# Chapter 1

# Fiber Bundles

**Definition 1.1** *A* fiber bundle *is a mapping $\pi$ from a manifold $E$ onto a manifold $M$ subject to the following properties:*

1. *$\pi$ is smooth and surjective.*
2. *There exist a manifold $F$, called the* fiber *of $\pi$, and an open cover $\mathcal{U}$ of $M$ along with a corresponding family of mappings $\psi_U : \pi^{-1}(U) \to U \times F$, $U \in \mathcal{U}$, such that*
   - (a) *$\psi_U$ is a diffeomorphism and*
   - (b) *If $\pi_U$ is the projection of $U \times F$ onto $U$, then $\pi_U(\psi_U(y)) = \pi(y)$ for all $y \in \pi^{-1}(U)$.*

*Condition (2b) is usually expressed by saying that the diagram*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi_U} & U \times F \\ {}_{\pi}\searrow & & \swarrow_{\pi_U} \\ & U & \end{array}$$

*is commutative. Moreover, the mappings $\{\psi_U\}_{U \in \mathcal{U}}$ are said to be* local trivializing mappings *of the bundle.*

Notice that if $u_0 \in U, U \in \mathcal{U}$, then $\pi^{-1}(u_0)$ is a submanifold of $\pi^{-1}(U) \subseteq E$ which is diffeomorphic to the fiber $F$ of $\pi$. To see that this is so, observe that $y \in \pi^{-1}(U)$ is mapped to $u_0$ by $\pi$ iff $\pi_U(\psi_U(y)) = u_0$, and this is true iff $\psi_U(y) = (u_0, f)$ for some $f \in F$. Thus,

$$\pi^{-1}(u_0) = \{y \in \pi^{-1}(U) | y = \psi_U^{-1}(u_0, f), f \in F\}$$

and

$$\pi^{-1}(u_0) = \psi_U^{-1}(\{u_0\} \times F).$$

**Definition 1.2** *If $\pi : E \to M$ is a fiber bundle, then $E$ is called the* bundle space *or simply the* bundle *of $\pi$ and $M$ is called the* base space *or* base *or $\pi$.*

**Definition 1.3** *If $\pi : E \to M$ is a fiber bundle, then $s$ is a* local section *of $\pi$ iff $s$ is a smooth mapping from some open subset $U \subseteq M$ into $E$ such that $\pi \circ s = \mathrm{id}_U$. The local section $s$ is called a* global section *of $\pi$ iff $U = M$.*

**Exercise 1.1** Show that $s(U)$ is a submanifold of $E$ which intersects each fiber $\pi^{-1}(u)$ over points $u \in U$ in one and only one point.

Observe that *every* point $m \in M$ is in the domain of some local section of $\pi$. To prove this, choose a local trivializing mapping $\psi_U : \pi^{-1}(U) \to U \times F$ such that $m \in U$. Let $f_0$ denote *any* element of $F$, the fiber of $\pi$, and define $s : U \to E$ by

$$s(x) = \psi_U^{-1}(x, f_0)$$

for each $x \in U$. Clearly $s$ is smooth and $\pi_U(\psi_U(s(x)) = \pi(x, f_0) = x$ and thus $\pi(s(x)) = x$ for all $x \in U$.

It follows that there is a family of local sections $\{s_U\}_{U \in \mathcal{U}}$ of $\pi$ whose domains cover the base space $M$. For many mappings $\pi : E \to M$, having such a family $\{s_U\}$ of local sections implies the existence of a family $\{\psi_U\}_{U \in \mathcal{U}}$ of local trivializing mappings and thus implies that $\pi$ is a fiber bundle. This need not hold in general, however.

**Definition 1.4** *If $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ are fiber bundles with fibers $F_1$ and $F_2$, respectively, then the pair of functions $(\Phi, \phi)$ is a* bundle isomorphism *from* $\pi_1$ *to* $\pi_2$ *iff* $\Phi$ *is a diffeomorphism from* $E_1$ *to* $E_2$, $\phi$ *is a diffeomorphism from* $M_1$ *to* $M_2$, *and the diagram*

$$\begin{array}{ccc} E_1 & \xrightarrow{\Phi} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{\phi} & M_2 \end{array}$$

*is commutative. In this case, the fiber* $\pi_1^{-1}(x)$ *of* $E_1$ *over* $x \in M_1$ *is mapped diffeomorphically by* $\Phi$ *onto the fiber* $\pi_2^{-1}(\phi(x))$ *of* $E_2$ *over* $\phi(x)$. *In particular,* $F_1$ *is diffeomorphic to* $F_2$. *A fiber bundle* $\pi : E \longrightarrow M$ *with fiber* $F$ *is said to be* trivial *iff it is bundle isomorphic to the product bundle* $\pi_M : M \times F \longrightarrow M$ *(note that the product bundle possesses a single trivializing mapping with* $U = M, \mathcal{U} = \{U\}, \psi_U = \mathrm{id_{M\times F}}$*).*

Finally, observe that if $\pi : E \longrightarrow M$ is any fiber bundle with local trivializing mappings $\{\psi_U\}_{U\in\mathcal{U}}$ then $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \longrightarrow U$ is itself a fiber bundle with a single local trivializing mapping $\psi_U$, and in fact $\pi|_{\pi^{-1}(U)}$ is isomorphic to a trivial fiber bundle, namely $\pi_U : U \times F \longrightarrow U$. Moreover, $(\psi_U, \mathrm{id_U})$ is a bundle isomorphism from $\pi|_{\pi^{-1}(U)}$ to $\pi_U$. Thus, *every fiber bundle is locally trivial in this sense,* but most interesting fiber bundles are nontrivial.

