GRAPHING HOMEWORK SOLn

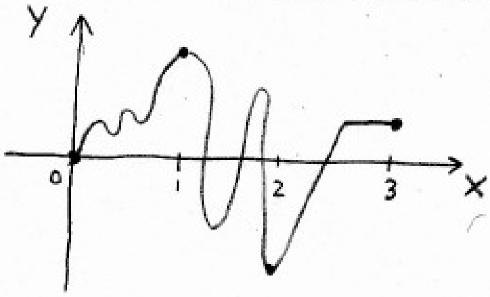
§4.2 #6 p.277

The absolute maximum is 5 at (7, 5)
The abs. min. is 0 at (1, 0)
Local max's at $x = 0$, $x = 3$, $x = 5$
Local min's at $x = 1$, $x = 4$, $x = 6$

(See text for graph)

§4.2 #8 p.277

Sketch a continuous function on $[0, 3]$ with an abs. max at 1 an abs. min at 2,



many possible answers!

§4.2 #24–28 p.277

Find the critical #'s of the function. Remember for fnct. f c is a critical # if $f'(c) = 0$ or $f'(c)$ d.n.e.

24.) $f(x) = x^3 + x^2 - x \Rightarrow f'(x) = 3x^2 + 2x - 1$

$$f'(x) = 0 \Rightarrow x = \frac{-2 \pm \sqrt{4 + 12}}{6} = \frac{-2 \pm 4}{6} = \frac{2}{6} \text{ or } \frac{-10}{6}$$

Thus the critical #'s are $\boxed{x = \frac{1}{3} \text{ OR } \frac{-5}{3}}$

26.) $g(t) = |3t - 4| = \begin{cases} 3t - 4 & t \geq 4/3 \\ 4 - 3t & t < 4/3 \end{cases}$

$g'(t) = \begin{cases} 3 & t > 4/3 \\ -3 & t < 4/3 \end{cases}$ Note $g'(0)$ d.n.e $\therefore$ $\boxed{0 \text{ is only critical } \#}$

28.) $f(z) = \dfrac{z+1}{z^2 + z + 1}$

$$f'(z) = \frac{(z^2 + z + 1) - (z+1)(2z+1)}{(z^2 + z + 1)^2} = \frac{z^2 + z + \cancel{1} - 2z^2 - 3z - \cancel{1}}{(z^2 + z + 1)^2}$$

Is $z^2 + z + 1 = 0$? Only when $z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$ so there are no real zeroes, thus $f'(z)$ exists for all $z \in \mathbb{R}$.

$$f'(z) = 0 \Leftrightarrow -z^2 - 2z = 0 \Rightarrow z(z+2) = 0$$

Thus the only critical #'s are $\boxed{z = 0 \text{ AND } -2}$

why did I check out the behaviour of the denominator of $f'(z)$?

§4.2 #30-34 p. 277-278	Find critical #'s of functions below

30.) $G(x) = \sqrt[3]{x^2 - x}$

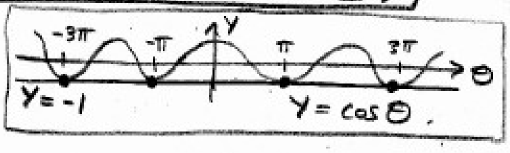
$$G'(x) = \frac{1}{3}(x^2-x)^{-\frac{2}{3}} \cdot (2x-1) = \frac{2x-1}{3(x^2-x)^{2/3}} = G'(x)$$

Notice $x^2 - x = x(x-1) = 0$ when $x = 0$ or $x = 1$, thus $G'(0)$ & $G'(1)$ d.n.e. Finally $2x-1 = 0 \Rightarrow x = \frac{1}{2}$ so critical #'s are $\boxed{0, \frac{1}{2}, 1}$

32.) $g(\theta) = \theta + \sin\theta \Rightarrow g'(\theta) = 1 + \cos\theta$ which is certainly defined for all $\theta \in \mathbb{R}$. Critical #'s come from $1 + \cos\theta = 0$ that is

$$\cos\theta = -1 \Rightarrow \boxed{\theta = \pi + 2\pi k \text{ for } k \in \mathbb{Z}.}$$

Meaning $\theta = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$



34.) $f(x) = xe^{2x}$

$f'(x) = e^{2x} + xe^{2x} \cdot 2 = e^{2x}(1+2x)$ for all $x \in \mathbb{R}$.

