MA 241 - SECTION 7, FINAL EXAM, INSTRUCTOR = JAMES COOK          NAME:

Work the problems on a seperate sheet, show your work and box your answers.

① (40pts.) Calculate the following integrals (Choose 8)

(a.) $\int x^3(1-x^4)\,dx$

(b.) $\int \frac{x+4}{x^2+5x+6}\,dx$

(c.) $\int \frac{1}{(x^2-1)^{3/2}}\,dx$

(d.) $\int_0^1 \sin(\pi x)\,dx$

(e) $\int x^5 \ln(x)\,dx$

(f) $\int \sin^3(2x+1)\,dx$

(g) $\int x \sin(x)\,dx$

(h) $\int \sec(x)\,dx$

(i) $\int \sqrt{a^2-x^2}\,dx$

(j) $\int 5^{2x}\,dx$

② (10pts) Find wether the improper integrals below converge or diverge. If it converges calculate the value it converges to. Explicitly show any limits involved in the integration.

(a.) $\int_0^\infty \frac{1}{x^2}\,dx$ (b.) $\int_0^\infty \left(\frac{1}{1+x^2}\right)dx$

③ (8pts) Find the length of the arc given by $y = \ln(\cos(x))$ for $0 \le x \le \pi/4$.

④ (12pts.) Find area of region bounded by $y = x$ and $y = x^2$. Then find the volume of solid obtained by rotating that region around $y$-axis.

⑤ (20pts.) Solve the differential equations below:

(a.) $\frac{dy}{dx} = e^{x+y}$ ; given $y(0) = 1$.

(b.) $\frac{dy}{dx} = \frac{\sin(x)}{y^4+y^2-23}$ ; find an implicit solⁿ and all equilibrium solⁿ's.

(c.) $y'' - y = \sin(x)$ ; find general solⁿ.

(d.) $\frac{d^2y}{dt^2} + 4y = 0$ ; given intial conditions $y(0) = 0$ and $y'(0) = 1$.

⑥ (20pts.) Use the known Maclaurin Series or binomial series to calculate

(a.) $\frac{1}{x}\sin(x)$ : find the complete power series expansion in $\Sigma$ notation.

(b.) $\cos^2(x)$ : find the first 3 non-zero terms in it's power series rep.

(c.) $\frac{x}{(1+3x)^7}$ : find the first 3 non-zero terms in it's power series rep.

(d.) $\int \frac{\sin(x)}{x}\,dx$ : use a to find a series solⁿ.

① a.) $\int x^3(1-x^4)\,dx = -\frac{1}{4}\int u\,du$

$u = 1-x^4$
$du = -4x^3\,dx$

$= -\frac{1}{8}u^2 + C$

$= \boxed{-\frac{1}{8}(1-x^4)^2 + C}$

b.) $\frac{x+4}{x^2+5x+6} = \frac{A}{x+3} + \frac{B}{x+2}$

$\Rightarrow x+4 = A(x+2) + B(x+3) \Rightarrow$ $x=-2 \,|\; 2 = B$; $x=-3 \,|\; 1 = -A$

$\int \frac{x+4}{x^2+5x+6}\,dx = \int \frac{-1}{x+3}\,dx + \int \frac{2}{x+2}\,dx$

$= \boxed{-\ln(x+3) + 2\ln(x+2) + C}$

c.) $\int \frac{1}{(x^2-1)^{3/2}}\,dx = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta}$

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$\sec^2\theta - 1 = \tan^2\theta$

$x = \sec\theta$
$dx = \sec\theta\tan\theta\,d\theta$
$(x^2-1)^{3/2} = (\tan^2\theta)^{3/2} = \tan^3\theta$

$= \int \frac{\sec\theta\,d\theta}{\tan^2\theta}$

$= \int \frac{1}{\cos\theta}\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$