## Examples

1. If $M$ is a manifold, then $\pi : TM \longrightarrow M$ is a fiber bundle. To see this, notice that if $(U, x)$ is any admissible chart of $M$, then

   $$\pi^{-1}(U) = TU = \{(m, v) | m \in U, v \in T_m M\}.$$

   Let $dx : TU \longrightarrow x(U) \times \mathbf{R}^n$ be the mapping defined by $dx(m, v) = (x(m), d_m x^i(v) r_i)$. Local trivializing mappings $\{\psi_U\}$ may be defined in terms of these charts $(TU, dx)$ of $TM$ by

   $$\psi_U = (x^{-1} \circ \mathrm{id}_{\mathbf{R}^n}) \circ dx : \pi^{-1}(U) \rightarrow U \times \mathbf{R}^n.$$

   Thus, the diagram

   $$\begin{array}{ccccc} \pi^{-1}(U) & \xrightarrow{dx} & x(U) \times \mathbf{R}^n & \xrightarrow{x^{-1}\times \mathrm{id}_{\mathbf{R}^n}} & U \times \mathbf{R}^n \\ & {}_{\pi}\searrow & & \swarrow_{\pi_U} & \\ & & U & & \end{array}$$

is commutative and defines the bundle structure of $\pi$. Note that the fiber of the bundle is $\mathbf{R}^n$, where $n = \dim M$. Observe that if $M$ does not have a well-defined dimension (if it varies from component to component) then $TM$ is not a fiber bundle.

2. If $M$ is a manifold and $T^*M$ is the cotangent bundle, then the projection $\pi^* : T^*M \to M$ is a fiber bundle (assuming $M$ is $n$-dimensional). Its fiber is $(\mathbf{R}^n)^*$, and trivializing mappings $\psi_U$ may be defined by

$$\psi_U(m, \alpha) = \left(m, \alpha\left(\left.\frac{\partial}{\partial x^i}\right|_m\right) r^i\right),$$

where $(U, x)$ is an admissible chart of $M$.

3. For each $1 \leq k \leq n$, $\pi : \bigwedge^k M \to M$ is a fiber bundle. The fiber is $\bigwedge^k \mathbf{R}^n$, and trivializing mappings are defined by

$$\psi_U(m, \alpha) = \left(m, \alpha\left(\left.\frac{\partial}{\partial x^{i_1}}\right|_m, \ldots, \left.\frac{\partial}{\partial x^{i_k}}\right|_m\right)\left(r^{i_1} \wedge \cdots \wedge r^{i_k}\right)\right).$$

4. Let $M$ be a manifold. We define a fiber bundle called the *frame bundle* of $M$. The bundle space is denoted $\mathcal{F}M$; it is the set of all ordered pairs $(m, \{e_i\})$ where $m \in M$ and $\{e_i\}$ is a basis of $T_mM$. Such a basis is called a *frame* at $m$ and thus $\mathcal{F}M$ is a bundle of frames of $M$. The fiber bundle mapping is $\pi : \mathcal{F}M \to M$ defined by $\pi(m, \{e_i\}) = m$; it designates the point at which the frame $\{e_i\}$ is attached. We show that $\mathcal{F}M$ is a manifold and that $\pi : \mathcal{F}M \to M$ is a fiber bundle with fiber the group $Gl(\mathbf{R}^n)$ of all nonsingular $n \times n$ real matrices. We elaborate in some detail the structure of $\pi : \mathcal{F}M \to M$.

   First observe that if $m \in M$ then $\pi^{-1}(m)$ is the set of all frames at $m$. If $(m, \{e_i\})$ and $(m, \{f_i\})$ are two points in the fiber $\pi^{-1}(m)$ then they are related via a unique $n \times n$ matrix $A$ such that

$$f_j = A^i_j e_i.$$

   This suggests that the fiber is $Gl(\mathbf{R}^n)$ and how to get charts and local trivializing mappings. Choose any admissible chart $(U, x)$ of $M$. Let

$$\mathcal{F}U = \{(m, \{e_i\}) | m \in U\}$$

and let $\mathcal{F}x : \mathcal{F}U \to x(U) \times Gl(\mathbf{R}^n)$ be defined by

$$(\mathcal{F}x)(m, \{e_i\}) = (x(m), (d_m x^j(e_i))).$$

Thus, $(\mathcal{F}x)(m, \{e_i\}) = (x(m), A)$ where $A$ is the $n \times n$ matrix defined by

$$A_i^j = d_m x^j(e_i).$$

$A$ is invertible since both $\{e_i\}$ and $\left\{ \frac{\partial}{\partial x^j}\Big|_m \right\}$ are bases of $T_m M$ and

$$e_i = A_i^j \left( \frac{\partial}{\partial x^j}\Big|_m \right).$$

Moreover, $\mathcal{F}x$ maps $\mathcal{F}U$ onto all of $x(U) \times Gl(\mathbf{R}^n)$. We leave it as an exercise to show that if $\mathcal{A}_M$ is an admissible atlas of $M$ then

$$\mathcal{A} = \{(\mathcal{F}U, \mathcal{F}x) | (U, x) \in \mathcal{A}\}$$

is an atlas of $\mathcal{F}M$. Moreover, $Gl(\mathbf{R}^n)$ is an open subset of $gl(\mathbf{R}^n)$, which may be identified with $\mathbf{R}^{n^2}$. Finally $\psi_U : \pi^{-1}(U) \to U \times Gl(\mathbf{R}^n)$ is a local trivializing mapping if we define it by

$$\psi_U = (x^{-1} \circ \mathrm{id}_{Gl(\mathbf{R}^n)}) \circ \mathcal{F}x.$$

5. Let $M$ be a manifold and $g$ a metric on $M$. Then $g$ is a type $\binom{0}{2}$ tensor field on $M$ which is symmetric and nondegenerate and which has constant index $k = n - p$. For each $m \in M$, $g_m$ is a metric on $T_m M$ and thus there is a $g$-orthonormal basis $\{e_i\}$ of $T_m M$ such that $\{j | g_m(e_j, e_j) = -1\}$ has $k$ elements in it. By reordering this basis if necessary we obtain

$$g_m(e_i, e_j) = G_{ij},$$

where

$$G_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j, 1 \leq i \leq p \\ -1 & i = j, p+1 \leq i \leq n. \end{cases}$$