$f'(x) = 0 \Rightarrow 1 + 2x = 0 \Rightarrow \boxed{x = -\frac{1}{2} \text{ only critical \#}}$

§4.2 #36 p. 278	Find the absolute max/min for $f(x) = x^3 - 3x + 1$ on $[0,3]$

$f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ are critical #'s

Notice only $1 \in [0,3]$ of course $-1 \notin [0,3]$.

$$f(0) = 1$$
$$f(1) = 1 - 3 + 1 = -1$$
$$f(3) = 3^3 - 3(3) + 1 = 27 - 9 + 1 = 19$$

Thus by the "closed interval method" Thm we have (See 67 notes)

$$\boxed{\begin{array}{l} \text{absolute max is } 19 \text{ at } x = 3 \quad (\text{for } f \text{ on } [0,3]) \\ \text{absolute min is } -1 \text{ at } x = 1 \quad (\text{for } f \text{ on } [0,3]) \end{array}}$$

§4.2 # 38 p. 278	find abs. min/max of $f(x) = \sqrt{9-x^2}$ on $[-1, 2]$

$f'(x) = \dfrac{-2x}{2\sqrt{9-x^2}}$ $\Rightarrow$ critical #'s are $0, -3, 3$

only $0 \in [-1,2]$.

Evaluate them,

$$f(-1) = \sqrt{9-1} = \sqrt{8}$$
$$f(0) = \sqrt{9} = \sqrt{9} = 3$$
$$f(2) = \sqrt{9-4} = \sqrt{5}$$

Notice that f is continuous on $[-1,2]$ (the. V.A. are at $x = \pm 3$) thus by closed int. method we find

$\boxed{\text{abs. min is } \sqrt{5} \text{ at } x=2 \text{ \& abs. max is } 3 \text{ at } x=0}$

§4.2 #56a p. 279	Let $f(x) = ax^3 + bx^2 + cx + d$ where $a \neq 0$. (its a cubic fnct.) Show that a cubic can have one, two or no critical #'s in general. Give examples of such cubics.

$$f'(x) = 3ax^2 + 2bx + c \qquad \forall x \in \mathbb{R}$$

The critical #'s come from $3ax^2 + 2bx + c = 0$. This is a quad. eq$^{\underline{n}}$ with sol$^{\underline{n}}$,

$$x = \frac{-2b \pm \sqrt{\boxed{(2b)^2 - 4(3a)c}}}{6a} \longleftarrow \text{"discriminant"}$$

The sol$^{\underline{n}}$ can be 3 distinct types

1.) Distinct real roots, discriminant > 0

2.) Repeated real roots, discriminant $= 0$

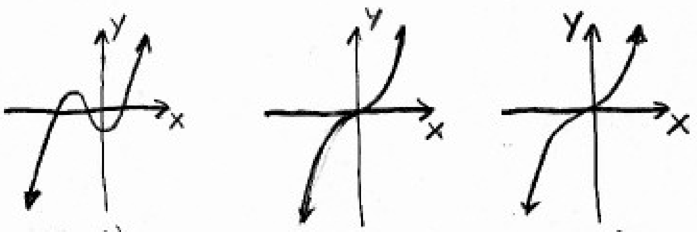
3.) Complex roots, discriminant < 0.

These correspond respectively to the possibilities

1.) 2 critical #'s

2.) 1 critical #

3.) no critical #.



case 1.) $Y = x^3 + x^2$

case 2.) $Y = x^3$

case 3.) $Y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x$

#6

a) $(2,4), (6,9)$ Because $f'(x)$ is positive in these intervals.

b) Local maximum: $x = 4$ Because $f'(x) = 0$ at this point, and $f'(x)$ changes from +ve to −ve thr. this pt.
Local minimum: $x = 2, 6$ Because $f'(x) = 0$ at these pts, and $f(x)$ changes from −ve to +ve thr. these pts.

c) Concave up: $(1,2), (5,7), (8,9)$ ∵ the slope of the graph of $f'(x)$ is positive. (or $f'(x)$ is increasing)
Concave down: $(0,1), (3,5), (7,8)$ ∵ the slope of the graph of $f'(x)$ is negative. (or $f'(x)$ is decreasing)

d) Inflection pts: $x = 1, 3, 5, 7, 8$ Because the slope of the graph of $f'(x)$ is zero, and the slope changes either from +ve to −ve or −ve to +ve thr. these pts.

#8 $f(x) = 1 + 8x - x^8$

$f'(x) = 8 - 8x^7$

$f''(x) = -56x^6$

a) f is increasing $\Rightarrow f' > 0 \Rightarrow 8 - 8x^7 > 0 \Rightarrow 8 > 8x^7 \Rightarrow 1 > x^7 \Rightarrow x < 1$.
f is decreasing $\Rightarrow f' < 0 \Rightarrow 8 - 8x^7 < 0 \Rightarrow x > 1$

b) Extremum $\Rightarrow f'(x) = 0 \Rightarrow 8 - 8x^7 = 0 \Rightarrow x = 1$ $f''(1) = -56 < 0 \Rightarrow x = 1, f(1) = 8$ $(1, 8)$ is the maximum.
there is no minimum for this function.

c) Concave up $\Rightarrow f''(x) > 0 \Rightarrow -56x^6 > 0 \Rightarrow$ no solution
concave down $\Rightarrow f''(x) < 0 \Rightarrow -56x^6 < 0 \Rightarrow x^6 > 0 \Rightarrow$ for all $x \Rightarrow$ graph is concave down for all x.
Since the graph is concave down for all x, there is no point of inflection.

