HOMEWORK SOLUTION : FUNCTIONS

Taken from "Calculus Concepts and Contexts" 2nd Ed., STEWART. I remind you the answers to the odd #'d problems are in the book.

§1.2 #2 pg. 35	Classify the functions, give the name of the type, $Y = f(x)$ and f is the function in question for each of the below,

a.) $Y = \frac{X-6}{X+6}$ is a rational (and algebraic function.)

b.) $Y = X + \frac{X^2}{\sqrt{X-1}}$ is an algebraic function

c.) $Y = 10^x$ is an exponential function (also transcendental)

d.) $Y = X^{10}$ is a power function (and a 10^{th} deg. polynomial fnct.) and yes its algebraic as well.

e.) $Y = 2t^6 + t^4 - \pi$ is a 6^{th} degree polynomial function. (also algebraic)

f.) $Y = \cos\Theta + \sin\Theta$ is a trigonometric function (also transcendental)

I have underlined the best answer for each of theabove.

§1.3 #2 p. 46	Explain how to obtain the graphs below from $Y = f(x)$

a.) $Y = 5f(x)$: stretch $Y = f(x)$ vertically by factor of 5.

b.) $Y = f(x-5)$: shift $Y = f(x)$ 5 units to the RIGHT.

c.) $Y = -f(x)$: invert $Y = f(x)$ over the x-axis,

d.) $Y = -5f(x)$: invert $Y = f(x)$ over the x-axis then stretch that by 5 vert.

e.) $Y = f(5x)$: compress $Y = f(x)$ by a factor of 5.

f.) $Y = 5f(x) + 3$; stretch $Y = f(x)$ by factor of 5 vertically THEN shift that graph up by 3-units.

These ideas are very useful to construct new graphs from old graphs. I did not cover this in lecture because you are supposed to have covered it before. If not don't panick just look over the book and ask me about it in office hours. (this advice is applicable to many ?'s)

§1.3 #32 p. 48 find $f+g$, $f-g$, fg and f/g. Additionally state the domains of these functions. Let

$$f(x) = \sqrt{1+x} \qquad \text{(notice } 1+x \geq 0 \text{ for squareroot to give real \#)}$$
$$g(x) = \sqrt{1-x} \qquad (1-x \geq 0 \text{ for the sqrt. to give real \#})$$

Notice that $\text{dom}(f) = [-1, \infty)$ while $\text{dom}(g) = (-\infty, 1]$. Notice,

$$(f+g)(x) = f(x) + g(x) = \boxed{\sqrt{1+x} + \sqrt{1-x} = (f+g)(x)}$$

$$(f-g)(x) = f(x) - g(x) = \boxed{\sqrt{1+x} - \sqrt{1-x} = (f-g)(x)}$$

$$(fg)(x) = f(x) \cdot g(x) = \sqrt{1+x}\sqrt{1-x} = \sqrt{(1+x)(1-x)} = \boxed{\sqrt{1-x^2} = (fg)(x)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \boxed{\frac{\sqrt{1+x}}{\sqrt{1-x}} = \left(\frac{f}{g}\right)(x)}$$

The domains of $f+g$, $f-g$ and fg are all the same,

$$\text{dom}(f) \cap \text{dom}(g) = [-1, \infty) \cap (-\infty, 1] = \boxed{[-1, 1] = \text{dom}(f \pm g) = \text{dom}(fg)}$$

We must throw out $x = 1$ for f/g because $g(1) = \sqrt{1-1} = 0$ and we cannot ÷ by zero, thus $\boxed{\text{dom}(f/g) = [-1, 1)}$

- Basically you just have to avoid taking sqrt. of negative number and division by zero, thats all there is to domains.

§1.3 #40 p. 48 Let $f(x) = \frac{2}{x+1}$ and $g(x) = \cos(x)$ and $h(x) = \sqrt{x+3}$. Now find $f \circ g \circ h$.