*Define* $\mathcal{O}_g M = \{(m, \{e_i\}) \in \mathcal{F}M | g_m(e_i, e_j) = G_{ij}\}$. We claim $\mathcal{O}_g M \xrightarrow{\pi} M, \pi(m, \{e_i\}) = m$, is a fiber bundle. This is not difficult to prove, given the following Theorem.

**Theorem 1.1** *If $M$ is a manifold and $g$ is a metric on $M$ with index $k = n - p$, then for each $m_0 \in M$ there exist an open set $U$ about $m_0$ and vector fields $\{\mathfrak{x}_i\}$ on $U$ such that*

$$g_m(\mathfrak{x}_i(m), \mathfrak{x}_j(m)) = G_{ij}$$

*for all $m \in U$.*

We first show how to use the theorem to prove that $\mathcal{O}_g M \xrightarrow{\pi} M$ is a fiber bundle, after which we will prove the Theorem.

Let $\mathcal{O}(p, k) = \left\{ A \in Gl(\mathbf{R}^n) | A^T G A = G \right\}$. We leave it as an exercise to be proven later that $\mathcal{O}(p, k)$ is a manifold. We show that $\pi : \mathcal{O}_g M \to M$ is locally trivial with fiber $\mathcal{O}(p, k)$. By the Theorem there is an open cover $\mathcal{U}$ of $M$ such that for each $U \in \mathcal{U}$ there exist vector fields $\{\mathfrak{x}_i\}_{i=1}^n$ defined on $U$ such that

$$g_m(\mathfrak{x}_i(m), \mathfrak{x}_j(m)) = G_{ij}$$

for all $m \in U$. Define a mapping

$$\psi_U : \pi^{-1}(U) \longrightarrow U \times \mathcal{O}(p, k)$$

by

$$\psi_U(m, \{e_i\}) = (m, (\Xi^j(m)(e_i)))$$

where $\Xi^j$ is the differential form defined on $U$ by

$$\Xi^j(m)(\mathfrak{x}_i(m)) = \delta_i^j.$$

To show that the matrix $\lambda$ whose components are

$$\lambda_i^j = \Xi^j(m)(e_i)$$

is actually in $\mathcal{O}(p, k)$, observe that $e_i = \lambda_i^k \mathfrak{x}_k(m)$ and

$$G_{ij} = g_m(e_i, e_j) = \lambda_i^k \lambda_j^l g_m(\mathfrak{x}_k(m), \mathfrak{x}_l(m))$$

and

$$G_j^i = \sum_{k,l} (\lambda_i^k \lambda_j^l) G_l^k = (\lambda^T G \lambda)_j^i.$$

So $G = \lambda^T G \lambda$ and $\lambda \in \mathcal{O}(p,k)$ as required. It follows that $\psi_U$ maps $\pi^{-1}(U)$ into $U \times \mathcal{O}(p,k)$. Moreover, $\psi_U$ has an inverse and in fact

$$\psi_U^{-1}(m, (\lambda_i^j)) = (m, \lambda_i^j \mathfrak{x}_j(m)).$$

The mappings $\{\psi_U\}_{U \in \mathcal{U}}$ have the formal requirements of local trivializing mappings but they must also be smooth. So one needs a manifold structure on $\mathcal{O}_g M$ such that the maps $\{\psi_U\}$ are diffeomorphisms. One defines such a structure on $\mathcal{O}_g M$ as follows.

First observe that it is no loss of generality to assume that for each $U \in \mathcal{U}$, $U$ is a subset of the domain of some chart of $M$. Let $\mathcal{A}(p,k)$ denote an atlas of admissible charts of $\mathcal{O}(p,k)$. For each $U \in \mathcal{U}$ and each chart $y \in \mathcal{A}(p,k)$ let $x$ denote an admissible chart of $M$ defined on $U$ and let $U(y) = \psi_U^{-1}(U \times V_y)$ where $V_y \subseteq \mathcal{O}(p,k)$ is the domain of $y$. Finally define a chart $\eta_y : U(y) \to x(U) \times y(V_y)$ by

$$\eta_y = (x \times y) \circ \psi_U.$$

It is easy to show that $\mathcal{A} = \{(U(y), \eta_y) | U \in \mathcal{A}_M, y \in \mathcal{A}(p,k)\}$ is an atlas on $\mathcal{O}_g(M)$ and this defines a differentiable structure on $\mathcal{O}_g(M)$. Moreover, relative to this structure the mappings $\{\psi_U\}_{U \in \mathcal{U}}$ are all smooth. Indeed, if one chooses a point of $\pi^{-1}(U)$ for some $U \in \mathcal{U}$, then that point is in $U(y) = \psi_U^{-1}(U \times V_y)$ for some $V_y$, and one can show that $\psi_U$ restricted to $U(y)$ is smooth by considering its local representatives. We see from the commutative diagram

$$\begin{array}{ccc} U_y & \xrightarrow{\psi_U} & U \times V_y \\ \eta_y \downarrow & & \downarrow x \times y \\ x(U) \times y(V_y) & \xrightarrow{\text{identity}} & x(U) \times y(V_y) \end{array}$$

that the identity mapping is the local representative of $\psi_U$ relative to the charts $\eta_y$ and $x \times y$ and so $\psi_U$ is indeed smooth.