$= \int \frac{\cos\theta\,d\theta}{\sin^2\theta}$

$u = \sin\theta$
$du = \cos\theta\,d\theta$

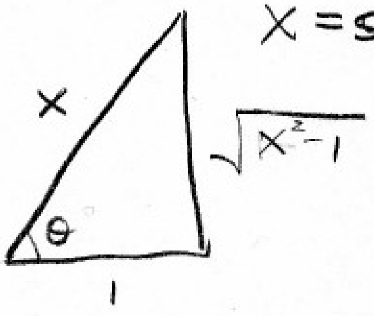
$= \int \frac{du}{u^2}$

$= \frac{-1}{u} + C$

$= \frac{-1}{\sin\theta} + C$

$= \boxed{\frac{-x}{\sqrt{x^2-1}} + C}$

$x = \sec\theta$



① d.) $\int_0^1 \sin(\pi x)\,dx = \int_0^{\pi} \frac{1}{\pi}\sin(u)\,du$

$u = \pi x$
$u(0) = 0$
$u(1) = \pi$
$du = \pi dx$

$= -\frac{1}{\pi}\cos(u)\Big|_0^{\pi}$

$= \frac{-1}{\pi}\left(\cos(\pi)^{-1} - \cos(0)^{1}\right)$

$= \boxed{\frac{2}{\pi}}$

e.) $\int x^5 \ln(x)\,dx = \frac{x^6}{6}\ln(x) - \int \frac{x^6}{6}\frac{dx}{x}$

$= \frac{1}{6}x^6\ln(x) - \frac{1}{6}\int x^5\,dx$

$= \boxed{\frac{1}{6}x^6\ln(x) - \frac{1}{36}x^6 + C}$

$u = \ln(x)$
$du = \frac{dx}{x}$
$dV = x^5 dx$
$V = \frac{x^6}{6}$

(f.) $\int \sin^3(2x+1)\,dx = \int \frac{1}{2}\sin^3(u)\,du$

$u = 2x+1$
$du = 2dx$

$= \frac{1}{2}\int (1-\cos^2(u))\sin(u)\,du$

$w = \cos(u)$
$dw = -\sin(u)du$

$= \frac{1}{2}\int (w^2 - 1)\,dw$

$= \frac{1}{2}\left(\frac{w^3}{3} - w\right) + C$

$= \boxed{\frac{1}{6}\cos^3(2x+1) - \frac{1}{2}\cos(2x+1) + C}$

g.) $\int x\sin(x)\,dx = -x\cos(x) + \int \cos(x)\,dx$

$= \boxed{-x\cos(x) + \sin(x) + C}$

$u = x$
$dV = \sin(x)dx$
$V = -\cos(x)$
$du = dx$

① h.) $\int \sec(x)dx = \int \frac{du}{u}$ 

$u = \sec(x) + \tan(x)$

$du = [\sec(x)\tan(x) + \sec^2(x)]dx$

$= \sec(x) u \, dx$

$\therefore \frac{du}{u} = \sec(x)dx$

$= \ln|u| + C$

$= \boxed{\ln|\sec(x) + \tan(x)| + C}$

i.) $\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2\cos^2\theta}\, a\cos\theta \, d\theta$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$a^2 - x^2 = a^2(1 - \sin^2\theta)$

$= a^2\cos^2\theta$

$= a^2 \int \cos^2\theta \, d\theta$

$= \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta$

$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$

$= \boxed{\frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{a}\right)\right)\right) + C}$

j.) $\int 5^{2x}\, dx = \frac{1}{2}\int 5^u \, du$

$u = 2x$

$du = 2dx$

$= \frac{1}{2\ln(5)} 5^u + C$

$= \boxed{\frac{1}{2\ln(5)} 5^{2x} + C}$

② a.) $\displaystyle\int_0^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to 0}\left(\int_t^1 \frac{1}{x^2}\,dx\right) + \lim_{s\to\infty}\left(\int_1^s \frac{1}{x^2}\,dx\right)$

$$= \lim_{t\to 0} \left.\frac{-1}{x}\right|_t^1 + \lim_{s\to\infty} \left.\frac{-1}{x}\right|_1^s$$

$$= \lim_{t\to 0} \underbrace{\left(-1+\frac{1}{t}\right)}_{\to\infty} + \lim_{s\to\infty} \underbrace{\left(\frac{-1}{s}+1\right)}_{\to 1.} \quad \therefore \text{ diverges.}$$

b.) $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_0^t \frac{1}{1+x^2}\,dx = \lim_{t\to\infty}\left(\tan^{-1}(t) - \tan^{-1}(0)\right) = \frac{\pi}{2}.$

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③ $y = \ln(\cos(x)) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos(x)}(-\sin(x)) = -\tan(x)$

$$s = \int_0^{\pi/4} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^{\pi/4} \sqrt{1+\tan^2(x)}\,dx$$

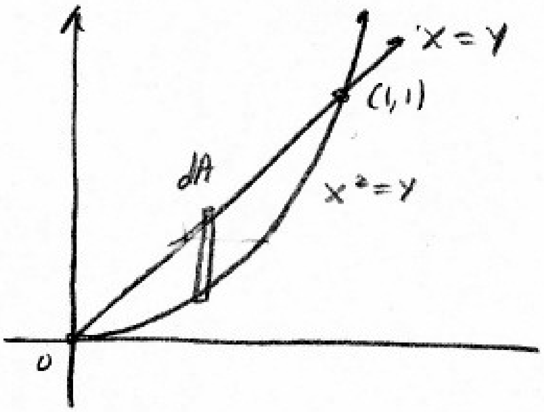
$$= \int_0^{\pi/4} \sec(x)\,dx$$

$$= \ln|\sec(x)+\tan(x)|\Big|_0^{\pi/4} \quad \text{using th.}$$

$$= \ln\left|\frac{1}{\cos(\pi/4)} + \tan(\pi/4)\right| - \underbrace{\cancel{\ln|\underbrace{\sec(0)}_{1} + \underbrace{\tan(0)}_{0}|}}_{\to 0}$$