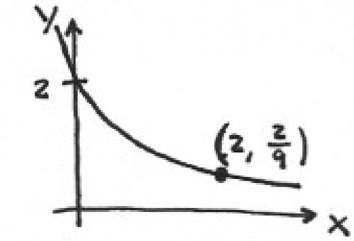
$$\begin{aligned}
(f \circ g \circ h)(x) &= f(g(h(x))) && : \text{definition of composite fncts.} \\
&= f(g(\sqrt{x+3})) && : \text{using } h(x) = \sqrt{x+3} \\
&= f(\cos(\sqrt{x+3})) && : \text{replacing } x \text{ with } \sqrt{x+3} \text{ in } g(x) = \cos(x). \\
&= \boxed{\frac{2}{\cos(\sqrt{x+3}) + 1}} && : \text{inserting } \cos\sqrt{x+3} \text{ for } x \text{ in } f(x) = \frac{2}{x+1}.
\end{aligned}$$

<u>Remark</u>: Don't confuse $f \circ g \circ h$ with fgh they're quite different,

$$(fgh)(x) = f(x)g(x)h(x) = \frac{2}{x+1}\cos(x)\sqrt{x+3}.$$

function multiplication and function composition are different.

§1.5 #16 p. 63 Assume the graph below is an exponential function, this means that $f(x)$ is of the form $f(x) = Ca^x$. Find C and a.



We can read from the graph where $Y = f(x)$ that

$$2 = f(0)$$
$$\tfrac{2}{9} = f(2)$$

But we also know that $f(x) = Ca^x$ thus

$$f(0) = Ca^0 = \boxed{C = 2}$$
$$f(2) = Ca^2 = 2a^2 = \tfrac{2}{9} \Longrightarrow a^2 = \tfrac{1}{9} \Rightarrow a = \pm\tfrac{1}{3}$$

Since its an exponential function we know $a > 0$ thus $\boxed{a = 1/3}$.
($f(x) = (-2)^x$ is very messy and discontinuous, need the base positive.)

§1.6 #2 p. 73 We answer the questions posed in text.

a.) Let f be a 1-1 function with $\text{dom}(f) = A$ and $\underbrace{f(A)}_{\text{range of } f.} = B$. Then we have the following picture: note $f^{-1}(B)$ is the range of f^{-1}

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

$\text{dom}(f^{-1}) = B$

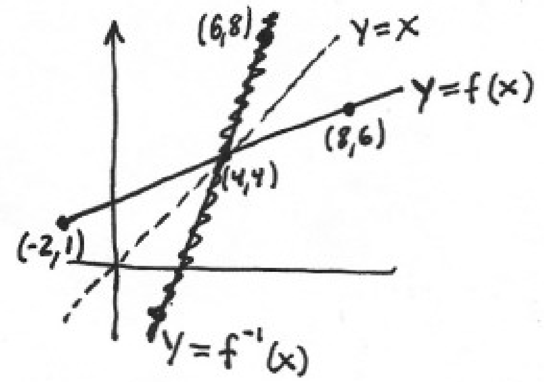
$f^{-1}(B) = f^{-1}(f(A)) = A$.

Where $f^{-1}(x)$ is defined by $f^{-1}(f(x)) = x$, and $f(f^{-1}(x)) = x$.

b.) Given $Y = f(x)$ I can find $f^{-1}(x)$ by the following steps, ex. $f(x) = 3x - 2$.

i.) write $x = f(Y)$	i.) $x = 3Y - 2$
ii.) Solve the above for $Y =$ stuff with x.	ii.) $Y = \frac{1}{3}(x+2)$
iii.) $f^{-1}(x) =$ stuff with x from ii.)	iii.) $f^{-1}(x) = \frac{1}{3}(x+2)$

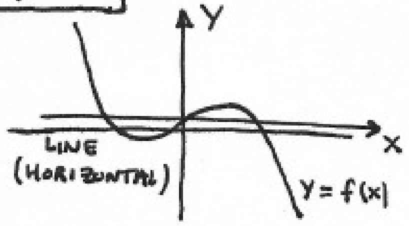
c.) You can construct $Y = f^{-1}(x)$'s graph by selecting a # of points on $Y = f(x)$ an flipping x & Y for the f^{-1} graph. For example:



I draw $Y = f(x)$, $Y = X$ and then construct $Y = f^{-1}(x)$ by flipping x & Y on f's graph.

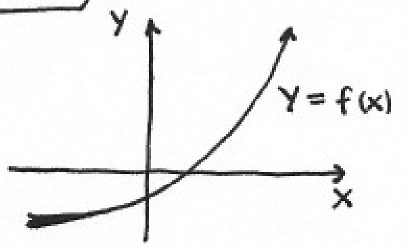
We say the graph of f^{-1} is obtained by reflecting the graph of f about $Y = X$.

§1.6 #6 p. 73 A function f has graph shown below is it one-to-one (1-1) ?