To complete the proof one needs to prove Theorem 2.1 above.

**Proof of Theorem 2.1.** The proof requires a number of steps. Throughout the proof let $m_0 \in M$ and let $(W, \bar{x})$ denote an admissible chart of $M$ such that $m_0 \in W$.

**Step I:** The chart $\bar{x}$ may be modified to obtain a new admissible chart $x$ defined on an open subset of $W$ such that

$$g_{m_0}\left(\left.\frac{\partial}{\partial x^i}\right|_{m_0}, \left.\frac{\partial}{\partial x^j}\right|_{m_0}\right) = G_{ij}.$$

To see this, first choose any frame $\{e_i\}$ at $m_0$ such that $g_{m_0}(e_i, e_j) = G_{ij}$. Let $A$ be any matrix such that $\left.\frac{\partial}{\partial \bar{x}^k}\right|_{m_0} = A^i_k e_i$. Define $x^i = A^i_k \bar{x}^k$ on all of $W$; then

$$\left.\frac{\partial}{\partial x^i}\right|_{m_0} = \frac{\partial \bar{x}^k}{\partial x^i}\left(\left.\frac{\partial}{\partial \bar{x}^k}\right|_{m_0}\right) = \overset{-1}{A}{}^k_i\left(\left.\frac{\partial}{\partial \bar{x}^k}\right|_{m_0}\right) = e_i$$

and consequently

$$g_{m_0}\left(\left.\frac{\partial}{\partial x^i}\right|_{m_0}, \left.\frac{\partial}{\partial x^j}\right|_{m_0}\right) = G_{ij}.$$

This proves Step I.

Notice that a consequence of Step I is that

$$g_{m_0}\left(\left.\frac{\partial}{\partial x^i}\right|_{m_0}, \left.\frac{\partial}{\partial x^j}\right|_{m_0}\right) = \delta_{ij}$$

for $1 \le i, j \le p$. We eventually show that this holds for all $m$ in some open set about $m_0$ and we characterize a maximal subset on which $g_m$ is positive definite.

**Step II:** Let $T^+_m M = \left\{\sum_{i=1}^{p} \lambda^i \left(\left.\frac{\partial}{\partial x^i}\right|_{m}\right) | \lambda^i \in \mathbf{R}\right\}$ for each $m \in W$. We show that there is an open subset $\mathcal{O}_{m_0} \subseteq W$ containing $m_0$ such that for each $m \in \mathcal{O}_{m_0}$, $g_m$ restricted to $T^+_m M$ is positive definite.

*Proof of Step II.* Let $S$ denote the unit sphere in $\mathbf{R}^p$. Thus $\vec{\lambda} \in S$ iff $\sum_{i=1}^{p}(\lambda^i)^2 = 1$. Define a function $H : S \times W \to \mathbf{R}$ by

$$H(\vec{\lambda}, m) = g_m\left(\sum_{i=1}^{p} \lambda^i \left(\left.\frac{\partial}{\partial x^i}\right|_{m}\right), \sum_{j=1}^{p} \lambda^j \left(\left.\frac{\partial}{\partial x^j}\right|_{m}\right)\right).$$

The mapping $H$ is continuous, and

$$\begin{aligned} H(\vec{\lambda}, m_0) &= g_m\left(\sum_{i=1}^{p}\lambda^i\left(\frac{\partial}{\partial x^i}\bigg|_{m_0}\right), \sum_{j=1}^{p}\lambda^j\left(\frac{\partial}{\partial x^j}\bigg|_{m_0}\right)\right) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}\lambda^i\lambda^j\delta_{ij} \\ &= \sum_{k=1}^{p}(\lambda^k)^2 = 1. \end{aligned}$$

For each $\vec{\lambda} \in S$, let $U_{\vec{\lambda}}$ be open about $\vec{\lambda}$ in $S$ and $\mathcal{O}_{\vec{\lambda}}$ open about $m_0$ in $M$ such that $H$ is positive on $U_{\vec{\lambda}} \times \mathcal{O}_{\vec{\lambda}}$. There exists a finite number of the sets $U_{\vec{\lambda}}$ which covers $S$, $U_{\vec{\lambda}_1}, \ldots, U_{\vec{\lambda}_N}$. Let

$$U_\alpha = U_{\vec{\lambda}_\alpha} \text{ and } \mathcal{O}_\alpha = \mathcal{O}_{\vec{\lambda}_\alpha}.$$

Let $\mathcal{O}_{m_0} = \bigcap_{\alpha=1}^{N}\mathcal{O}_\alpha$. For $(\vec{\lambda}, m) \in S \times \mathcal{O}_{m_0}$ we see that $\vec{\lambda} \in U_{\alpha_0}$ for some $\alpha_0$ and, since $m \in \mathcal{O}_\alpha$ for all $\alpha$, we see that $(\vec{\lambda}, m) \in U_{\alpha_0} \times \mathcal{O}_{\alpha_0}$ and thus $H(\vec{\lambda}, m) > 0$. So $H$ is positive on $S \times \mathcal{O}_{m_0}$. We claim $g_m$ is positive definite on $T_m^+M$ for all $m \in \mathcal{O}_{m_0}$. To see this, let $m \in \mathcal{O}_{m_0}$ and $V \in T_{m_0}^+M$ such that $v \neq 0$. Then $v = \sum_{i=1}^{p}\lambda^i\left(\frac{\partial}{\partial x^i}\Big|_{m_0}\right)$ and $\sum_{i=1}^{p}(\lambda^i)^2 \neq 0$. Let