$$= \boxed{\ln\left|\frac{2}{\sqrt{2}} + 1\right|}$$

④ Find area of region bounded by $y = x$ and $y = x^2$




$dA = (x - x^2)\,dx$

Intersection Points

$y = y$

$x = x^2$

$x - x^2 = 0$

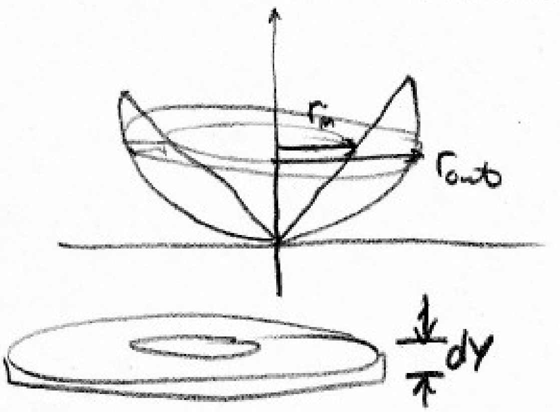
$x(1-x) = 0 \Rightarrow x = 0$

$x = 1$

where $y = 1$

$$A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$$

Find Volume obtained from rotating around $y$-axis




$r_{in} = x = y$

$r_{out} = x = \sqrt{y}$

$dV = \pi(r_{out}^2 - r_{in}^2)\,dy$

add washers from $y = 0$ to $y = 1$

$dV = \pi(y - y^2)\,dy$

$$V = \int_0^1 \pi(y - y^2)\,dy = \pi\left(\frac{y^2}{2} - \frac{y^3}{3}\right|_0^1 = \boxed{\frac{\pi}{6}}$$

(5) a.) $\frac{dy}{dx} = e^{x+y} \Rightarrow e^{-y}dy = e^{x}dx$

$$\Rightarrow -e^{-y} = e^{x} + c$$

$$\Rightarrow e^{-y} = k - e^{x}$$

$$\Rightarrow -y = \ln(k - e^{x})$$

$$\Rightarrow y = \ln\left(\frac{1}{k-e^{x}}\right)$$

Now $y(0) = 1 \Rightarrow -1 = \ln(k-1) \Rightarrow e^{-1} = k-1$
$\Rightarrow k = \frac{1}{e} + 1$

Thus $\boxed{y = \ln\left(\frac{1}{\frac{1}{e}+1-e^{x}}\right)}$

(b.) $\frac{dy}{dx} = \frac{\sin(x)}{y^4+y^2-23} \Rightarrow (y^4+y^2-23)dy = \sin(x)dx$

$$\Rightarrow \boxed{\frac{1}{5}y^5 + \frac{1}{3}y^3 - 23y = -\cos(x) + C}$$ implicit sol$^{n}$'s.

$$\frac{\sin(x)}{y^4+y^2-23} = 0 \Leftrightarrow \sin(x) = 0 \Leftrightarrow \boxed{x = n\pi, \; n \in \mathbb{Z}}$$ equilibrium sol$^{n}$'s

(c.) $y'' - y = \sin(x)$.

$\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \Rightarrow y_h = c_1 e^{x} + c_2 e^{-x}$

Thus $y_p = A\sin(x) + B\cos(x)$

$y_p' = A\cos(x) - B\sin(x)$

$y_p'' = -A\sin(x) - B\cos(x) = -y_p$

$$y_p'' - y_p = -y_p - y_p = -2A\sin(x) - 2B\cos(x) = \sin(x)$$

So we can read off $A = -\frac{1}{2}$ & $B = 0$. Thus the general sol$^{n}$ is

$$\boxed{y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2}\sin(x)}$$

(5) (d.) $\frac{d^2y}{dt^2} + 4y = 0$ with $y(0) = 0$ and $y'(0) = 1$

$\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow y = C_1 \cos(2t) + C_2 \sin(2t)$

$y' = -2C_1 \sin(2t) + 2C_2 \cos(2t)$

$y(0) = C_1 \cos(0) + C_2 \sin(0) = C_1 = 0$

$y'(0) = -2C_1 \sin(0) + 2C_2 \cos(0) = 2C_2 = 1 \quad \therefore \quad C_2 = 1/2$

Hence $\boxed{y = \frac{1}{2}\sin(2t)}$

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(6a) $$\frac{1}{x}\sin(x) = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{\frac{1}{x}x^{2n+1}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}}$$

(6b) $$\cos^2(x) = \frac{1}{2}(1+\cos(2x))$$

$$= \frac{1}{2}\left(1 + 1 - \frac{1}{2}(2x)^2 + \frac{1}{4!}(2x)^4 + \cdots\right)$$

$$= 1 - x^2 + \frac{1}{48}16x^4$$

$$= \boxed{1 - x^2 + \frac{1}{3}x^4}$$

(6c) $$\frac{x}{(1+3x)^7} = x(1+3x)^{-7} = x(1+u)^k = x(1 + ku + \tfrac{1}{2}k(k-1)u^2 + \cdots)$$

$$= x(1 - 7(3x) + \tfrac{1}{2}(-7)(-8)(3x)^2 + \cdots)$$

$$= \boxed{x - 21x^2 + \tfrac{1}{2}(56)\cdot 9x^3}$$

$\frac{56}{2} = 28 \cdot 9 = 252$

$$= \boxed{x^2 - 21x^2 + 252x^3}$$

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(6d) $$\int \frac{\sin(x)}{x}dx = \int \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int x^{2n}dx + C$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} + C}$$