Clearly this function is <u>not</u> 1-1 because it fails the horizontal line test. As you can see the horiz. line intersects the graph of f at more than one point.

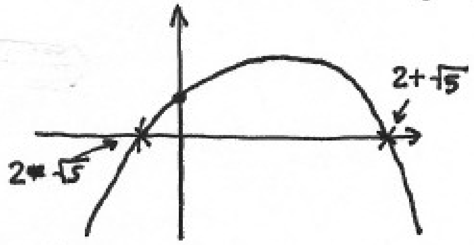
§1.6 # 8 p. 73 A function f has graph shown below is it 1-1 ?



Clearly this function is 1-1. No two x-values return the same Y - value for f(x). That is to say it passes the horiz. line. test. Equivalently, $f(a) = f(b) \implies a = b. \ \forall a, b \in \text{dom}(f)$.

§1.6 #10 p. 73 Let $f(x) = 1 + 4x - x^2$ is f one-to-one ?

Intuitively no, this is a parabola when graphed.



find roots by quad. eqⁿ, $f(x) = 0$;

$$x = \frac{-4 \pm \sqrt{16+4}}{-2} = 2 \pm \frac{\sqrt{20}}{2}$$

So the graph $y = f(x)$ intersects the x-axis at two points, $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$

Using the quadratic formula we see that

$$f(2 + \sqrt{5}) = 0 \quad \text{and} \quad f(2 - \sqrt{5}) = 0$$

So two different x-values map to the same Y-value, namely zero. Thus f is <u>not</u> 1-1.

§1.6 #12 p. 73 Let $f(x) = \sqrt{x}$ is f one-to-one ?

I think so, thus we let a and b be two <u>arbitrairy</u> #'s in the domain of f. (That just means $a, b \geq 0$, correct ?). Now set $f(a) = f(b)$ and try to <u>show</u> that $a = b$ thru <u>algebra</u>

$$f(a) = f(b) \implies \sqrt{a} = \sqrt{b}$$

$$\implies (\sqrt{a})^2 = (\sqrt{b})^2$$

$$\implies a = b.$$

were done. that proves f is one-to-one.

- Alternatively you could graph $y = \sqrt{x}$ and apply the horiz. line test.

§1.6 #22
p. 73

Let $f(v) \equiv \gamma \equiv \dfrac{1}{\sqrt{1-v^2/c^2}}$. Can you find $f^{-1}(v)$? Why is the book wrong on this one? We can fix it by simply changing the meaning of v from "velocity" to "speed", why?

Here's the problem, f is not 1-1 if we allow v to be positive and negative. For example $v = \pm c/2$ give same output,

$$f(c/2) = \frac{1}{\sqrt{1-(1/2)^2}} = \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}}$$

$$f(-c/2) = \frac{1}{\sqrt{1-(-1/2)^2}} = \frac{1}{\sqrt{3/4}} = \frac{2}{\sqrt{3}}$$

typically people allow velocity to be negative so this is a problem. However speed is the "magnitude" of the velocity and is positive or zero.
So define $f(v) = \dfrac{1}{\sqrt{1-v^2/c^2}}$ for $v \geq 0$ where v is the <u>speed</u>.
then find $f^{-1}(v)$. Following the advice of #26 where x becomes v and y becomes γ for this problem,

$$v = \frac{1}{\sqrt{1-\gamma^2/c^2}} \Rightarrow v^2 = \frac{1}{1-\gamma^2/c^2}$$

$$\Rightarrow v^2(1-\gamma^2/c^2) = 1$$

$$\Rightarrow \frac{-\gamma^2}{c^2}v^2 = 1 - v^2$$

$$\Rightarrow \gamma^2 = \frac{c^2(v^2-1)}{v^2}$$

$$\Rightarrow \gamma = \boxed{c\sqrt{\frac{v^2-1}{v^2}} = f^{-1}(v)}$$

<u>Lets check our answer:</u>

$$f(f^{-1}(v)) = \frac{1}{\sqrt{1-\frac{1}{c^2}\left(c\sqrt{\frac{v^2-1}{v^2}}\right)^2}}$$

$$= \left(\sqrt{1-\frac{v^2-1}{v^2}}\right)^{-1}$$

$$= \left(\sqrt{\frac{v^2-v^2+1}{v^2}}\right)^{-1}$$

$$= \left(\sqrt{1/v^2}\right)^{-1}$$

$$= \left(\frac{1}{v}\right)^{-1}$$

$$= v$$

<u>Remark</u>:
many physicsists would not say the mass changes with velocity. The real "mass" is m_o in the books problem not m. The quantity m_o does not change. Sorry, it just bugs me. It is γ not "m" which is important. In most books m_o is m.