$$\|\vec{\lambda}\| = \left[\sum_{i=1}^{p}(\lambda^i)^2\right]^{1/2}$$

and observe that

$$\frac{1}{\|\vec{\lambda}\|}v = \sum_{i=1}^{p}\left(\frac{\lambda^i}{\|\vec{\lambda}\|}\right)\left(\frac{\partial}{\partial x^i}\bigg|_m\right),$$

where

$$\sum_{i=1}^{p}\left(\frac{\lambda^i}{\|\vec{\lambda}\|}\right)^2 = \sum_{i=1}^{p}\left[\frac{(\lambda^i)^2}{\sum_{j=1}^{p}(\lambda^j)^2}\right] = 1.$$

Thus $g_m(\frac{1}{\|\vec{\lambda}\|}v, \frac{1}{\|\vec{\lambda}\|}v) = \frac{1}{\|\vec{\lambda}\|^2}g_m(v,v)$ and $\frac{1}{\|\vec{\lambda}\|^2}g_m(v,v) = H(\frac{\vec{\lambda}}{\|\vec{\lambda}\|}, m) > 0$. Thus $g_m(v,v) > 0$ as required. So $g_m$ is positive definite on $T_m^+M$ for all $m \in \mathcal{O}_{m_0}$ and Step II follows.

Notice that for each $m \in \mathcal{O}_{m_0}$, $\left\{ \frac{\partial}{\partial x^i}\Big|_m \right\}$ is a basis of $T_m^+ M$. We may apply Gram-Schmit orthogonalization to this basis to obtain a $g_m\Big|(T_m^+ M \times T_m^+ M)$ orthogonal basis of $T_m^+ M$. Let $\{\xi_i(m)\}$ denote this basis. An examination of the orthogonalization process shows that the resulting vector fields $\{\xi_i\}$ on $\mathcal{O}_{m_0}$ are in fact smooth and so one has vector fields $\{\xi_i\}_{i=1}^p$ on $\mathcal{O}_{m_0}$ such that

$$g_m\left(\xi_i(m), \xi_j(m)\right) = \delta_{ij}$$

for all $m \in \mathcal{O}_{m_0}$, $1 \leq i, j \leq p$.

**Step III:** Let $T_m^- M$ denote the $g_m$ orthogonal complement of $T_m^+ M$ in $T_m M$ for each $m \in \mathcal{O}_{m_0}$. We claim that $T_m M = T_m^+ M \oplus T_m^- M$ for all $m \in \mathcal{O}_{m_0}$ and that the restriction of $g_m$ to $T_m^- M$ is negative definite.

*Proof.* Let $v \in T_m M$, $m \in \mathcal{O}_{m_0}$. We show that $v = v^+ + v^-$ for some $v^+ \in T_m^+ M$, $v^- \in T_m^- M$. Define $v^+$ by

$$v^+ = \sum_{j=1}^{p} g_m(v, \xi_j(m))\xi_j(m).$$

Note that

$$\begin{aligned} g_m(v - v^+, \xi_i(m)) &= g_m(v, \xi_i(m)) - \sum_{j=1}^{p} g_m(v, \xi_j(m)) g_m(\xi_j(m), \xi_i(m)) \\ &= g_m(v, \xi_i(m)) - \sum_{j=1}^{p} g_m(v, \xi_j(m))\delta_{ji} = 0. \end{aligned}$$

Since this holds for all $\xi_i(m)$ and since $\{\xi_i(m)\}$ is a basis of $T_m^+ M$ we see that $v - v^+$ is in the $g_m$-orthogonal complement of $T_m^+ M$ in $T_m M$ and thus $v - v^+ \in T_m^- M$. If we let $v^- = v - v^+$ we have $v = v^+ + v^-$ as we require. To see that the sum is a direct sum note that if $v \in T_m^+ M \cap T_m^- M$ then $v \in T_m^+ M$ is such that $g_m(v, v) = 0$ and since $g_m$ is positive definite on $T_m^+ M$, $v = 0$. Thus $T_m M = T_m^+ M \oplus T_m^- M$.

We now show that $g_m$ restricted to $T_m^- M$ is negative definite. Assume this is not so; then an orthonormal basis $\{f_j\}_{j=p+1}^n$ of $T_m^- M$

exists for which there is at least one $p+1 \leq j \leq n$ such that $g_m(f_j, f_j) = 1$. It follows that

$$\xi_1(m), \xi_2(m), \ldots, \xi_p(m), f_{p+1}, f_{p+2}, \ldots, f_m$$

is a $g_m$-orthonormal basis of $T_mM$ such that $g_m\left(\xi_i(m), \xi_i(m)\right) = 1$, $1 \leq i \leq p$, and $g_m(f_j, f_j) = 1$. This implies that the index of $g_m$ is less than $n-p$, contrary to hypothesis. It follows that $g_m$ restricted to $T_m^-M$ is negative definite.