#10 $f(x) = x/(1+x)^2$

$$f'(x) = \frac{(1+x)^2 - x(2)(1+x)}{(1+x)^4} = \frac{(1+x) - 2x}{(1+x)^3} = \frac{1-x}{(1+x)^3}$$

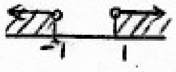
$$f''(x) = \frac{(1+x)^3(-1) - (1-x)(3)(1+x)^2}{(1+x)^6} = \frac{-(1+x) - (1-x)(3)}{(1+x)^4} = \frac{2x-4}{(1+x)^4}$$

a) f is increasing $\Rightarrow f' > 0 \Rightarrow \frac{(1-x)}{(1+x)^3} > 0 \Rightarrow$ either both ① $(1-x)$ & $(1+x) > 0$ OR both ② $(1-x)$ & $(1+x) < 0$

① $1 - x > 0 \Rightarrow x < 1$
$1 + x > 0 \Rightarrow x > -1$ $\Rightarrow$

$-1 < x < 1$

② $1 - x < 0 \Rightarrow x > 1$
$1 + x < 0 \Rightarrow x < -1$



No overlap $\Rightarrow$ No solution.

∴ f is increasing when $-1 < x < 1$.
f is decreasing when $x < -1$ or $x > 1$.

b) Extremum $\Rightarrow f'(x) = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1$
$f''(1) = \frac{-2}{2^4} = -\frac{1}{8} < 0 \Rightarrow x = 1, f(1) = \frac{1}{4}$ is a maximum.
There is no minimum for this function.

c) Concave up $\Rightarrow f''(x) > 0 \Rightarrow 2x - 4 > 0$ (∵ $(1+x)^4$ is alway non-negative, so we don't need to worry about it) $\Rightarrow x > 2$.
Concave down $\Rightarrow f''(x) < 0 \Rightarrow 2x - 4 < 0 \Rightarrow x < 2$.
Point of inflection $\Rightarrow f''(x) = 0 \Rightarrow x = 2, f(2) = \frac{2}{9}$ is the point of inflection.

#12 $f(x) = x^2e^x$; $f'(x) = 2xe^x + x^2e^x = xe^x(2+x)$; $f''(x) = e^x(2+4x+x^2)$ 50

a) $f(x)$ is increasing $\Rightarrow f'(x) > 0 \Rightarrow x(2+x) > 0$ ($\because e^x$ is also positive; we don't need to worry about it in deciding signs) $\Rightarrow x > 0$ or $x < -2$.

$f(x)$ is decreasing $\Rightarrow f'(x) < 0 \Rightarrow -2 < x < 0$.

b) Extremum $\Rightarrow f'(x) = 0 \Rightarrow x(2+x) = 0 \Rightarrow x = 0$ or $x = -2$

$f''(0) = 2 > 0 \Rightarrow x = 0, f(0) = 0$ is a local minimum

$f''(-2) = -2e^{-2} < 0 \Rightarrow x = -2, f(-2) = 4e^{-2}$ is a local maximum

c) Concave up $\Rightarrow f''(x) > 0 \Rightarrow 2+4x+x^2 > 0 \Rightarrow (x-(-2+\sqrt{2}))(x-(-2-\sqrt{2})) > 0 \Rightarrow x > -2+\sqrt{2}$ or $x < -2-\sqrt{2}$

Concave down $\Rightarrow f''(x) < 0 \Rightarrow 2+4x+x^2 < 0 \Rightarrow (x-(-2+\sqrt{2}))(x-(-2-\sqrt{2})) < 0 \Rightarrow -2-\sqrt{2} < x < -2+\sqrt{2}$

Point of inflection $\Rightarrow f''(x) = 0 \Rightarrow x = -2+\sqrt{2}$ or $-2-\sqrt{2}$

#15 $f(x) = x + \sqrt{1-x} \Rightarrow f'(x) = 1 - \frac{1}{2\sqrt{1-x}}$; $f''(x) = -\frac{1}{4(1-x)^{3/2}}$ Domain $x < 1$

$f'(x) = 0 \Rightarrow 1 = \frac{1}{2\sqrt{1-x}} \Rightarrow (1-x)^{\frac{1}{2}} = \frac{1}{2} \Rightarrow 1-x = \frac{1}{4} \Rightarrow x = \frac{3}{4}$.

1st derivative test: $f'(x) < 0$ when $1 > x > \frac{3}{4}$

$f'(x) > 0$ when $x < \frac{3}{4}$

$\therefore$ $x = \frac{3}{4}$, $f(\frac{3}{4}) = \frac{3}{4} + \sqrt{\frac{1}{4}} = \frac{5}{4}$ is a maximum.

2nd derivative test: $f''(\frac{3}{4}) = -\frac{1}{4(1-\frac{3}{4})^{3/2}} = -2 < 0 \Rightarrow (\frac{3}{4}, \frac{5}{4})$ is a maximum.

I prefer either test in my convenience. If $f''(x)$ is awefully hard to obtain, then I'll just go for 1st one. It really depends on what function you have. :)