \times W_{yH}$

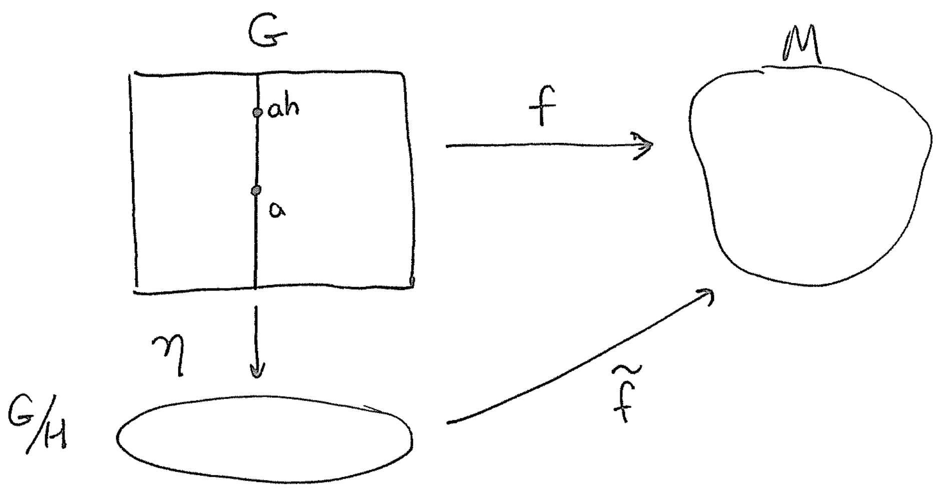
$$\mu = \eta \circ \mu_G \circ (s_{xH} \times s_{yH})$$

Now all these things are smooth. The proof for the inverse mapping is similar; $i = \eta \circ i_G \circ s_{xH}$

Corollary 2: If H is a closed subgroup of a Lie group G and $f : G \longrightarrow M$ is smooth for some manifold M and if $\forall a \in G$ and $h_1, h_2 \in H$, (f is constant over each coset, for fixed a)

$$f(ah_1) = f(ah_2)$$

Then $\exists$ smooth function $\tilde{f} : G/H \longrightarrow M$ such that $f = \tilde{f} \circ \eta$



$\tilde{f}(\eta(a)) = f(a)$

now if f weren't constant then

$\tilde{f}(\eta(ah)) = f(ah) = f(a)$

ill-defined if ~~each~~ $f(ah) \neq f($

Proof of Corollary 2:

Let $\eta(x) = xH \in G/H$. Choose a section $\Delta_x : W_{xH} \longrightarrow G$, $W_{xH} \subseteq G/H$ open

Choose W small enough so we get a section...

$$\tilde{f} = f \circ \Delta_x \quad //$$

Discussion: If we want to define $\tilde{f} : G/H \longrightarrow M$ we'll define $f : G \longrightarrow M$ which is constant along the cosets. The converse is trivial $\hat{f} \circ \eta = f$ smooth. The only way to construct a function on the coset space is to find a function which is constant on the fibers.