*Proof of the Theorem itself.* Let $\rho_m : T_mM \to T_m^-M$ denote the orthogonal projection of $T_mM$ onto $T_m^-M$. Recall this may be defined by $\rho_m(v) = v^-$ where $v = v^+ + v^-$ is the decomposition in Step III. Let $w \in T_mM$ and write $w = w^+ + w^-$. Since $w^- \in T_mM$, $w^- = \sum_{i=1}^n \mu_i \left( \left.\frac{\partial}{\partial x^i}\right|_m \right)$ and

$$\begin{aligned}
\rho_m(w^-) &= \sum_{i=1}^n \mu_i \rho_m \left( \left.\frac{\partial}{\partial x^i}\right|_m \right) \\
&= \sum_{i=1}^p \mu_i \rho_m \left( \left.\frac{\partial}{\partial x^i}\right|_m \right) + \sum_{i=p+1}^n \mu_i \rho_m \left( \left.\frac{\partial}{\partial x^i}\right|_m \right) \\
&= \sum_{i=p+1}^n \mu_i \rho_m \left( \left.\frac{\partial}{\partial x^i}\right|_m \right).
\end{aligned}$$

So $w^- = \rho_m(w^-) = \sum_{i=p+1}^n \mu_i \rho_m \left( \left.\frac{\partial}{\partial x^i}\right|_m \right)$. The metric $-g_m$ is positive definite on $T_m^-M$ and so we can apply Gram-Schmit orthogonalization to the vector fields

$$m \mapsto \rho_m \left( \left.\frac{\partial}{\partial x^i}\right|_m \right) \quad , \quad p+1 \leq i \leq n$$

on $\mathcal{O}_{m_0}$. We obtain vector fields $\xi_{p+1}, \ldots, \xi_n$ on $\mathcal{O}_{m_0}$ such that

$$\left(-g_m\right)\left(\xi_i(m), \xi_j(m)\right) = \delta_{ij} \quad , \quad p+1 \leq i, j \leq n$$

for all $m \in \mathcal{O}_{m_0}$. Thus we have vector fields $\xi_1, \xi_2, \ldots, \xi_n$ on $\mathcal{O}_{m_0}$ such that

$$g_m\left(\xi_i(m), \xi_j(m)\right) = G_{ij}$$

for all $m \in \mathcal{O}_{m_0}$.

**Definition 1.5** *A fiber bundle $\pi : E \longrightarrow M$ is called a* vector bundle *iff*

1. *the fiber of the bundle is a vector space $V$,*

2. *there is a family of local trivializing mappings $\psi_U : \pi^{-1}(U) \rightarrow U \times V$, $U \in \mathcal{U}$ such that if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$, then for each $m \in U_1 \cap U_2$ the mapping from $V$ to $V$ defined by*

$$x \longmapsto \pi_V\left(\psi_{U_2}\left(\psi_{U_1}^{-1}(m, x)\right)\right)$$

*is a vector space isomorphism.*

*Observe that in this case there exist well-defined continuous operations + and $\cdot$ on each fiber $\pi^{-1}(m)$, $m \in M$. These operations are defined by*

$$\begin{aligned} v + w &= \psi_U^{-1}\left(m, \pi_V\left(\psi_U(v)\right) + \pi_V\left(\psi_U(w)\right)\right) \\ cv &= \psi_U^{-1}\left(m, c \cdot \pi_V\left(\psi_U(v)\right)\right). \end{aligned}$$

**Exercise 1.2** Show that $TM, T^*M, \Lambda^k M$ are vector bundles.

**Definition 1.6** *Two vector bundles $(E_1, M_1, \pi_1)$ and $(E_2, M_2, \pi_2)$ are* vector bundle isomorphic *iff there exists a fiber bundle isomorphism $(\Phi, \phi)$ from $\pi_1$ to $\pi_2$ such that for each $m \in M$ the restriction of $\Phi$ to $\pi_1^{-1}(m)$ is a vector space isomorphism from $\pi_1^{-1}(m)$ onto $\pi_2^{-1}(\phi(m))$.*

**Examples**

1. Let $\mathcal{N}$ denote Newtonian space, i.e. $\mathcal{N}$ is a manifold with an atlas $\mathcal{A}$ such that

   (a) if $x, y \in \mathcal{A}$ then $y \circ x^{-1}$ is a rigid motion of $\mathbf{R}^3$

   (b) if $x \in \mathcal{A}$ and $\phi$ is a rigid motion of $\mathbf{R}^3$ then $\phi \circ x \in \mathcal{A}$.

Let $\mathcal{SN} = \mathbf{R} \times \mathcal{N}$ denote the bundle space of the trivial bundle $\pi_T : \mathcal{SN} \longrightarrow \mathbf{R}, \pi_T(t,x) = t$. Observe that trajectories of objects in Newtonian space are described by local sections of this bundle: $\hat{\gamma}(t) = (t, \gamma(t))$ where $\gamma(t) \in$ is the position of the object at time $t$. The velocity of the object is $\frac{d}{dt}\pi_{\mathcal{A}}(\hat{\gamma}(t)) = \frac{d}{dt}(\gamma(t))$. We thus see that Newtonian spacetime is a fiber bundle over time-axis but Minkowski spacetime is not.

2. Let $Q$ be the configuration space of a system of particles. The time evolution of the system is a section of the trivial bundle $\mathbf{R} \times TQ \longrightarrow \mathbf{R}$.

3. Let $M$ denote Minkowski spacetime. The electromagnetic field tensor is a section of the bundle $\Lambda^2 M \longrightarrow M$, a trivial fiber bundle which is not obviously trivial. Similarly vector potentials are sections of the bundle $\Lambda^1 M \longrightarrow M$.

4. Let $M$ denote Minkowski space and $\psi : M \longrightarrow \mathbf{C}^2$ a spin field. Note that this defines a section $\hat{\psi}(x) = (x, \psi(x))$ of the trivial bundle $M \times \mathbf{C}^2 \longrightarrow M$.

These examples show that most dynamical fields in physics may be viewed as (local) sections of some fiber bundle.

It is our intent to formulate a theory in which all Lagrangians have domain an appropriate fiber bundle.