§1.6 # 28 p. 73 Let $f(x) = \frac{1+e^x}{1-e^x}$ find the inverse for appropriately restricted values of x. (Let assume $x > 0$ is domain of f).

Begin by switching $y = f(x)$ to $x = f(y)$,

$$x = \frac{1+e^y}{1-e^y}$$

$$\Rightarrow (1-e^y)x = 1+e^y$$

$$\Rightarrow x - e^y x = 1 + e^y$$

$$\Rightarrow e^y(1+x) = x - 1$$

$$\Rightarrow e^y = \frac{x-1}{1+x}$$ (I can divide by $1+x$ because $x > 0$ means $1+x \neq 0$ correct?)

Now I have isolated y sort of, how do I get rid of the exponential though? Remember? Take natural log of both sides.

$$\ln(e^y) = \ln\left(\frac{x-1}{1+x}\right)$$

$$\therefore \quad y = \boxed{\ln\left(\frac{x-1}{1+x}\right) = f^{-1}(x)}$$

Bonus Point: find the domain of f^{-1} if $\mathrm{dom}(f) = (0, \infty)$

Chapter 1 Review #6 p. 84 | Examples of some function types (no unique answers!)

(You can give less general examples)

a.) $f(x) = mx + b$: is a "linear" function (m, b constants, $m \neq 0$)

b.) $f(x) = x^{\pi}$: is a power function

c.) $f(x) = \left(\frac{3}{2}\right)^x$: is an exponential function

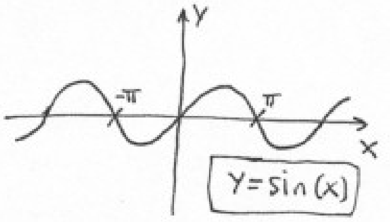
d.) $f(x) = Ax^2 + Bx + C$: is a quadratic function (A, B, C constants, $A \neq 0$)

e.) $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$: is a "quintic" function, aka its a polynomial of degree 5.

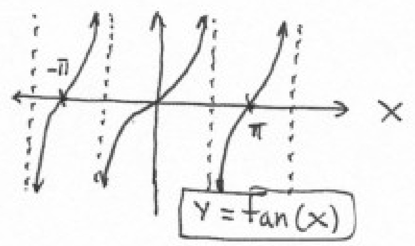
f.) $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials, is a rational function.

Chapter 1 Review #8 p. 84 | Draw a rough (but neat) sketch of the graph of each function

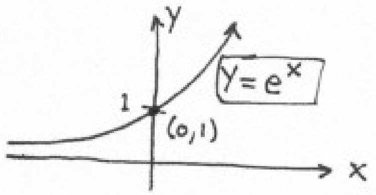
a.)



b.)

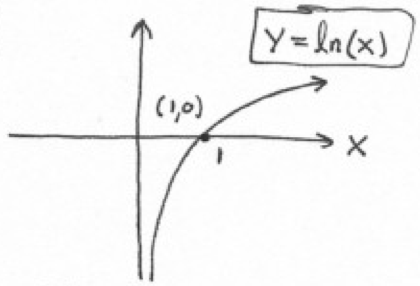


c.)



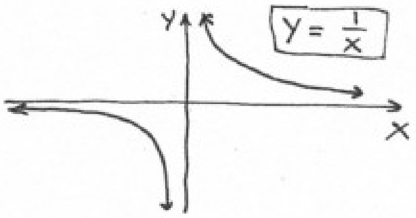
Notice ↕ are related. Just

d.)

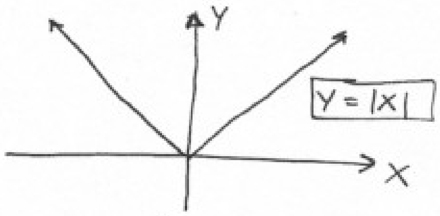


reflect across $Y = X$. Remember e^x and $\ln(x)$ are inverse functions.

e.)



f.)



$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Notice $|x| \geq 0 \;\; \forall x \in \mathbb{R}$.

g.)