**Definition 1.7** *If $\pi : E \longrightarrow M$ is a fiber bundle with fiber $F$ and $(U, \bar{y})$ is an admissible chart of $E$ then we say that this chart is* adapted *to the bundle $\pi$ iff $\pi(U)$ is open in $M$ and there is a chart $\bar{x}$ of $M$ defined on $\pi(U)$ such that $\bar{y}^\mu = \bar{x}^\mu \circ \pi$ for $1 \le \mu \le n$, $n = \dim M$. In this case we often write $x^\mu = \bar{x}^\mu \circ \pi$, $1 \le \mu \le n$, and $y^a = \bar{y}^{a+n}$ for $1 \le a \le N$ where $N = \dim F$.*

**Exercise 1.3** If $\pi : E \longrightarrow M$ is a fiber bundle and $y \in E$ then there is an adapted coordinate system at $y$.

Note that if $u \in E$ and $w \in T_u E$ such that $d_u\pi(w) = 0$ then

$$w = \sum_{a=1}^{N} w^a \left( \frac{\partial}{\partial y^a}\bigg|_w \right).$$

Indeed, in general, $w = \sum_{\mu=1}^{n} w^\mu \left( \left.\frac{\partial}{\partial x^\mu}\right|_w \right) + \sum_{a=1}^{N} w^a \left( \left.\frac{\partial}{\partial y^a}\right|_w \right)$. But $d_u x^\mu(w) = d\bar{x}^\mu \left(d_u \pi(w)\right) = 0$ and also

$$d_u x^\mu(w) = dx^\mu \left( \sum_\nu w^\nu \left( \left.\frac{\partial}{\partial x^\nu}\right|_w \right) + \sum_a w^a \left( \left.\frac{\partial}{\partial y^a}\right|_w \right) \right) = w^\mu.$$

Thus $w^\mu = 0$ for $1 \leq \mu \leq n$ and

$$w = \sum_{a=1}^{N} w^a \left( \left.\frac{\partial}{\partial y^a}\right|_w \right)$$

as asserted.

**Definition 1.8** *If $\pi : E \longrightarrow M$ is a fiber bundle then a tangent vector $w \in T_uE$ at $u \in E$ is* vertical *iff $d_u\pi(w) = 0$. A curve $\gamma : I \longrightarrow E$ in $E$ is* vertical *iff $\gamma'(t) \in T_{\gamma(t)}E$ is vertical for all $t \in I$.*

**Exercise 1.4** Show that a curve $\gamma : I \longrightarrow E$ is vertical iff the image of $\gamma$ lies in a single fiber of $E$.

**Definition 1.9** *If $\pi : E \longrightarrow M$ is a fiber bundle and $y_0 \in E$ then $J_{y_0}E$ denotes the set of all linear mappings $\gamma : T_{\pi(y_0)}M \longrightarrow T_{y_0}E$ such that*

$$d_{y_0}\pi \circ \gamma = \mathrm{id}_{T_{\pi(y_0)}M}.$$

*If $JE = \left\{ (y, \gamma) \middle| y \in E \text{ and } \gamma \in J_yE \right\}$ then we will show that the mapping $\pi_E : JE \longrightarrow E$ defined by $\pi_E(y, \gamma) = y$ defines a fiber bundle structure. This fiber bundle is called the* first order jet bundle *of the fiber bundle $\pi : E \longrightarrow M$.*

**Theorem 1.2** *If $\pi : E \longrightarrow M$ is a fiber bundle and $\gamma \in J_yE$, then $T_yE = \mathrm{Im}\,\gamma \oplus \mathrm{Ker}\, d_y\pi$. Moreover there is a local section $s : U \longrightarrow E$ of $\pi$ such that $s(x) = y$ and $\gamma = d_x s$ where $x = \pi(y)$.*

*Proof.* First we show that $T_yE = \operatorname{Im}\gamma \oplus \operatorname{Ker} d_y\pi$. Let $w \in T_yE$ and notice that

$$\begin{aligned} d_y\pi\left(w - \gamma\left(d_y\pi(w)\right)\right) &= d_y\pi(w) - d_y\pi\left(\gamma\left(d_y\pi(w)\right)\right) \\ &= d_y\pi(w) - d_y\pi(w) = 0. \end{aligned}$$

So $w - \gamma\left(d_y\pi(w)\right) \in \operatorname{Ker} d_y\pi$ and if we let $k = w - \gamma\left(d_y\pi(w)\right)$ then $w = \gamma\left(d_y\pi(w)\right) + k \in \operatorname{Im}\gamma + \operatorname{Ker} d_y\pi$. Also observe that if $w \in \operatorname{Im}\gamma \cap \operatorname{Ker} d_y\pi$ then $w = \gamma(v)$ for some $v \in T_{\pi(y)}M$ and $d_y\pi(w) = 0$. Thus $v = d_y\pi\left(\gamma(v)\right) = d_y\pi(w) = 0$ and $w = \gamma(v) = 0$. So $w \in \operatorname{Im}\gamma \cap \operatorname{Ker} d_y\pi = 0$ and $T_yE = \operatorname{Im}\gamma \oplus \operatorname{Ker} d_y\pi$.

To prove the second part of the theorem, let $\left(\tilde{U}, x^\mu, y^a\right)$ denote an admissible chart of $E$ which is adapted to the fiber bundle structure $\pi : E \longrightarrow M$. If $(\bar{x}^\mu)$ is a chart of $M$ on $\pi\left(\tilde{U}\right)$ such that $x^\mu = \bar{x}^\mu \circ \pi$ then $\gamma$ has the property that

$$\gamma\left(\left.\frac{\partial}{\partial \bar{x}^\mu}\right|_x\right) = \left(\left.\frac{\partial}{\partial x^\mu}\right|_y\right) + \gamma_\mu^a\left(\left.\frac{\partial}{\partial y^a}\right|_y\right) \tag{1.1}$$

for some set of numbers $\gamma_\mu^a \in \mathbf{R}$. This follows ¿from the fact that $\gamma$ maps $T_xM$ to $T_yE$ and thus

$$\gamma\left(\left.\frac{\partial}{\partial \bar{x}^\mu}\right|_x\right) = a_\mu^\nu\left(\left.\frac{\partial}{\partial x^\nu}\right|_y\right) + \gamma_\mu^a\left(\left.\frac{\partial}{\partial y^a}\right|_y\right).$$

On the other hand the fact that $d_y\pi\left(\gamma_x\left(\left.\frac{\partial}{\partial \bar{x}^\mu}\right|_x\right)\right) = \left.\frac{\partial}{\partial \bar{x}^\mu}\right|_x$ implies that $a_\mu^\nu = \delta_\mu^\nu$ and (??) holds. To show that there exists a mapping $s$ satisfying the conclusion of the theorem we prescribe the components of $s$ in the chart $(x^\mu, y^a)$ and thus prescribe $s$ itself. We want $s(x) = y$ so we must require that$x^\mu(s(x)) = x^\mu(y)$ and $y^a(s(x)) = y^a(y)$ for all $1 \le \mu \le n, 1 \le a \le N$. Moreover we require that $d_xs = \gamma$ and since for every local section $s$

$$\begin{aligned} d_us\left(\left.\frac{\partial}{\partial \bar{x}^\nu}\right|_u\right) &= \partial_\nu\left(x^\mu \circ s \circ \bar{x}^{-1}\right)\left(\left.\frac{\partial}{\partial x^\mu}\right|_u\right) + \partial_\nu\left(y^a \circ s \circ \bar{x}^{-1}\right)\left(\left.\frac{\partial}{\partial y^a}\right|_u\right) \\ &= \left(\left.\frac{\partial}{\partial x^\nu}\right|_u\right) + \frac{\partial(y^a \circ s)}{\partial \bar{x}^\nu}\left(\left.\frac{\partial}{\partial y^a}\right|_u\right) \end{aligned}$$

we must have

$$\gamma_\nu^a = \frac{\partial(y^a \circ s)}{\partial \bar{x}^\nu}(x).$$

So, essentially we must prescribe the terms of the "first order Taylor polynomial" of $s$ in order to obtain the desired properties: $s(x) = y$, $d_x s = \gamma$. Thus we want $s$ to satisfy the conditions:

$$\begin{cases} x^\mu(s(u)) &= \bar{x}^\mu(\pi(s(u))) = \bar{x}^\mu(u) \\ y^a(s(u)) &= y^a(y) + \gamma^a_\nu\left(\bar{x}^\nu(u) - \bar{x}^\nu(x)\right). \end{cases} \tag{1.2}$$

When these conditions hold we see that

$$\begin{aligned} x^\mu(s(x)) &= \bar{x}^\mu(x) = \bar{x}^\mu(\pi(y)) = x^\mu(y), \\ y^a(s(x)) &= y^a(y) \end{aligned}$$

and so $s(x) = y$. Moreover

$$\begin{aligned} \frac{\partial(y^a \circ s)}{\partial \bar{x}^\mu} &= \partial_\mu\left(y^a \circ s \circ \bar{x}^{-1}\right) \\ &= \partial_\mu\left\{y^a(y) + \gamma^a_\nu\left((\bar{x}^\nu \circ \bar{x}^{-1}) - (\bar{x}^\nu \circ \bar{x}^{-1})(\bar{x}(x))\right)\right\} \\ &= \gamma^a_\nu \delta^\nu_\mu = \gamma^a_\mu. \end{aligned}$$

It follows that

$$\frac{\partial(y^a \circ s)}{\partial \bar{x}^\mu}(u) = \gamma^a_\mu$$

for all $u$ and thus for $u = x$. Finally observe that the equations (??) may indeed be imposed on $s$ on some open subset $U$ about $x$ lying in the image of $\pi(\tilde{U})$ since they hold at $u = x$ itself. The theorem follows.

**Definition 1.10** *If $\pi : E \longrightarrow M$ is a fiber bundle and $x_0 \in M$ then two local sections $s_1$ and $s_2$ of $\pi$, each defined on an open subset of $M$ containing $x_0$, are said to be* 1-jet equivalent *iff $s_1(x_0) = s_2(x_0)$ and $d_{x_0}s_1 = d_{x_0}s_2$. Thus $s_1$ and $s_2$ must define the same point $y_0 = s_1(x_0) = s_2(x_0)$ of $E$ and the same element $d_{x_0}s_1 = d_{x_0}s_2 \in J_{y_0}E$. Observe that for fixed $x_0 \in M$ the notion of 1-jet equivalence defines an equivalence relation on the set of all local sections of $\pi$ defined at $x_0$. Given such a local section $s$ denote the equivalence class determined by $s$ by $(js)(x_0)$. Observe that one obtains a mapping $js$ from the domain of $s$ into $JE$ and moreover that $(js)(x) = (s(x), d_x s)$ for each $x \in \operatorname{dom} s$. We show that the mappings $\pi_E : JE \longrightarrow E$ and $\pi_M : JE \longrightarrow M$ defined by $\pi_E(y, \gamma) = y$, $\pi_M(y, \gamma) = \pi(y)$ are fiber bundles and that $js$ is a local section of $\pi_M$ for each local section $s$ of $\pi : E \longrightarrow M$.*